Comparing non-standard axioms with axioms of second-order arithmetic

Keita Yokoyama

Mathematical Institute, Tohoku University

March 9, 2010

Outline

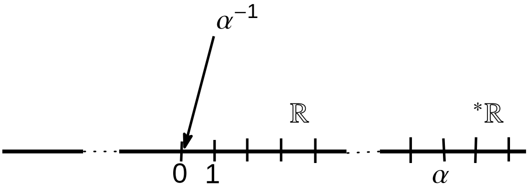
1. Introduction
   - Non-standard analysis
   - Related topics
   - Subsystems of second-order arithmetic
2. Non-standard second-order arithmetic
   - Non-standard second-order arithmetic
   - Axioms for non-standard analysis
   - Non-standard analysis in NSOA
3. Non-standard axioms and Reverse Mathematics
   - Reverse Mathematics for non-standard analysis
   - Transfer principle
   - Standard part principle

Outline

Introduction

Non-standard analysis was introduced by Abraham Robinson in 1960s (based on model theory).

- Expanding the universe ($\mathbb{N} \subseteq \mathbb{N}^*$, $\mathbb{R} \subseteq \mathbb{R}^*$), we can use infinitesimals (infinitely large and small numbers).



Introduction

Example (Bolzano Weierstraß theorem).
Let $\langle a_n \mid n \in \mathbb{N}\rangle$ be a real sequence.
Let $\langle a^*_n \mid n \in \mathbb{N}^*\rangle$ be the non-standard expansion of $\langle a_n \mid n \in \mathbb{N}\rangle$.
Then, for any infinitely large number $\omega \in \mathbb{N}^* \setminus \mathbb{N}$, there exists a subsequence $\langle a_{n_i} \mid i \in \mathbb{N}\rangle$ which converges to $r := \mathrm{st}(a^*_\omega)$.

We can do mathematics only by using bounded formulas or less complicated ($\Sigma^0_1 \cup \Pi^0_1$) formulas.

Related topics

- Comparing axioms of non-standard arithmetic and second-order arithmetic (Keisler, Henson, Kaufmann,. . .).
- Comparing axioms of non-standard arithmetic and weak axioms of arithmetic (Inpense, Sanders).
- (Model theoretic) non-standard arguments for reverse mathematics in WKL_0 and ACA_0 (Tanaka, Yamzaki, Sakamoto, Y).
- Reverse Mathematics for analysis.

Question

Which axioms are essentially needed for non-standard analysis?

language of second order arithmetic ($\mathcal{L}_2$)

Definition (language of second order arithmetic)

number variables: $x, y, z, \ldots$

set variables: $X, Y, Z, \ldots$

constant symbols: $0, 1$

function symbols: $+, \cdot$

relation symbols: $=, <, \in$

Classes of formulas

bounded formula: all quantifiers are of the form $\forall x < y$, $\exists x < y$.

arithmetical formulas:(θ: bounded formula)

- Σ^0_n formula: $\exists x_1 \forall x_2 \ldots x_n \theta$
- Π^0_n formula: $\forall x_1 \exists x_2 \ldots x_n \theta$

analytic formula:(φ: arithmetical formula)

- Σ^1_n formula:$\exists X_1 \forall X_2 \ldots X_n \varphi$
- Π^1_n formula:$\forall X_1 \exists X_2 \ldots X_n \varphi$

Subsystems of second-order arithmetic

- RCA_0: "discrete ordered semi ring"+Σ^0_1 induction+recursive comprehension.
- WWKL_0: RCA_0 + weak weak König's lemma.
- WKL_0: RCA_0 + weak König's lemma.
- ACA_0: RCA_0 + arithmetical comprehension.
- ATR_0: RCA_0 + arithmetical transfinite recursion.
- $\Pi^1_1\mathsf{CA}_0$: RCA_0 + Π^1_1 comprehension.

Outline

Language of non-standard second order arithmetic

Definition (Language $\mathcal{L}_2^*$)

Language of non-standard second order arithmetic ($\mathcal{L}_2^*$) are the following:

- s number variables: $x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots,$
- $*$ number variables: $x^*, y^*, \ldots,$
- s set variables: $X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots,$
- $*$ set variables: $X^*, Y^*, \ldots,$
- s symbols: $0^{\mathrm{s}}, 1^{\mathrm{s}}, =^{\mathrm{s}}, +^{\mathrm{s}}, \cdot^{\mathrm{s}}, <^{\mathrm{s}}, \in^{\mathrm{s}},$
- $*$ symbols: $0^*, 1^*, =^*, +^*, \cdot^*, <^*, \in^*,$
- function symbol: $\surd$.

s-structure and $*$-structure

M^{s}: range of $x^{\mathrm{s}}, y^{\mathrm{s}}, \ldots$,
M^{*}: range of $x^{*}, y^{*}, \ldots$,
S^{s}: range of $X^{\mathrm{s}}, Y^{\mathrm{s}}, \ldots$,
S^{*}: range of $X^{*}, Y^{*}, \ldots$.

$V^{\mathrm{s}} = (M^{\mathrm{s}}, S^{\mathrm{s}}; 0^{\mathrm{s}}, 1^{\mathrm{s}}, \ldots)$: s-$\mathcal{L}_2$ structure.
$V^{*} = (M^{*}, S^{*}; 0^{*}, 1^{*}, \ldots)$: $*$-$\mathcal{L}_2$ structure.
$\surd : M^{\mathrm{s}} \cup S^{\mathrm{s}} \to M^{*} \cup S^{*}$: embedding.

We usually regard M^{s} as a subset of M^{*}.

Notations:

Let φ be an $\mathcal{L}_2$-formula.

- φ^{s} : $\mathcal{L}_2^*$ formula constructed by adding $^{\mathrm{s}}$ to any $\mathcal{L}_2$ symbols in φ.
- φ^* : $\mathcal{L}_2^*$ formula constructed by adding * to any $\mathcal{L}_2$ symbols in φ.
- $\check{x}^{\mathrm{s}} := \surd(x^{\mathrm{s}})$.
- $\check{X}^{\mathrm{s}} := \surd(X^{\mathrm{s}})$.

We usually omit $^{\mathrm{s}}$ and * of relations $=, \leq, \in$.

We often say "φ holds in V^{s} (in V^*)" when φ^{s} (φ^*) holds.

classes of formulas

Σ_0^S formula: all quantifiers are of the form $\forall x^s < y^s$, $\exists x^s < y^s$, $\forall x^* < y^*$, $\exists x^* < y^*$.

S formulas:(θ: Σ_0^S formula)

- Σ_n^S formula: $\exists x_1^s \forall x_2^s \ldots x_n^s \theta$
- Π_n^S formula: $\forall x_1^s \exists x_2^s \ldots x_n^s \theta$

$\dagger$ formula:(φ: S formula)

- $\Sigma_n^\dagger$ formula:$\exists x_1^* \forall x_2^* \ldots x_n^* \varphi$
- $\Pi_n^\dagger$ formula:$\forall x_1^* \exists x_2^* \ldots x_n^* \varphi$

Axioms for non-standard analysis

Definition (Saturation principles)

$$\Sigma^i_j\mathrm{WSAT}^0: \quad (\forall x^{\mathrm{s}}\exists y^{*}\varphi(\check{x}^{\mathrm{s}},y^{*})^{*} \to \exists y^{*}\forall x^{\mathrm{s}}\varphi(\check{x}^{\mathrm{s}},y^{*})^{*})$$

for any Σ^i_j formula $\varphi(x,y)$.

$$\Sigma^i_j\mathrm{WSAT}^1: \quad (\forall x^{\mathrm{s}}\exists Y^{*}\varphi(\check{x}^{\mathrm{s}},Y^{*})^{*} \to \exists Y^{*}\forall x^{\mathrm{s}}\varphi(\check{x}^{\mathrm{s}},Y^{*})^{*})$$

for any Σ^i_j formula $\varphi(x,Y)$.

Definition (Transfer principles)

$$\Sigma^i_j\mathrm{EQ}: \quad (\varphi^{\mathrm{s}} \leftrightarrow \varphi^{*})$$

for any Σ^i_j sentence φ.

$$\Sigma^i_j\mathrm{TP}: \quad \forall x^{\mathrm{s}}\forall X^{\mathrm{s}}(\varphi(x^{\mathrm{s}},X^{\mathrm{s}})^{\mathrm{s}} \leftrightarrow \varphi(\check{x}^{\mathrm{s}},\check{X}^{\mathrm{s}})^{*})$$

for any Σ^i_j formula $\varphi(x,X)$.

Definition (Standard part principles)

$$\mathrm{fst}: \quad \forall X^*(\mathrm{card}(X^*) \in M^{\mathrm{s}} \rightarrow \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}}(x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in X^*).$$

$$\mathrm{ST}: \quad \forall X^* \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}}(x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \check{x}^{\mathrm{s}} \in X^*).$$

$$\mathrm{LMP}: \quad \forall H^* \in \mathbb{N}^* \setminus \mathbb{N}^{\mathrm{s}} \, \forall T^* \subseteq 2^{<H^*} \; \mathrm{st}\left(\frac{\mathrm{card}(\{\sigma^* \in T^* \mid \mathrm{lh}(\sigma^*) = H^*\})}{2^{H^*}}\right) > 0 \rightarrow \exists \sigma^* \in T^* \mathrm{lh}(\sigma^*) = H^* \wedge \sigma^* \cap \mathbb{N}^{\mathrm{s}} \in V^{\mathrm{s}}.$$

(An NS-tree which has a positive measure has a standard path.)

Definition (Basic axioms)

$$\mathrm{emb}: \quad \text{“}\surd \text{ is an injective homomorphism”}.$$

$$\mathrm{e}: \quad \forall x^* \forall y^{\mathrm{s}}(x^* < \check{y}^{\mathrm{s}} \rightarrow \exists z^{\mathrm{s}}(x^* = \check{z}^{\mathrm{s}})).$$

HKK transformation

Definition (Henson, Kaufmann, Keisler)

Let φ be an $\mathcal{L}_2$ formula. Then, $\varphi^\dagger$ is an $\mathcal{L}_2^*$ formula obtained by replacing each x_i by x_i^s, each X_i by x_i^*, and

$$x \in Y \mapsto x^s \in \mathrm{code}(y^*)$$

where $\mathrm{code}(a)$ is a finite set coded by a.

$\dagger$ maps Σ^0_j formulas to Σ^S_j formulas, and Σ^1_j formulas to $\Sigma^\dagger_j$ formulas.

We define

$$\Sigma^\dagger_j\text{-CA} = (\Sigma^1_j\text{-CA})^\dagger, \qquad \Sigma^\dagger_j\text{-SEP} = (\Sigma^1_j\text{-SEP})^\dagger, \ldots$$

Systems of non-standard second-order arithmetic

We define systems of non-standard second order arithmetic as follows.

Definition

$$\mathsf{RCA_0}^{\mathsf{ns}} = (\mathsf{RCA_0})^s + (\mathsf{RCA_0})^* + \mathrm{emb} + \mathrm{e} + \mathrm{fst} + \Sigma^0_1\mathrm{WSAT}^0 + \Sigma^1_1\mathrm{EQ} + \Sigma^0_0\mathrm{TP}.$$

$$\mathsf{WWKL_0}^{\mathsf{ns}} = \mathsf{RCA_0}^{\mathsf{ns}} + \mathrm{LMP}.$$

$$\mathsf{WKL_0}^{\mathsf{ns}} = \mathsf{RCA_0}^{\mathsf{ns}} + \mathrm{ST}.$$

$$\mathsf{ACA_0}^{\mathsf{ns}} = \mathsf{RCA_0}^{\mathsf{ns}} + \mathrm{ST} + \Sigma^1_1\mathrm{TP} + \Sigma^1_0\mathrm{WSAT}^0.$$

$$\mathsf{ATR_0}^{\mathsf{ns}} = \mathsf{ACA_0}^{\mathsf{ns}} + \Sigma^{\dagger}_1\mathrm{SEP} + \Sigma^1_2\mathrm{TP}.$$

$$\Pi^1_1\mathsf{CA_0}^{\mathsf{ns}} = \mathsf{ACA_0}^{\mathsf{ns}} + \Sigma^{\dagger}_1\mathrm{CA} + \Sigma^1_2\mathrm{TP} + \Sigma^1_1\mathrm{WSAT}^1.$$

Non-standard analysis in NSOA

- Within ${\mathsf{RCA}_0}^{\mathrm{ns}}$, we can define real numbers, open sets, continuous functions, complete separable metric spaces,... in both s-structure and $*$-structure.
- We can also define ‘non-standard concepts’ such as standard part, monad, s-continuous, ...
- Then, ...

- $\mathrm{WWKL}_0{}^{\mathrm{ns}}$ proves:
  - Lebesgue measure is the standard part of Loeb measure.
  - Riemann integral can be infinitesimally approximated by hyperfinite Riemann sum.
- $\mathrm{WKL}_0{}^{\mathrm{ns}}$ proves:
  - Every continuous function can be infinitesimally approximated by hyperfinite broken line.
  - Every compact separable metric space is a standard part of a non-standard metric space whose points are all standard.

$\vdots$

- $\mathrm{ACA_0}^{\mathrm{ns}}$ proves:
  - If $\{a_n^s\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a_\omega^*)$ is an accumulation value of $\{a_n^s\}$.
  - If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f_\omega^*)$ is an accumulation value of $\{f_n^s\}$.
- $\Pi^1_1\mathrm{CA_0}^{\mathrm{ns}}$ proves:
  - For every closed set $\mathbb{C}$ of a complete separable metric space, there exists a non-standard sequence $\{c_n^*\}$ such that for any non-standard ω, $\mathrm{st}(\{c_i^* \mid i < \omega\}) = C$.

$\vdots$

Conservativity

Moreover,

Theorem

T^{ns} is a conservative extension of T for $T \in \{\mathrm{RCA}_0, \mathrm{WWKL}_0, \mathrm{WKL}_0, \mathrm{ACA}_0, \mathrm{ATR}_0, \Pi^1_1\mathrm{CA}_0\}$, i.e., $T^{\mathrm{ns}} \vdash \varphi^{\mathrm{s}}$ implies $T \vdash \varphi$ for any $\mathcal{L}_2$ sentence φ.

By this theorem, we can use non-standard analysis for standard analysis in SOA.

Outline

Question

Which axioms are essentially needed for non-standard analysis?

Reverse Mathematics for non-standard analysis

- $\mathsf{WWKL}_0{}^{\mathrm{ns}}$ proves:
  - Lebesgue measure is the standard part of Loeb measure.
  - Riemann integral can be infinitesimally approximated by hyperfinite Riemann sum.

Reverse Mathematics for non-standard analysis

- Within $\mathsf{RCA_0}^{ns}$,
  - "Lebesgue measure is the standard part of Loeb measure" $\Leftrightarrow \mathsf{WWKL_0}^{ns}$.
  - "Riemann integral can be infinitesimally approximated by hyperfinite Riemann sum" $\Leftrightarrow \mathsf{WWKL_0}^{ns}$.

This means that LMP is equivalent to "approximation for measure and integral".

Reverse Mathematics for non-standard analysis

- Within $\mathsf{RCA}_0{}^{\mathrm{ns}}$,
  - "Lebesgue measure is the standard part of Loeb measure" $\Leftrightarrow$ $\mathsf{WWKL}_0{}^{\mathrm{ns}}$.
  - "Riemann integral can be infinitesimally approximated by hyperfinite Riemann sum" $\Leftrightarrow$ $\mathsf{WWKL}_0{}^{\mathrm{ns}}$.

This means that LMP is equivalent to "approximation for measure and integral".

Reverse Mathematics for non-standard analysis

- $\mathrm{WKL_0}^{\mathrm{ns}}$ proves:
  - Every continuous function can be infinitesimally approximated by hyperfinite broken line.
  - Every compact separable metric space is a standard part of a non-standard metric space whose points are all standard.

Reverse Mathematics for non-standard analysis

- Within $\mathrm{RCA_0}^{\mathrm{ns}}$,
  - Every continuous function can be infinitesimally approximated by hyperfinite broken line.
    $\Leftrightarrow$ $\mathrm{WKL_0}^{\mathrm{ns}}$.
  - Every compact separable metric space is a standard part of a non-standard metric space whose points are all standard.
    $\Leftrightarrow$ $\mathrm{WKL_0}^{\mathrm{ns}}$.

This means that ST is equivalent to “approximation for continuous functions”.

Reverse Mathematics for non-standard analysis

- Within $\mathrm{RCA_0}^{\mathrm{ns}}$,
  - Every continuous function can be infinitesimally approximated by hyperfinite broken line.
    $\Leftrightarrow \mathrm{WKL_0}^{\mathrm{ns}}$.
  - Every compact separable metric space is a standard part of a non-standard metric space whose points are all standard.
    $\Leftrightarrow \mathrm{WKL_0}^{\mathrm{ns}}$.

This means that ST is equivalent to “approximation for continuous functions”.

Reverse Mathematics for non-standard analysis

- $\mathsf{ACA_0}^{\mathrm{ns}}$ proves:
  - If $\{a_n^s\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a_\omega^*)$ is an accumulation value of $\{a_n^s\}$.
  - If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f_\omega^*)$ is an accumulation value of $\{f_n^s\}$.
- $\Pi_1^1\mathsf{CA_0}^{\mathrm{ns}}$ proves:
  - For every closed set $\mathbb{C}$ of a complete separable metric space, there exists a non-standard sequence $\{c_n^*\}$ such that for any non-standard ω, $\mathrm{st}(\{c_i^* \mid i < \omega\}) = C$.

Reverse Mathematics for non-standard analysis?

- Within $\mathrm{RCA_0}^{\mathrm{ns}}$,
  - "If $\{a_n^s\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a_\omega^*)$ is an accumulation value of $\{a_n^s\}$"
    $\not\Rightarrow \mathrm{ACA_0}^{\mathrm{ns}}$.
  - "If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f_\omega^*)$ is an accumulation value of $\{f_n^s\}$"
    $\not\Rightarrow \mathrm{ACA_0}^{\mathrm{ns}}$.
- Within $\mathrm{RCA_0}^{\mathrm{ns}}$,
  - For every closed set $\mathbb{C}$ of a complete separable metric space, there exists a non-standard sequence $\{c_n^*\}$ such that for any non-standard ω, $\mathrm{st}(\{c_i^* \mid i < \omega\}) = C$.
    $\not\Rightarrow \Pi^1_1\mathrm{CA_0}^{\mathrm{ns}}$.

Reverse Mathematics for non-standard analysis?

- Within $\mathrm{RCA_0}^{\mathrm{ns}}$,
  - "If $\{a_n^{\mathrm{s}}\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a^*_\omega)$ is an accumulation value of $\{a_n^{\mathrm{s}}\}$"
    $\not\Rightarrow \mathrm{ACA_0}^{\mathrm{ns}}$.
  - "If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f^*_\omega)$ is an accumulation value of $\{f_n^{\mathrm{s}}\}$"
    $\not\Rightarrow \mathrm{ACA_0}^{\mathrm{ns}}$.

Why?
$\Rightarrow$ These are provable within $\mathrm{WKL_0}^{\mathrm{ns}} + \Sigma^1_0\mathrm{TP}$, but $\Sigma^1_1\mathrm{TP}$ is not provable from $\mathrm{WKL_0}^{\mathrm{ns}} + \Sigma^1_0\mathrm{TP}$.
In fact, $\Sigma^1_1\mathrm{TP}$ is non-standard over $\mathrm{WKL_0}^{\mathrm{ns}} + \Sigma^1_0\mathrm{TP}$, *i.e.*, $\mathrm{WKL_0}^{\mathrm{ns}} + \Sigma^1_0\mathrm{TP} + (\mathrm{TA}^2)^{\mathrm{s}}$ does not prove $\Sigma^1_1\mathrm{TP}$.

Reverse Mathematics for non-standard analysis?

- Within ${\mathsf{RCA}_0}^{\mathsf{ns}}$,
  - "If $\{a_n^{\mathrm{s}}\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a_\omega^*)$ is an accumulation value of $\{a_n^{\mathrm{s}}\}$"
    $\not\Rightarrow {\mathsf{ACA}_0}^{\mathsf{ns}}$.
  - "If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f_\omega^*)$ is an accumulation value of $\{f_n^{\mathrm{s}}\}$"
    $\not\Rightarrow {\mathsf{ACA}_0}^{\mathsf{ns}}$.

Why?
$\Rightarrow$ These are provable within ${\mathsf{WKL}_0}^{\mathsf{ns}} + \Sigma^1_0\mathsf{TP}$, but $\Sigma^1_1\mathsf{TP}$ is not provable from ${\mathsf{WKL}_0}^{\mathsf{ns}} + \Sigma^1_0\mathsf{TP}$.
In fact, $\Sigma^1_1\mathsf{TP}$ is non-standard over ${\mathsf{WKL}_0}^{\mathsf{ns}} + \Sigma^1_0\mathsf{TP}$, *i.e.*, ${\mathsf{WKL}_0}^{\mathsf{ns}} + \Sigma^1_0\mathsf{TP} + (\mathsf{TA}^2)^{\mathrm{s}}$ does not prove $\Sigma^1_1\mathsf{TP}$.

Reverse Mathematics for non-standard analysis?

- Within ${\sf RCA_0}^{\rm ns}$,
  - "If $\{a_n^s\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a_\omega^*)$ is an accumulation value of $\{a_n^s\}$"
    $\not\Rightarrow$ ${\sf ACA_0}^{\rm ns}$.
  - "If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f_\omega^*)$ is an accumulation value of $\{f_n^s\}$"
    $\not\Rightarrow$ ${\sf ACA_0}^{\rm ns}$.

Why?
$\Rightarrow$ These are provable within ${\sf WKL_0}^{\rm ns} + \Sigma^1_0$TP, but Σ^1_1TP is not provable from ${\sf WKL_0}^{\rm ns} + \Sigma^1_0$TP.
In fact, Σ^1_1TP is non-standard over ${\sf WKL_0}^{\rm ns} + \Sigma^1_0$TP, *i.e.*, ${\sf WKL_0}^{\rm ns} + \Sigma^1_0TP+({\sf TA}^2)^s$ does not prove Σ^1_1TP.

Transfer principle

We need to compare TP with axioms of second-order arithmetic. However, . . .

Theorem

$\mathrm{WKL_0}^{\mathrm{ns}} + \Sigma^1_0\mathrm{TP} + (\mathrm{TA}^2)^{\mathrm{s}}$ *does not prove* $\Sigma^1_1\mathrm{TP}$.

Theorem

1. *(Impense, Sanders)* $\mathrm{PRA} \vdash \mathrm{Con}(\mathrm{ERNA} + \Sigma^0_2\mathrm{TP})$.
   (ERNA*: Elementary Recursive Non-standard Analysis)*
2. $\mathrm{RCA_0}^{\mathrm{ns}} + \mathrm{ST} + \Sigma^0_2\mathrm{TP} \vdash \mathrm{ACA_0}$.

Transfer principle

Similarly,

Theorem

1. ${\mathrm{RCA}_0}^{\mathrm{ns}} + \mathrm{ST} + \Sigma^1_2\mathrm{TP} \equiv_{\Pi^1_2} \mathrm{ACA}_0$.
2. ${\mathrm{RCA}_0}^{\mathrm{ns}} + \mathrm{ST} + \Sigma^1_2\mathrm{TP} + (\mathrm{I}\Sigma^1_1)^s \vdash \Pi^1_1\mathrm{CA}_0$.

It seems to be impossible to compare the strength of TP with axioms of second-order arithmetic.

(As far as I know,) there is no theorem in non-standard analysis which implies some transfer principle.

Transfer principle

Similarly,

Theorem

1. $\mathrm{RCA_0}^{\mathrm{ns}} + \mathrm{ST} + \Sigma^1_2\mathrm{TP} \equiv_{\Pi^1_2} \mathrm{ACA_0}$.
2. $\mathrm{RCA_0}^{\mathrm{ns}} + \mathrm{ST} + \Sigma^1_2\mathrm{TP} + (\mathrm{I}\Sigma^1_1)^s \vdash \Pi^1_1\mathrm{CA_0}$.

It seems to be impossible to compare the strength of TP with axioms of second-order arithmetic.

(As far as I know,) there is no theorem in non-standard analysis which implies some transfer principle.

Transfer principle

Similarly,

Theorem

1. $\mathsf{RCA_0}^{\mathsf{ns}} + \mathrm{ST} + \Sigma^1_2\mathrm{TP} \equiv_{\Pi^1_2} \mathsf{ACA_0}$.
2. $\mathsf{RCA_0}^{\mathsf{ns}} + \mathrm{ST} + \Sigma^1_2\mathrm{TP} + (\mathrm{I}\Sigma^1_1)^{\mathrm{s}} \vdash \Pi^1_1\mathsf{CA_0}$.

It seems to be impossible to compare the strength of TP with axioms of second-order arithmetic.
(As far as I know,) there is no theorem in non-standard analysis which implies some transfer principle.

Reverse Mathematics for non-standard analysis?

- Within ${\mathrm{RCA}_0}^{\mathrm{ns}}$,
  - "If $\{a^s_n\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a^*_\omega)$ is an accumulation value of $\{a^s_n\}$"
    $\not\Rightarrow {\mathrm{ACA}_0}^{\mathrm{ns}}$.
  - "If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f^*_\omega)$ is an accumulation value of $\{f^s_n\}$"
    $\not\Rightarrow {\mathrm{ACA}_0}^{\mathrm{ns}}$.

What is essential?

Reverse Mathematics for non-standard analysis!

Definition (Standard part principles)

$$\Sigma_i^\dagger \mathrm{ST} : \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \varphi(x^{\mathrm{s}}))$$

for any $\Sigma_i^\dagger$ formula $\varphi(x^{\mathrm{s}})$.

Then,

- Within $\mathrm{RCA}_0{}^{\mathrm{ns}}$,
  - "If $\{a_n^{\mathrm{s}}\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a_\omega^*)$ is an accumulation value of $\{a_n^{\mathrm{s}}\}$"
    $\Leftrightarrow \Sigma_0^\dagger \mathrm{ST}$.
  - "If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f_\omega^*)$ is an accumulation value of $\{f_n^{\mathrm{s}}\}$"
    $\Leftrightarrow \Sigma_0^\dagger \mathrm{ST}$.

This means that $\Sigma_0^\dagger \mathrm{ST}$ is equivalent to "approximation for limits".

Reverse Mathematics for non-standard analysis!

Definition (Standard part principles)

$$\Sigma^{\dagger}_{i}\mathrm{ST} : \exists Y^{\mathrm{s}} \forall x^{\mathrm{s}} (x^{\mathrm{s}} \in Y^{\mathrm{s}} \leftrightarrow \varphi(x^{\mathrm{s}}))$$

for any $\Sigma^{\dagger}_{i}$ formula $\varphi(x^{\mathrm{s}})$.

Then,

- Within $\mathrm{RCA}_0{}^{\mathrm{ns}}$,
  - "If $\{a^{\mathrm{s}}_n\}$ is a bounded sequence of standard real numbers, then for some non-standard ω, $\mathrm{st}(a^{*}_{\omega})$ is an accumulation value of $\{a^{\mathrm{s}}_n\}$"
    $\Leftrightarrow \Sigma^{\dagger}_{0}\mathrm{ST}$.
  - "If $\{f_n\}$ is a normal family of holomorphic functions on a bounded domain, then for some non-standard ω, $\mathrm{st}(f^{*}_{\omega})$ is an accumulation value of $\{f^{\mathrm{s}}_n\}$"
    $\Leftrightarrow \Sigma^{\dagger}_{0}\mathrm{ST}$.

This means that $\Sigma^{\dagger}_{0}\mathrm{ST}$ is equivalent to "approximation for limits".

Reverse Mathematics for non-standard analysis?

Similarly,

- Within $\mathrm{RCA_0}^{\mathrm{ns}}$,
  - For every closed set $\mathbb{C}$ of a complete separable metric space, there exists a non-standard sequence $\{c_n^*\}$ such that for any non-standard ω, $\mathrm{st}(\{c_i^* \mid i < \omega\}) = C$.
    $\Leftrightarrow \Sigma^{\dagger}_1\mathrm{ST}$.

This means that $\Sigma^{\dagger}_1\mathrm{ST}$ is equivalent to "approximation for closed sets".

Reverse Mathematics for non-standard analysis?

Similarly,

- Within ${\mathrm{RCA}_0}^{\mathrm{ns}}$,
  - For every closed set $\mathbb{C}$ of a complete separable metric space, there exists a non-standard sequence $\{c_n^*\}$ such that for any non-standard ω, $\mathrm{st}(\{c_i^* \mid i < \omega\}) = C$.
    $\Leftrightarrow \Sigma_1^\dagger\mathrm{ST}$.

This means that $\Sigma_1^\dagger\mathrm{ST}$ is equivalent to “approximation for closed sets”.

Systems of non-standard second-order arithmetic

We should redefine systems of non-standard second order arithmetic as follows.

Definition

$$\begin{aligned}
\mathrm{RCA_0}^{\mathrm{ns}} &= (\mathrm{RCA_0})^{s} + (\mathrm{RCA_0})^{*} + \mathrm{emb} + \mathrm{e} + \mathrm{fst} + \Sigma^0_1\mathrm{WSAT}^0 \\
&\quad + \Sigma^1_1\mathrm{EQ} + \Sigma^0_0\mathrm{TP}. \\
\mathrm{WWKL_0}^{\mathrm{ns}} &= \mathrm{RCA_0}^{\mathrm{ns}} + \mathrm{LMP}. \\
\mathrm{WKL_0}^{\mathrm{ns}} &= \mathrm{RCA_0}^{\mathrm{ns}} + \mathrm{ST}. \\
\mathrm{ACA_0}^{\mathrm{ns}} &= \mathrm{RCA_0}^{\mathrm{ns}} + \Sigma^{\dagger}_0\mathrm{ST}. \\
\mathrm{ATR_0}^{\mathrm{ns}} &= \mathrm{ACA_0}^{\mathrm{ns}} + \Sigma^{\dagger}_1\mathrm{SEP}(?). \\
\Pi^1_1\mathrm{CA_0}^{\mathrm{ns}} &= \mathrm{ACA_0}^{\mathrm{ns}} + \Sigma^{\dagger}_1\mathrm{ST}.
\end{aligned}$$

Conclusion

- Standard part principles provide the infinitesimal approximation principles.
- Standard part principles correspond to the comprehension axioms in SOA in the viewpoint of Reverse Mathematics.
- The strength of transfer principles is unstable (it depends heavily on the base system).
- Propositions in non-standard analysis usually do NOT require transfer principles.

Question

What is the role of transfer principle in non-standard analysis?