Rewriting Techniques in Complexity Analysis

on-going work

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joint work with

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Reference

this talk is based on

Nao Hirokawa and Georg Moser
Automated Complexity Analysis Based on the Dependency Pair Method
Proceedings of 4th IJCAR, LNAI 5195, pp. 364-379, 2008

Term Rewriting

Definition

- pair of terms $\ell \to r$ is rewrite rule if $\ell \notin \mathcal{V}$ & $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\ell)$
- term rewrite system (TRS) $\mathcal{R}$ is set of rewrite rules
- rewrite relation $\to_{\mathcal{R}}$ (or simply $\to$)

$$\frac{\ell \to r \in \mathcal{R}}{\ell\sigma \to_{\mathcal{R}} r\sigma} \qquad \frac{s \to_{\mathcal{R}} t}{f(\dots, s, \dots) \to_{\mathcal{R}} f(\dots, t, \dots)}$$

Example

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

rewriting

$$\mathsf{s}(0) \times \mathsf{s}(0) \to_{\mathcal{R}} \mathsf{s}(0) \times 0 + \mathsf{s}(0) \to_{\mathcal{R}} 0 + \mathsf{s}(0)$$
$$\to_{\mathcal{R}} \mathsf{s}(0 + 0) \qquad \to_{\mathcal{R}} \mathsf{s}(0)$$

how many steps do we need to compute $\mathsf{s}^m(0) + \mathsf{s}^n(0)$ or $\mathsf{s}^m(0) \times \mathsf{s}^n(0)$?

Runtime Complexity

Definition

for TRS $\mathcal{R}$ over signature $\mathcal{F}$

- $\mathcal{D} = \{\mathrm{root}(\ell) \mid \ell \to r \in \mathcal{R}\}$ — defined symbols
- $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$ — constructor symbols

Definition — runtime complexity

$\mathsf{rc}_{\mathcal{R}}(n) = \max\{\mathsf{dh}(t, \to_{\mathcal{R}}) \mid t \text{ is basic term of size } n\}$, where

- $\mathsf{dh}(t, \to) = \max\{k \mid t \to^{k} u \text{ for some } u\}$
- $f(t_1, \ldots, t_n)$ is basic term if $t_1, \ldots, t_n$ are constructor terms

Example

$\mathsf{s}(\mathsf{s}(0)) + \mathsf{s}(x)$ is basic but $\mathsf{s}(0 + \mathsf{s}(0)) \times \mathsf{s}(x)$ is not basic

Quiz

find smallest d with $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^d)$ for next TRSs $\mathcal{R}$

1.

$$x - \mathsf{0} \to x$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y$$

$\mathsf{rc}_{\mathcal{R}}(n) \in O(n)$

2.

$$\mathsf{0} + y \to y$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y)$$

$\mathsf{rc}_{\mathcal{R}}(n) \in O(n)$

3.

$$\mathsf{f}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{f}(\mathsf{s}(x), y)$$
$$\mathsf{f}(\mathsf{s}(x), \mathsf{0}) \to \mathsf{f}(x, x)$$

$\mathsf{rc}_{\mathcal{R}}(n) \in O(n^2)$

This Talk

AIM

develop techniques for estimating runtime complexity

OVERVIEW

- basic analysis
- usable replacement map (new)
- conclusion

Basic Analysis

Constructor-Restricted Polynomial Interpretations

Bonfante & Cichon & Marion & Touzet, JFP'01

DEFINITION constructor-restricted polynomial interpretation

CPI of degree d (d-CPI) is algebra $\mathcal{A}$ with

- carrier is $\mathbb{N}$
- $f_{\mathcal{A}}(x_1, \ldots, x_n)$ is polynomial of at most degree d if $f \in \mathcal{D}$
- $f_{\mathcal{A}}(x_1, \ldots, x_n) = x_1 + \cdots + x_n + c_f$ if constructor f

$\ell >_{\mathcal{A}} r$ if $[\alpha]_{\mathcal{A}}(\ell) > [\alpha]_{\mathcal{A}}(r)$ for all $\alpha : \mathcal{V} \to \mathbb{N}$

THEOREM

$\mathcal{R} \subseteq >_{\mathcal{A}}$ for some monotone d-CPI $\mathcal{A}$ $\quad\Longrightarrow\quad$ $\mathsf{rc}_{\mathcal{R}} \in O(n^d)$

PROOF

let $\alpha(x) = 0$ for all variables x

1. $[\alpha_0]_{\mathcal{A}}(s) \in O(|s|^d)$ for all basic term s
2. $s \to_{\mathcal{R}} t \quad\Longrightarrow\quad [\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$

□

Monotonicity is Essential

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{s}(x)) \to \mathsf{c}(\mathsf{f}(x), \mathsf{f}(x))$$

let $\mathbf{n} = \mathsf{s}^n(0)$. $\mathsf{rc}_{\mathcal{R}}(n)$ is exponential due to $\mathsf{f}(\mathbf{n})$; e.g.

$$\begin{aligned} \mathsf{f}(\mathbf{2}) &\to \mathsf{c}(\mathsf{f}(\mathbf{1}), \mathsf{f}(\mathbf{1})) \\ &\to \mathsf{c}(\mathsf{c}(0,0), \mathsf{f}(\mathbf{1})) \\ &\to \mathsf{c}(\mathsf{c}(0,0), \mathsf{c}(0,0)) \end{aligned}$$

- there is no suitable monotone d-CPI, but
- there is, if one ignores monotonicity:

$$\mathsf{f}_{\mathcal{A}}(x, y) = 1 \qquad \mathsf{c}_{\mathcal{A}}(x, y) = 0$$

Example for Basic Analysis

TRS $\mathcal{R}$

$$\begin{aligned} x + \mathsf{0} &\to x & \qquad x + \mathsf{s}(y) &\to \mathsf{s}(x + y) \\ x \times \mathsf{0} &\to \mathsf{0} & \qquad x \times \mathsf{s}(y) &\to x \times y + x \end{aligned}$$

monotone 2-CPI $\mathcal{A}$

$$\begin{aligned} \mathsf{0}_{\mathcal{A}} &= 1 & \qquad \times_{\mathcal{A}}(X, Y) &= 2XY + Y + 1 \\ \mathsf{s}_{\mathcal{A}}(X) &= X + 1 & \qquad +_{\mathcal{A}}(X, Y) &= X + 2Y \end{aligned}$$

is compatible with $\mathcal{R}$ because

$$\bigwedge \left\{ \begin{array}{llll} [\alpha]_{\mathbb{N}}(x + \mathsf{0}) & = X + 2 & > X & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = X + 2Y + 2 & > X + 2Y + 1 & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ [\alpha]_{\mathbb{N}}(x \times \mathsf{0}) & = 2X + 2 & > 1 & = [\alpha]_{\mathbb{N}}(\mathsf{0}) \\ [\alpha]_{\mathbb{N}}(x \times \mathsf{s}(y)) & = \cdots & > \cdots & = [\alpha]_{\mathbb{N}}(x \times y + x) \end{array} \right\}$$

where $X = \alpha(x)$ and $Y = \alpha(y)$. hence $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^2)$

Automation

TRS $\mathcal{R}$ — Contejean, Marché, Monate, and Urbain, JAR 2004

$$x + \mathsf{0} \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times \mathsf{0} \to \mathsf{0} \qquad x \times \mathsf{s}(y) \to x \times y + x$$

2-CPI $\mathcal{A}$ with unknown coefficients

$$\mathsf{0}_{\mathcal{A}} = a \qquad +_{\mathcal{A}}(X,Y) = dXY + eX + fY + g$$
$$\mathsf{s}_{\mathcal{A}}(X) = bX + c \qquad \times_{\mathcal{A}}(X,Y) = iXY + jX + kY + l$$

monotonicity and $\mathcal{R} \subseteq >_{\mathcal{A}}$ follow from

$$\exists a,\ldots,l \in \mathbb{N}.\ \forall X,Y \in \mathbb{N}.\ \ b,e,f,j,k > 0 \ \wedge\ a,c,g,h,l \geqslant 0 \ \wedge$$
$$\wedge \left\{ \begin{array}{c} (bd+e)X + af + g > X \\ bXY + eX + bfY + cf + g > bdX + beX + bfY + bg + c \\ \cdots \end{array} \right\}$$

shifting method approximates to QF-NIA — use SMT solver

History: Interpretation Orders

- CPI of degree d Bonfante & Cichon & Marion & Touzet, JFP'01

- TMI: $d \times d$ triangular matrix interpretation on $\mathbb{N}$
  Moser & Schnabl & Waldmann, FSTTCS'08

$$f_{\mathcal{A}}(\vec{x}, \vec{y}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- matrix interpretations on $\mathbb{N}$ / $\mathbb{Q}_{\geqslant 0}$ / $\mathbb{R}_{\geqslant 0}$ using spectral radius
  Neurauter & Zankl & Middeldorp, IJCAR'10

New: Usable Replacement Maps

Quiz (Difficult)

find smallest d with $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^d)$ for next TRS $\mathcal{R}$

$$x - \mathsf{0} \to x \qquad \mathsf{0} \div \mathsf{s}(y) \to \mathsf{0}$$

$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y \qquad \mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$$

PROBLEM

- no compatible monotone CPI
- all existing methods fail

Context-Sensitive Rewriting

DEFINITION context-sensitive rewriting

suppose $\mu(f) \subseteq \mathbb{N}$ for all $f \in \mathcal{F}$ replacement map

- μ-step

$$\frac{\ell \to r \in \mathcal{R}, \mu}{\ell\sigma \to_{\mathcal{R},\mu} r\sigma} \qquad \frac{s \to_{\mathcal{R}} t}{f(\dots,\underset{i}{s},\dots) \to_{\mathcal{R},\mu} f(\dots,\underset{i}{t},\dots)}\ i \in \mu(f)$$

- μ-replacing positions

$$\mathcal{P}\mathsf{os}_\mu(t) = \begin{cases} \{\epsilon\} & \text{if } t \text{ is variable} \\ \{\epsilon\} \cup \{ip \mid i \in \mu(f) \text{ and } p \in \mathcal{P}\mathsf{os}_\mu(t_i) & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

THEOREM

$\mathsf{rc}_{\mathcal{R},\mu}(n) \in O(n^d)$ if $\mathcal{R} \subseteq >_{\mathcal{A}}$ for some μ-monotone CPI

Replacing Position

suppose $\mathbf{n} = \mathsf{s}^n(0)$

$$
\begin{aligned}
\underline{\mathbf{4} \div \mathbf{2}} &\to \mathsf{s}(\underline{(\mathbf{3} - \mathbf{1})} \div \mathbf{2}) &&\to \mathsf{s}((\underline{\mathbf{2} - 0}) \div \mathbf{2}) \\
&\to \mathsf{s}(\underline{\mathbf{2} \div \mathbf{2}}) &&\to \mathsf{s}(\mathsf{s}(\underline{(\mathbf{1} - \mathbf{1})} \div \mathbf{2})) \\
&\to \mathsf{s}(\mathsf{s}(\underline{(0 - 0)} \div \mathbf{2})) &&\to \mathsf{s}(\mathsf{s}(\underline{0 \div \mathbf{2}})) \\
&\to \mathsf{s}(\mathsf{s}(0))
\end{aligned}
$$

replacement map for runtime complexity:

$$\mu(\mathsf{s}) = \{1\} \qquad \mu(-) = \varnothing \qquad \mu(\div) = \{1\}$$

there are efficient and good approximations

- for innermost termination analysis — Fernández, IPL'05
- for runtime complexity analysis — ☞ next slide

Usable Replacement Maps

Definition usable replacement map

μ_f is mapping induced from least fixed point of Υ:

$$\Upsilon(\mu) = \{(f,i) \mid \exists \text{ rule } \ell \to C[f(r_1,\dots,r_n)] \text{ and } \mathsf{CAP}^\ell_\mu(r_i) \neq r_i\}$$

where

$$\mathsf{CAP}^s_\mu(t) = \begin{cases} t & \text{if } \exists p \in \mathcal{P}\mathsf{os}(s) \setminus \mathcal{P}\mathsf{os}_\mu(s) : t = s|_p \\ u & \text{if } t = f(t_1,\dots,t_n),\ u \text{ and } \ell \text{ do not unify any rules } \ell \to r \\ y & \text{otherwise} \end{cases}$$

and $u = f(\mathsf{CAP}^s_\mu(t_1),\dots,\mathsf{CAP}^s_\mu(t_n))$ and y is fresh variable

Theorem Hirokawa and Moser

$s \to^n_\mathcal{R} t \iff s \to^n_{\mathcal{R},\mu_\mathsf{f}} t$ for all basic terms s

Solution to Quiz

TRS $\mathcal{R}$

$$x - 0 \to x \qquad 0 \div \mathsf{s}(y) \to 0$$
$$\mathsf{s}(x) - \mathsf{s}(y) \to x - y \qquad \mathsf{s}(x) \div \mathsf{s}(y) \to \mathsf{s}((x - y) \div \mathsf{s}(y))$$

usable replacement map

$$\mu_{\mathsf{f}}(\mathsf{s}) = \{1\} \qquad \mu_{\mathsf{f}}(-) = \{1\} \qquad \mu_{\mathsf{f}}(\div) = \{1\}$$

μ_{f}-monotone 1-CPI $\mathcal{A}$

$$\mathsf{0}_{\mathcal{A}} = 0 \quad \mathsf{s}_{\mathcal{A}}(X) = X + 2 \quad -_{\mathcal{A}}(X, Y) = X + 1 \quad \div_{\mathcal{A}}(X, Y) = 3X$$

$\mathcal{R} \subseteq >_{\mathcal{A}}$. hence $\mathsf{rc}_{\mathcal{R}}(n) \in O(n)$

To Conclude...

History: Implicit Computational Complexity

THEOREM safe algebra is variant of prim. rec. Bellantoni and Cook, CC 1992

functions in safe algebra are $\mathbf{P}$-computable (by Turing machine)

THEOREM LMPO is variant of RPO Marion, 1997, 2003

$\mathcal{R} \subseteq >_{\mathsf{LMPO}} \implies$ functions in $\mathcal{R}$ are $\mathbf{P}$-computable

for confluent constructor TRS $\mathcal{R}$ over simple constructor signature

THEOREM POP* is variant of RPO Avanzini & Moser, RTA 2009

$\mathcal{R} \subseteq >_{\mathsf{POP*}} \implies \mathsf{irc}_{\mathcal{R}}(n) \in \mathbf{P}$

for constructor TRS $\mathcal{R}$

THEOREM adequacy theorem Avanzini & Moser, RTA 2010

$\mathsf{rc}_{\mathcal{R}}(n) \in \mathbf{P}$ or $\mathsf{irc}_{\mathcal{R}}(n) \in \mathbf{P} \implies$ functions in $\mathcal{R}$ are $\mathbf{P}$-computable

Why Runtime Complexity?

- polynomial runtime complexity induces polynomial computability
- e.g. "quick-sort is $O(n^2)$" is

  about runtime complexity but not computability

Example: GCD

TRS

$$
\begin{aligned}
\mathsf{0} \leqslant y &\to \mathsf{true} & \mathsf{gcd}(\mathsf{0}, y) &\to y \\
\mathsf{s}(x) \leqslant \mathsf{0} &\to \mathsf{false} & \mathsf{gcd}(\mathsf{s}(x), \mathsf{0}) &\to \mathsf{s}(x) \\
\mathsf{s}(x) \leqslant \mathsf{s}(y) &\to x \leqslant y & \mathsf{gcd}(\mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{if}_{\mathsf{gcd}}(y \leqslant x, \mathsf{s}(x), \mathsf{s}(y)) \\
x - \mathsf{0} &\to x & \mathsf{if}_{\mathsf{gcd}}(\mathsf{true}, \mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(x - y, \mathsf{s}(y)) \\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y & \mathsf{if}_{\mathsf{gcd}}(\mathsf{false}, \mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{gcd}(y - x, \mathsf{s}(x))
\end{aligned}
$$

there is μ_{f}-monotone constructor-restricted 2×2 TMI,

hence $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^2)$ ☞ optimal?

Example: List Reversing

TRS $\mathcal{R}$

$$[\,] +\!\!\!+\; z \to z \qquad \mathsf{rev}([\,]) \to [\,]$$

$$(x : y) +\!\!\!+\; z \to x : (y +\!\!\!+\; z) \qquad \mathsf{rev}(x : y) \to \mathsf{rev}(y) +\!\!\!+\; (x : [\,])$$

we cannot conclude anything... ☞ further investigation is necessary

Experiments

- 2134 TRSs from termination problem Database (TPDB)
- constructor-restricted triangular matrix interpretations (TMI)
- 60 seconds timeout

$\mathsf{rc}_{\mathcal{R}}(n)$	monotone TMI	μ_{f}-monotone TMI
$O(1)$	19	22
$O(n)$	122	129
$O(n^2)$	122	176

Conclusion

- constructor-restricted polynomial interpretations
- new: usable replacement map

CURRENT STATUS

- not yet powerful enough

- international competition 2008– in TERMCOMP

  AProVE, CaT, Matchbox, Oops, TCT

- new participants & tools are welcome!

Thank You For Your Attention!