Join-Completions and Finite Embeddability

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Workshop on Logic and Computation:
from proof theory to program verification

Kanazawa, Japan
February 8-9, 2011

Abstract-Motivation

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W. J. Blok and C. J. Van Alten, The finite embeddability property for residuated lattices, pocrims and BCK-algebras, Algebra Univers. 48 (2002), 253-271.

W. J. Blok and C. J. Van Alten, The finite embeddability property for residuated ordered groupoids, Trans. Amer. Math. Soc. 357 (10) (2005), 4141-4157.

M. Okada, K. Terui, The finite model property for various fragments of intuitionistic linear logic, J. Symbolic Logic 64(2) (1999), 790-802.

H. Ono, Completions of Algebras and Completeness of Modal and Substructural Logics, Advances in Modal Logics 4 (2002), 1-20.

H. Ono, Closure Operators and Complete Embeddings of Residuated Lattices, Studia Logica 74 (3) (2003), 427-440.

C. J. van Alten, Completion and the finite embeddability property for residuated ordered algebras, Algebra Univers. 62 (2009), 419-451.

Featured Themes

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(2) The role of nuclei and co-nuclei in logic

(3) Interaction of residuals with join extensions

(4) The finite embeddability property and some of its consequences

Join-Completions

Join-Extensions and Join-Completions

A poset $\mathbf{Q}$ is called be an extension of a poset $\mathbf{P}$ provided $P \subseteq Q$ and the order of $\mathbf{Q}$ restricts to that of $\mathbf{P}$. In case every element of $\mathbf{Q}$ is a join (in $\mathbf{Q}$) of elements of $\mathbf{P}$, we say that $\mathbf{Q}$ is a join-extension of $\mathbf{P}$ and that $\mathbf{P}$ is join-dense in $\mathbf{Q}$. We use the term join-completion for a join-extension that is a complete lattice. The concepts of a meet-extension and a meet-completion are defined dually.

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Join-completions, introduced by B. Banaschewski (1956), are intimately related to representations of complete lattices studied systematically by J.R. Büchi (1952).

Preservation of Meets

If $\mathbf{Q}$ is a join-extension of $\mathbf{P}$, then the inclusion map $i : \mathbf{P} \to \mathbf{Q}$ preserves all existing meets.

Equivalently, if $X \subseteq P$ and $\bigwedge^{\mathbf{P}} X$ exists, then $\bigwedge^{\mathbf{Q}} X$ exists and $\bigwedge^{\mathbf{P}} X = \bigwedge^{\mathbf{Q}} X$

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Dually, If $\mathbf{Q}$ is a meet-extension of $\mathbf{P}$, then the inclusion map $i : \mathbf{P} \to \mathbf{Q}$ preserves all existing joins.

Lower and Upper Sets

A subset I of a poset $\mathbf{P}$ is said to be a lower set of $\mathbf{P}$ if whenever $y \in P$, $x \in I$, and $y \leq x$, then $y \in I$. Note that the empty set $\emptyset$ is a lower set.

A principal lower set is a lower set of the form

$$\downarrow a = \{x \in P \mid x \leq a\} \ (a \in P).$$

For $A \subseteq P$,

$$\downarrow A = \{x \in P \mid x \leq a, \text{ for some a} \in \text{A}\}$$

denotes the smallest lower set containing A.

The set $\mathcal{L}(\mathbf{P})$ of lower sets of $\mathbf{P}$ ordered by set inclusion is a complete lattice; the join is the set-union, and the meet is the set-intersection.

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The upper sets of $\mathbf{P}$ are defined dually. Further, we write $\mathcal{U}(\mathbf{P})$ for the lattice of upper sets of $\mathbf{P}$,

$\uparrow a = \{x \in P \mid a \leq x\}$ for $a \in P$, and

$\uparrow A = \{x \in P \mid a \leq x, \text{ for some a} \in \text{A}\}$ for $A \subseteq P$.

Canonical Representation of Join-Extensions

Each join-extension $\mathbf{Q}$ of a poset $\mathbf{P}$ can be identified with its canonical image $\dot{\mathbf{Q}}$:

$$\dot{\mathbf{Q}} = \{\downarrow x \cap P : x \in Q\}.$$

In particular, P can be identified with the poset $\dot{\mathbf{P}}$ of its principal lower sets:

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The largest join-extension of $\mathbf{P}$ is $\mathcal{L}(\mathbf{P})$. Thus, for any join-extension $\mathbf{Q}$ of $\mathbf{P}$, we have

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The smallest join-completion of $\mathbf{P}$, is the so called Dedekind-MacNeille completion $\mathcal{N}(\mathbf{P})$. Its canonical image consists of all lower sets that are intersections of principal lower sets.

Abstract Description of Join-Extensions

It is often desirable to look at the elements of a join-extension of $\mathbf{P}$ as just elements – such as the elements of $\mathbf{P}$ itself – rather than certain lower sets of $\mathbf{P}$.

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$\mathcal{N}(\mathbf{P})$ has this abstract description: it is the only join and meet-completion of $\mathbf{P}$ (Banaschewski). The inclusion $i : \mathbf{P} \to \mathcal{N}(\mathbf{P})$ preserves all existing joins and meets. The Crawley completion $\mathcal{C}(\mathbf{P})$ – with canonical image consisting of all complete lower sets of $\mathbf{P}$, that is, lower sets that are closed with respect to any existing joins of their elements – is the largest join-completion with this property.

Closure Operators and Closure Systems

Recall that a closure operator on a poset $\mathbf{P}$ is a map $\gamma : \mathbf{P} \to \mathbf{P}$ with the usual properties of being order-preserving ($x \le y \Rightarrow \gamma(x) \le \gamma(y)$), enlarging ($x \le \gamma(x)$), and idempotent ($\gamma(\gamma(x)) = \gamma(x)$). It is completely determined by its image $\mathbf{P}_\gamma$ by virtue of the formula

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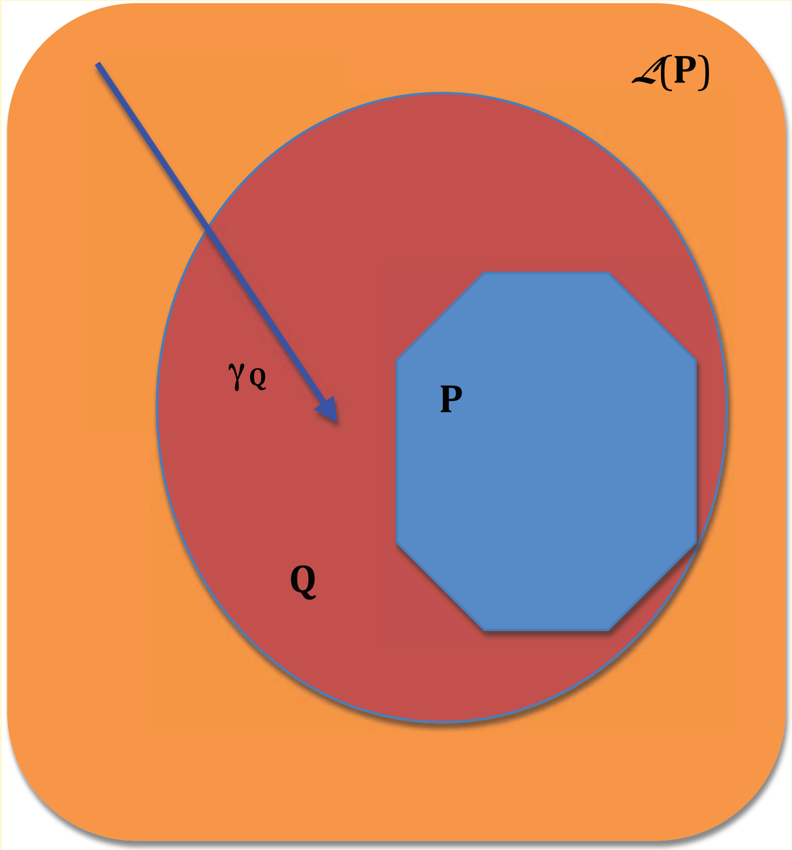
If $\mathbf{Q}$ is a join-completion of $\mathbf{P}$, then $\mathbf{Q}$ is a closure system on $\mathcal{L}(\mathbf{P})$. We write $\gamma_{\mathbf{Q}}$ for the associated closure operator on $\mathcal{L}(\mathbf{P})$.

Back to Join-Completions



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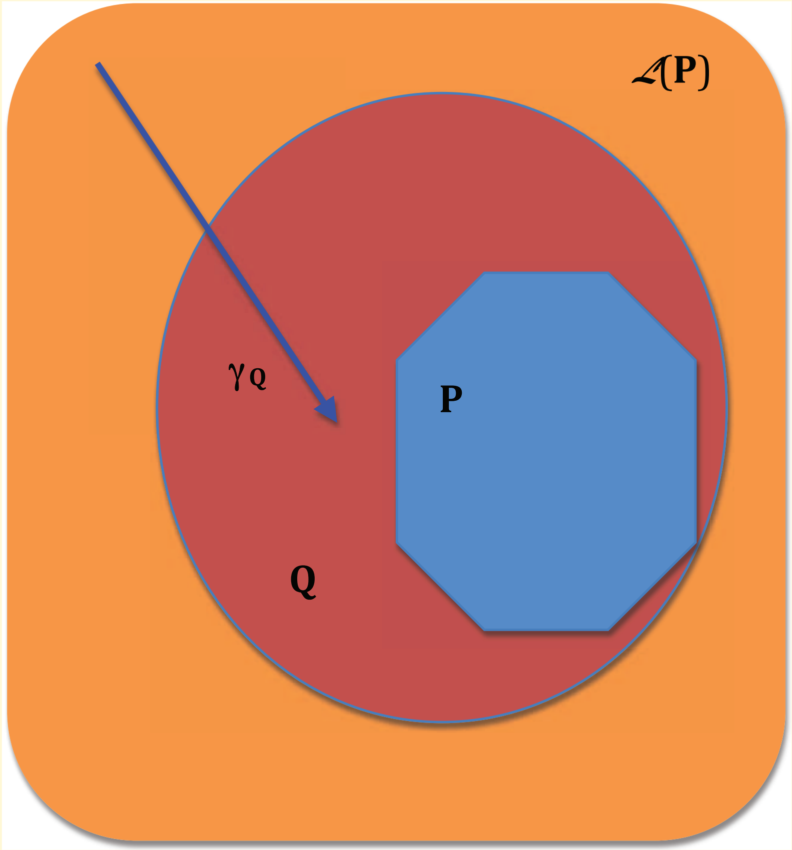

There is a bijective correspondence between join-completions of $\mathbf{P}$ and closure operators γ on $\mathcal{L}(\mathbf{P})$ with $P \subseteq \mathcal{L}(P)_\gamma$

Join-Extensions of Ordered Structures

Residuals in Partially Ordered Monoids

Given a partially ordered monoid – pom for short – $\mathbf{P}$, the following question arises:

Which join-completions of $\mathbf{P}$ are residuated lattices with respect to a, necessarily unique, multiplication that extends the multiplication of $\mathbf{P}$?

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$x \backslash z$ is read as "x under z"

z / x is read as "z over x"

Residuals in Partially Ordered Monoids

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Alternatively, a residuated pom $\mathbf{P}$ is one in which the binary operation $\cdot$ is residuated. This means that there exist binary operations $\backslash$ and $/$ on P such that for all $x, y, z \in P$,

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A residuated lattice is a residuated lattice-ordered monoid $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$

Residuals in Partially Ordered Monoids

Let $\mathbf{P}$ be a monoid. Then $\wp(\mathbf{P}) = \langle \wp(P), \cap, \cup, \cdot, \backslash, /, \{1\} \rangle$ is a residuated lattice where:

$$X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\},$$

$$X \backslash Y = \{z \mid X \cdot \{z\} \subseteq Y\}, \text{ and}$$

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$$X \backslash Y = \{z \mid X \cdot \{z\} \subseteq Y\}, \text{ and}$$

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Note: $(\downarrow x) \cdot (\downarrow y) = \downarrow (x \cdot y)$; hence $\dot{\mathbf{P}}$, is a submonoid of $\mathcal{L}(\mathbf{P})$

Nuclei and Closure Retractions

A nucleus on a pom $\mathbf{P}$ is a closure operator γ on $\mathbf{P}$ such that $\gamma(a)\gamma(b) \leq \gamma(ab)$ (equivalently, $\gamma(\gamma(a)\gamma(b)) = \gamma(ab)$), for all $a, b \in P$.

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Let γ be a closure operator on a residuated pom $\mathbf{P}$, and let $\mathbf{P}_\gamma$ be the closure system associated with γ. Then γ is a nucleus iff $\mathbf{P}_\gamma$ is a closure retraction of $\mathbf{P}$.

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A closure retraction $\mathbf{P}_\gamma$ of a residuated pom $\mathbf{P}$ is a residuated pom. The product of $x, y \in \mathbf{P}_\gamma$ is given by $x \circ_\gamma y = \gamma(x \cdot y)$, and the residuals are the restrictions on $\mathbf{P}_\gamma$ of the residuals of $\mathbf{P}$. In particular, if $\mathbf{P}$ is a residuated lattice, then so is $\mathbf{P}_\gamma$, with $x \vee_\gamma y = \gamma(x \vee y)$ and $x \wedge_\gamma y = x \wedge y$

Let $\mathbf{Q}$ be a join completion of a pom $\mathbf{P}$, and let $\delta_{\mathbf{Q}}$ be the corresponding closure operator on $\wp(\mathbf{P})$.

The following statements are equivalent:

(1) $\mathbf{Q}$ is a residuated lattice with respect to a multiplication extending the multiplication of $\mathbf{P}$.

(2) $a\backslash_{\mathcal{L}(\mathbf{P})} b \in Q$ and $b/_{\mathcal{L}(\mathbf{P})} a \in Q$, for all $a \in P$ and $b \in Q$.

(3) $\gamma_{\mathbf{Q}}$ is a nucleus on $\mathcal{L}(\mathbf{P})$

(4) $\mathbf{Q}$ is a closure retraction on $\mathcal{L}(\mathbf{P})$.

Furthermore, if the preceding conditions are satisfied, then the inclusion map $\mathbf{P} \hookrightarrow \mathbf{Q}$ preserves multiplication, all meets and all existing residuals.

Crucial Lemma

Let $\mathbf{P}$ be a pom and let $\mathbf{Q}$ be a join-completion of $\mathbf{P}$ that is a pom with respect to a multiplication that extends the multiplication of $\mathbf{P}$. Then for all $a, b \in P$, if $a \backslash_{\mathbf{P}} b$ exists, then $a \backslash_{\mathbf{Q}} b$ exists and

$$a \backslash_{\mathbf{P}} b = a \backslash_{\mathbf{Q}} b = a \backslash_{\mathcal{L}(\mathbf{P})} b.$$

Likewise for the other residual.

The Finite Embeddability Property

Definitions and Basic Properties

A class $\mathcal{K}$ of algebras is said to have the finite embeddability property (FEP) if every finite partial subalgebra of a member of $\mathcal{K}$ can be embedded into a finite member of $\mathcal{K}$. If $\mathcal{K}$ is a class of ordered algebras, the preceding embedding must be an ordered embedding.

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[Blok - van Alten]

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- If $\mathcal{K}$ is closed under finite products, then the (FEP) and the (SFMP) are equivalent.

[T. Evans]

- Any variety that satisfies the (FEP) has a solvable word problem. In particular, its equational theory is decidable.
- A finitely presented algebra in any variety satisfying the (FEP) is residually finite.

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Let $\mathbf{A} \in \mathcal{CIRL}$, and let $\mathbf{B}$ be any partial subalgebra of $\mathbf{A}$. We'll first show that $\mathbf{B}$ can be embedded into a lattice-complete algebra $\mathbf{C} \in \mathcal{CIRL}$ so that:
if $(b_i \mid i \in I)$ is a family of elements of $\mathbf{B}$ such that $\bigvee^{\mathbf{A}}_{i\in I} b_i \in B$, then $\bigvee^{\mathbf{A}}_{i\in I} b_i = \bigvee^{\mathbf{B}}_{i\in I} b_i = \bigvee^{\mathbf{C}}_{i\in I} b_i$.
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What do we expect to preserve for "$\cdot$" and "$\rightarrow$"?

Proof (I)

Let $\mathbf{M}$ be the submonoid of $\mathbf{A}$ generated by $\mathbf{B}$. We use the same notation $\mathbf{M}$ to denote the induced partial subalgebra of $\mathbf{A}$: $\mathbf{B} \leq \mathbf{M} \leq \mathbf{A}$
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Note that $B \subseteq \bar{C}$, since $1 \in M$ and $1 \to b = b$, for all $b \in B$.

Let $\mathbf{C}$ be the closure system generated by $\bar{C}$:
$\mathbf{C} = \{\bigwedge X \mid X \subseteq \bar{C}\}$ (Note that $\bigwedge \emptyset = 1 \in C$.)

Claim: $\mathbf{C}$ is a closure retraction of $\mathcal{L}(\mathbf{M})$. Indeed, let $a \in \mathcal{L}(\mathbf{M})$ and $x \in C$. We need to show that $a \to x \in C$. There exists a family $(m_i \mid i \in I)$ of elements of M, and a family $(m_j \to b_j \mid j \in J)$ of elements of $\bar{C}$ such that $a = \bigvee_{i \in I} m_i$ and $x = \bigwedge_{j \in J}(m_j \to b_j)$. We have:

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In view of the general theory, $\mathbf{C} \in \mathcal{CIRL}$, and the residuals and (arbitrary) meet operations in it agree with those in $\mathcal{L}(\mathbf{M})$. Note further the following:

Proof (II)

Claim: $\mathbf{C}$ is a closure retraction of $\mathcal{L}(\mathbf{M})$. Indeed, let $a \in \mathcal{L}(\mathbf{M})$ and $x \in C$. We need to show that $a \to x \in C$. There exists a family $(m_i \mid i \in I)$ of elements of M, and a family $(m_j \to b_j \mid j \in J)$ of elements of $\bar{C}$ such that $a = \bigvee_{i \in I} m_i$ and $x = \bigwedge_{j \in J}(m_j \to b_j)$. We have:

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In view of the general theory, $\mathbf{C} \in \mathcal{CIRL}$, and the residuals and (arbitrary) meet operations in it agree with those in $\mathcal{L}(\mathbf{M})$. Note further the following:

When multiplication or residuals (more precisely, restrictions of residuals of $\mathbf{A}$) are defined in $\mathbf{B}$, they agree with the corresponding operations in $\mathbf{C}$.

Let $(x_i \mid i \in I)$ is a family of elements of B such that $\bigwedge^{\mathbf{A}}_{i\in I} x_i \in B$. Then

$$\bigwedge^{\mathbf{B}}_{i\in I} x_i = \bigwedge^{\mathbf{A}}_{i\in I} x_i = \bigwedge^{\mathbf{M}}_{i\in I} x_i = \bigwedge^{\mathbf{C}}_{i\in I} x_i.$$

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Lastly, let $(x_i \mid i \in I)$ be a family in B such that $\bigvee^{\mathbf{A}}_{i\in I} x_i \in B$. Then $\bigvee^{\mathbf{B}}_{i\in I} x_i = \bigvee^{\mathbf{A}}_{i\in I} x_i = \bigvee^{\mathbf{M}}_{i\in I} x_i$.

Claim: $\bigvee^{\mathbf{B}}_{i\in I} x_i = \bigvee^{\mathbf{C}}_{i\in I} x_i$.

Clearly $\bigvee^{\mathbf{C}}_{i\in I} x_i \leq \bigvee^{\mathbf{B}}_{i\in I} x_i$.

Conversely, suppose that $m \to b$ $(m \in M, b \in B)$ is an upper bound of all the elements x_i in $\mathbf{C}$. For each i,

$x_i \leq m \to b \implies mx_i \leq b \implies \bigvee^{\mathbf{A}}_{i\in I} mx_i \leq b \implies m \bigvee^{\mathbf{A}}_{i\in I} x_i = m \bigvee^{\mathbf{B}}_{i\in I} x_i \leq b \implies \bigvee^{\mathbf{B}}_{i\in I} x_i \leq m \to b$. It follows that $\bigvee^{\mathbf{B}}_{i\in I} x_i \leq \bigvee^{\mathbf{C}}_{i\in I} x_i$.

Finiteness

Suppose now that $\mathbf{B}$ is finite. We claim that $\bar{C}$, and hence $\mathbf{C}$, is finite. Let $B = \{b_1, \ldots, b_n\}$. Let $\mathbf{F} = \langle x_1, \ldots, x_n \rangle$ be the free commutative monoid on n generators: $\mathbf{F} \cong (\mathbb{Z}^-)^n$. Endowing $\mathbf{F}$ with the cartesian product order, we get a member of $\mathcal{CIRL}$, which will also be denoted by $\mathbf{F}$. Note that $\mathbf{F}$ satisfies the (ACC), and every antichain in it is finite.

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Let $\varphi : \mathbf{F} \to \mathbf{M}$ be the monoid epimorphism that extends the assignment $x_i \mapsto b_i$. We'll think of φ as a map $\varphi : \mathbf{F} \to \mathcal{L}(\mathbf{M})$, but remember that $\varphi[F] = M$. An important observation here is that φ is an order homomorphism.

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To prove that $\bar{C}$ is finite, it will suffice to show that, for a fixed $b \in B$, the set $\{a \to b \mid a \in M\}$ is finite.

Finiteness

Now $\varphi^{-1}(\downarrow b) = \downarrow Y$, for some finite antichain $Y \subseteq F$. Further, by integrality, $b \leq a \to b$, and so $\downarrow Y = \varphi^{-1}(\downarrow b) \subseteq \varphi^{-1}(\downarrow a \to b)$. It follows that $\varphi^{-1}(\downarrow a \to b) = \downarrow Z$, for some $Z \subseteq \uparrow Y$. Since $\uparrow Y$ is finite, and $\downarrow a_1 \to b \neq \downarrow a_2 \to b$ implies $\varphi^{-1}(\downarrow a_1 \to b) \neq \varphi^{-1}(\downarrow a_2 \to b)$, we can conclude that the set $\{a \to b \mid a \in M\}$ is finite. The proof is now complete.