Completion and Finite Embeddability Property for Residuated Ordered Algebras

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A classical question regarding lattices and other ordered structures is whether one can embed such a structure into a *complete* lattice. A number of constructions exist that achieve this. If the ordered structure has additional operations on it, then the question arises of whether the complete extension preserves the original operations, and whether it is of the correct type. In this talk, we consider this question for ordered structures with additional *residuated* operations. For example, $\boldsymbol{A} = \langle A, \circ, \rightarrow, \leq \rangle$, where $\leq$ is a partial order (or lattice order) and $\circ$ is a binary commutative residuated operation with residual $\rightarrow$, i.e.,

$$x \circ y \leq z \quad \text{iff} \quad y \leq x \rightarrow z.$$

This class of objects includes all types of *commutative residuated lattices* as well as other algebraic structures that arise in logic. We consider also unary operations $\Diamond$ that are residuated, meaning that there exists a unary operation $\Diamond^*$ such that

$$\Diamond x \leq y \quad \text{iff} \quad x \leq \Diamond^* y.$$

Thus, many types of modal algebras also fall within the scope of the talk.

A general completion for ordered algebras with residuated operations will be presented. In addition, by making suitable adjustments to the construction, we obtain a *finite embeddability property* for many such classes. The finite embeddability property for a class of algebras means that for every $\boldsymbol{A}$ in the class and every finite subset B of $\boldsymbol{A}$, there exists a finite algebra $\boldsymbol{C}$ in the class such that B embeds into $\boldsymbol{C}$ and all existing operations are preserved. The finite embeddability property implies the finite model property.

Lastly, we consider properties that are preserved by the construction. That is, if the original algebra satisfies an inequality $s \leq t$ for terms s, t, does the complete extension also satisfy this inequality? We shall present conditions on s and t in the style of the Sahlqvist formulas for modal logics under which one can show that $s \leq t$ is preserved.