Algebraic Characterizations of Several Logical Properties for Substructural Logics

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Contents

- What are substructural logics?
  - aims of substructural logics
  - relations between substructural logics and FL-algebras
- progress of 2006
  - Craig’s interpolation property and the amalgamation property
  - "Algebra & Substructural Logics III", November 2006
  - "Residuated lattices: An Algebraic Glimpse at Substructural Logics", Chapter 5
- summary and future works
Some problems

There are several differences between logics in human thinking and classical logic.

(Example 1) \( \neg \neg \phi \rightarrow \phi \) (the law of double negation)

- \( \phi \): "I want to go to the party."
- \( \neg \neg \phi \rightarrow \phi \): If it is not the case that "I don’t want to go to the party" then "I want to go to the party."
- \( \neg \phi \): It is not the case that "I don’t want to go to the party."

(?) "I want to go to the party."
(?) "I do not feel like going to the party very much."

"Intuitionistic logic"
Some problems

(Example 2) the role of implication $\rightarrow$

- $\phi$: "Fishes are plants."
- $\psi$: "Beethoven composed nine symphonies."

Then, the formula $\phi \rightarrow \psi$ is true in classical logic since $\psi$ is true. But,

(?) there is no relation between $\phi$ and $\psi$,

(?) $\phi \rightarrow \psi$ is equivalent to $\neg \phi \lor \psi$.

"Relevant logic"
The purpose of substructural logics

Until now, various nonclassical logics have been introduced by their own motivations.

The purpose of the study of substructural logics is to find common features among such nonclassical logics, and to introduce a uniform framework for them.
Examples of substructural logics

The following logics can be regarded as substructural logics;

- The set of all formulas \((\mathcal{F} m)\),
- Classical logic \((\text{CL})\),
- Intuitionistic logic \((\text{Int})\),
- Full Lambek calculus \((\text{FL})\), etc.
The lattice of all substructural logics

The set of all substructural logics forms a lattice with respect to set inclusion.
FL-algebras provide algebraic semantics for substructural logics.

(cf)

Boolean algebras $\Leftrightarrow$ classical logic
Heyting algebras $\Leftrightarrow$ intuitionistic logic
A class of FL-algebras is called a **variety** if it is closed under subalgebras, homomorphic image and direct product.

(Examples of variety)
- the class of all FL-algebras ($\mathcal{FL}$),
- the class of all Heyting algebras ($\mathcal{HA}$),
- the class of all Boolean algebras ($\mathcal{BA}$), etc.
The subvariety lattice of $\mathcal{FL}$

The set of all varieties of FL-algebras forms a lattice with respect to set inclusion.

\[
\begin{array}{c}
\mathcal{FL} \\
\mathcal{HA} \\
\mathcal{BA}
\end{array}
\]
Relations between logics and varieties
Craig interpolation property

A substructural logic $\mathcal{L}$ has the Craig interpolation property (CIP), if for any formulas $\phi$ and $\psi$, if $\phi \to \psi$ is provable on $\mathcal{L}$ then there exists a formula $\delta$ such that

- both of $\phi \to \delta$ and $\delta \to \psi$ are provable on $\mathcal{L}$, and
- $\text{Var}(\delta) \subseteq \text{Var}(\phi) \cap \text{Var}(\psi)$,

where $\text{Var}(\gamma)$ denotes the set of all propositional variables in a formula $\gamma$. 
Research on CIP

Until now, there have been many studies concerned with the CIP. For example:
- For a given logic, does it have the CIP?
- How many logics satisfying the CIP are there?

"Find an algebraic property to which the CIP corresponds."
Amalgamation property 1

A variety $V$ of FL-algebras has the **amalgamation property** (AP) if for all $A, B, C$ in $V$ and for all embeddings $f : A \to B$ and $g : A \to C$ there exist an algebra $D$ in $V$ and embeddings $h : B \to D$ and $k : C \to D$ such that

$$h \circ f = k \circ g.$$
Amalgamation property 2

\[ h \circ f = k \circ g \]
Intermediate logics

the CIP

\[ \iff \]

the AP
Commutative substructural logics

the CIP

\[ \Leftrightarrow \]

the super-AP

\( \mathcal{F} m \)

FL
Substructural logics

the CIP

\[ \Leftrightarrow \]

???
Strategy

the amalgamation property (AP)

\[ \downarrow \text{ (generalize)} \]

the generalized amalgamation property (GAP)

\[ \downarrow \text{ (add certain condition)} \]

the super generalized amalgamation property (superGAP)

\[ \uparrow \]

Craig’s interpolation property
Amalgamation property

\[ h \circ f = k \circ g \]
Another look at AP
\begin{equation}
\forall a \in A, \exists d \in D_1 \cap D_2 \quad (f(a) = i(d) \text{ and } g(a) = j(d))
\end{equation}
Generalized amalgamation property

\[\forall a \in A, \exists d \in D_1 \cap D_2 \quad (f(a) = i(d) \text{ and } g(a) = j(d))\]
CIP vs SuperGAP

THEOREM

For each substructural logic $L$, $L$ has the CIP iff $V(L)$ has the superGAP.

Here, $V(L)$ denotes the subvariety of the variety of FL-algebars which corresponds to a logic $L$. 
Summary

- Algebraic characterizations of interpolation properties
  - the CIP $\iff$ the superGAP

- Our methods can be applied also to other logical properties. For example,
  - Maksimova’s variable separation property,
  - Halldén completeness,
  - Pseudo-relevance property,
  - etc
Some questions

- Relations between the CIP and the DIP
  - Does the CIP always imply the DIP? If not, what is a counterexample of such a logic?

- How many substructural logics satisfying the CIP are there?

- Generalize our results in the context of universal algebra.