# A Case Study: Analyzing the One Dimensional Ising Model by Probabilistic Model Checking

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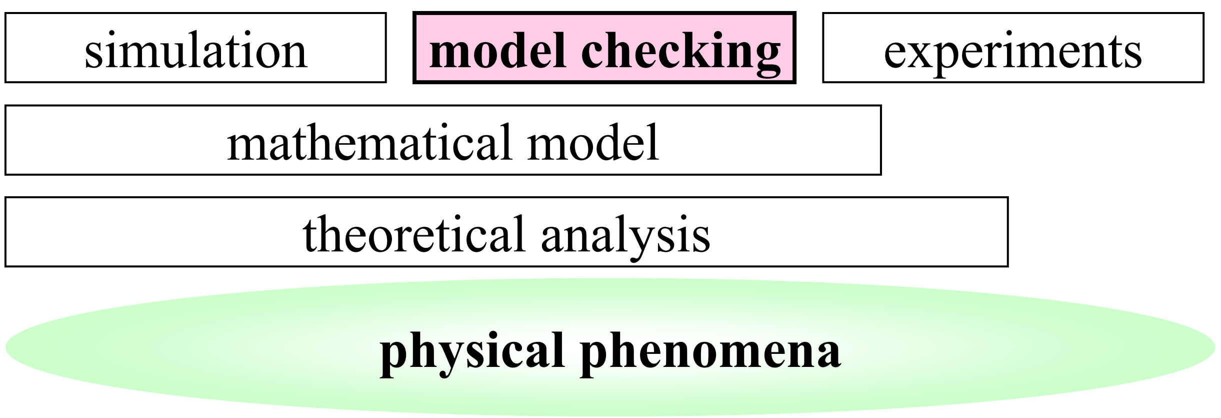
# Proposed Method

- Analyzing the Ising model using probabilistic model checking.

As an example, we analyze physical phenomena of the 1D Ising model.



# Outline

- The Ising model
- Probabilistic Model Checking
- Discussion

# The Ising model

The Ising model is:

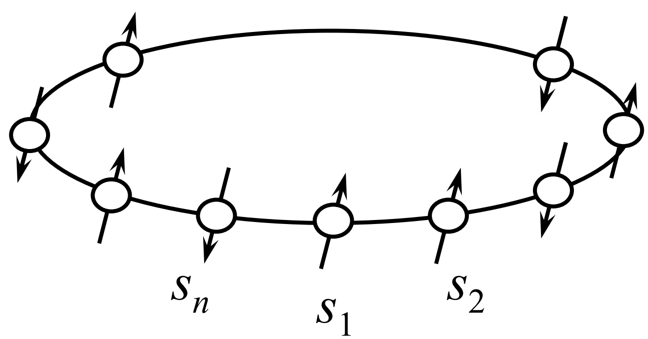
- a simplified model for magnets.
- $G = (S, E)$
  - spin $S = (s_1, s_2, \ldots), s_i = +1, -1$
    elementary microscopic objects.
    $s = +1$ represents up, and $s = -1$ represents down.
  - energy $E$
    macroscopic physical quantity.

# The one dimensional Ising model

The 1D Ising model has:

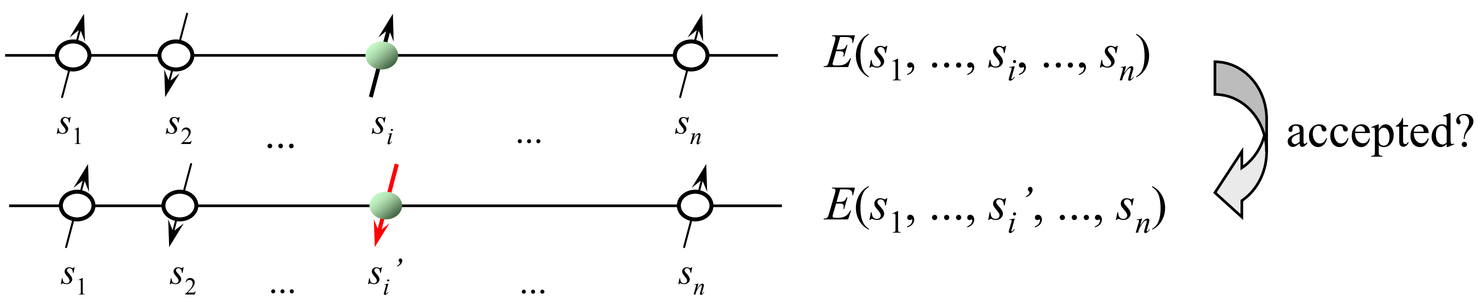
- $n$ spins, $s_1$, $s_2$, …, $s_n$, located on a line in order.
- boundary condition $s_{n+1} = s_1$.
- interactions restricted to nearby spins ($s_i$, $s_{i+1}$).
- energy $E(s_1, s_2, \ldots, s_n) = -J \sum_{i=1}^{n} s_i s_{i+1}$
- physical quantity
  - magnetization

$$M(s_1, s_2, \ldots, s_n) = \sum_{i=1}^{n} s_i$$



# Random spin flipping

1. Choose a spin $s_i$ randomly.
2. Fix other spins $s_{i \neq j}$ and evaluate the energy difference $\Delta E = E' - E$, where $E = E(s_1, \ldots, s_i, \ldots, s_n)$ and $E' = E(s_1, \ldots, s_i', \ldots, s_n)$
3. If $\Delta E < 0$, the spin flipping is accepted. Otherwise accepted with probability $e^{-\Delta E/T}$, where $T$ is temperature.
4. Repeat steps 1 to 3 sufficient number of times.



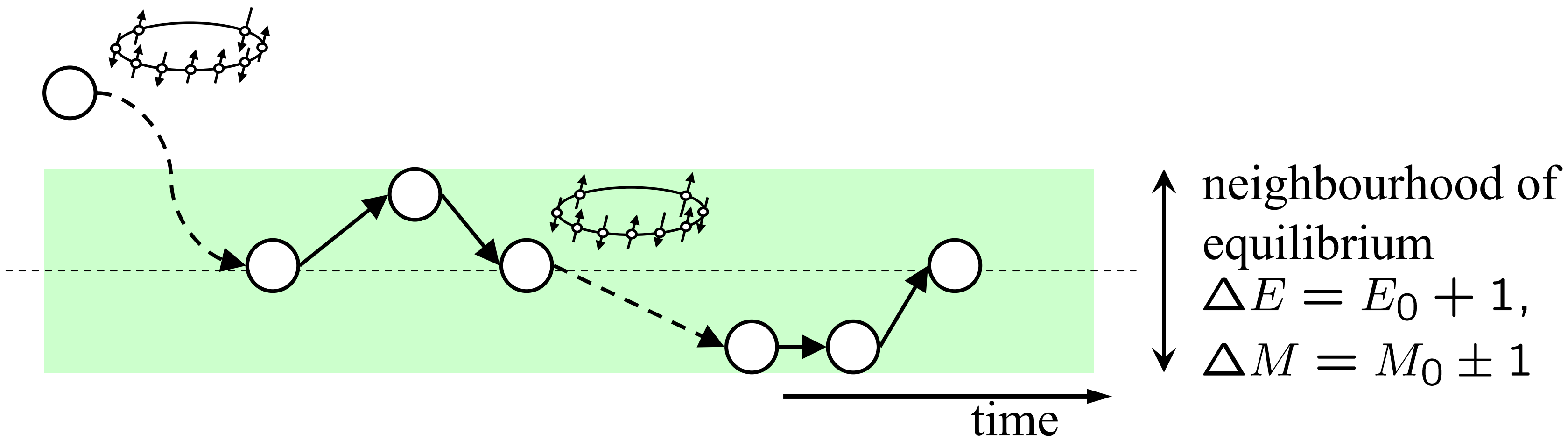
# Behaviour of the Ising model

We consider

- neighbourhood of equilibrium

  $\Delta E = E_0 + 1, \ \Delta M = M_0 \pm 1$

  where energy $E_0$ = -$n$, and magnetization $M_0$ = 0 at equilibrium.



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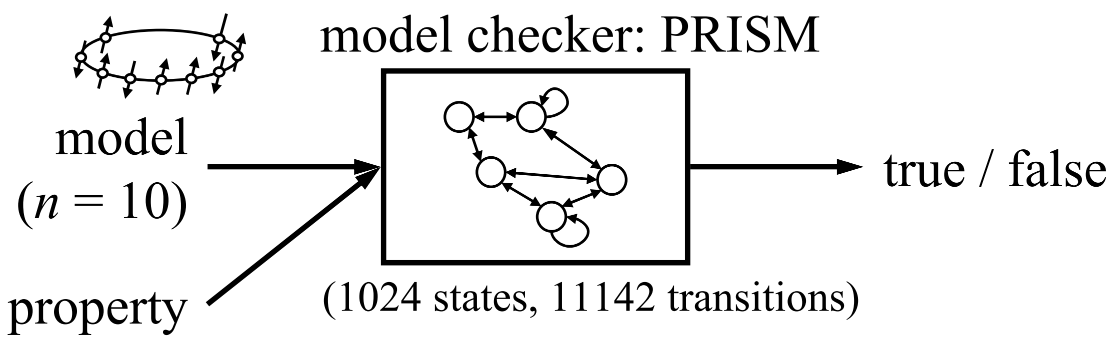
# Probabilistic model checking

A formal technique of verification.

- Input
  - a finite transition system (model)
    (DTMC: Discrete Time Markov Chain)
  - a property
    (PCTL: Probabilistic real time Computation Tree Logic)
- Output
  - true / false



# DTMC (Discrete Time Markov Chain)

Let AP be a set of atomic propositions.

A labelled DTMC (Discrete Time Markov Chain) is a tuple $\mathcal{M} = (V, v^i, \mathcal{T}, \mathcal{L})$:

- $V$, a finite set of states.
- $v^i \in V$, the initial state.
- $\mathcal{T} : V \times V \to [0, 1]$, a transition probability function such that $\forall v \in V, \sum_{v' \in V} \mathcal{T}(v, v') = 1$.
- $\mathcal{L} : V \to 2^{\mathtt{AP}}$, a labelling function.

# PCTL

PCTL (Probabilistic real time Computation Tree Logic) is a probabilistic extension of the temporal logic CTL.

[H. Hanson, et al. Formal Asp. Comput. 1994]

- Syntax:

$$\varphi ::= \top \mid \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \mathbb{P}_{\sim\lambda}(\psi)$$

$$\psi ::= \varphi \ \mathtt{U} \ \varphi \mid \varphi \ \mathtt{U}^{\leq t} \ \varphi$$

where $p$ is an atomic proposition,

$\sim \in \{<, \leq, \geq, >\}$ is a relational operator,

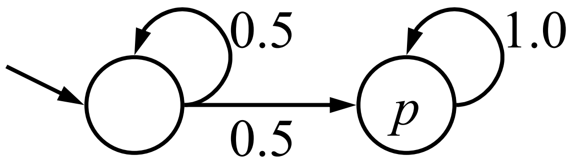
$\lambda \in [0, 1]$ is a probability, and

$t \in \mathtt{Nat}$ or $\infty$.

- Semantics

(*snip*)

# Examples of DTMC + PCTL




- "The probability of reaching a state where *p* holds within 10 steps is greater than 0.3."

$$\mathbb{P}_{>0.3}\left(\top \ \mathtt{U}^{\leq 10} \ p\right)$$

- "A state where *p* holds is reachable (with probability 100%)."

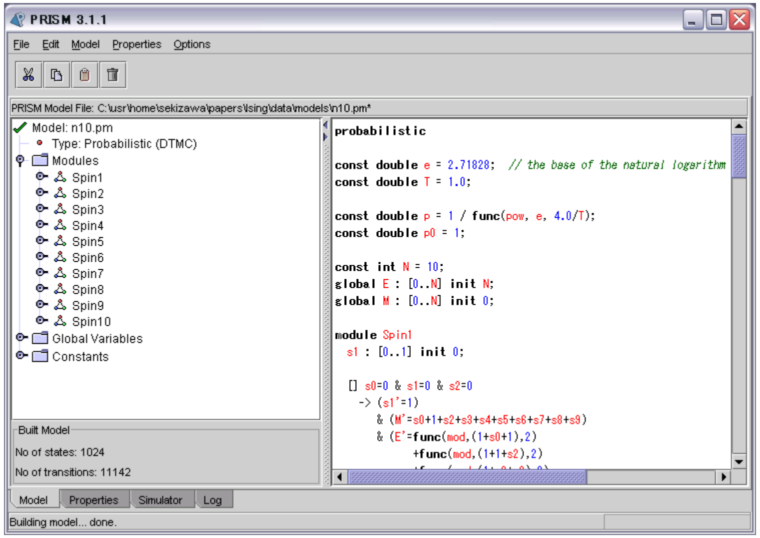
$$\mathbb{P}_{\geq 1}\left(\top \ \mathtt{U} \ p\right)$$

# PRISM

## Probabilistic Symbolic Model Checker

[http://www.prismmodelchecker.org/]

- input
  - a DTMC model
  - a property
    - PCTL formula
    - calculating probability
    - (transition) rewards
- output
  - true / false, probability, value of expected rewards



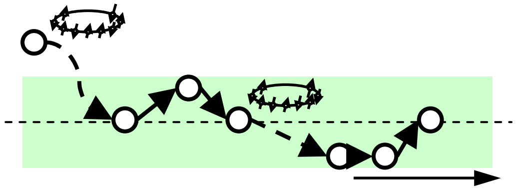
# Modelling

## Modelling the 1D Ising model

- 10 spins, $s_1$, …, $s_{10}$, located on a line in order.
- interaction coefficient $J$ = -1 (anti-ferromagnetic).
- boundary condition $s_{11} = s_1$.
- temperature $T$ is constant in a model.
- transition rule is the random spin flipping.
- each transition has value of reward 1.
- energy $E(s_1, s_2, \ldots, s_N) = -J \sum_{i=1}^{N} s_i s_{i+1}$
- magnetization $M(s_1, s_2, \ldots, s_N) = \sum_{i=1}^{N} s_i$

# Properties verified

"equilibrium is reachable from arbitrary state,
after reaching equilibrium,
the system stays in neighbourhood of equilibrium
within 100 times of spin flipping with probability
more than 70%."



$$\mathbb{P}_{\geq 1}(\top \ \mathtt{U} \ ((E = 0 \wedge M = 5) \wedge \psi_{\mathsf{in}}))$$

where $\psi_{\mathsf{in}} = \mathbb{P}_{\leq 0.3}(\chi_{\mathsf{lhs}} \ \mathtt{U}^{\leq 100} \ \chi_{\mathsf{rhs}})$,

$\chi_{\mathsf{lhs}} = (E \leq 2) \wedge (4 \leq M \wedge M \leq 6)$, and

$\chi_{\mathsf{rhs}} = (2 < E) \vee (M < 4 \vee 6 < M)$

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# Discussion

Probabilistic model checking and Computer simulation:

- probabilistic model checking
  based on exhaustive search.
  - advantage: formal definitions, reusability of models.
  - disadvantage: unsuitable to verify time dependency.
- computer simulation
  based on evaluation along time series.
  - advantage: suitable for statistic analysis.
  - disadvantage: unsuitable for formal methods.

# Conclusion

Probabilistic model checking

- is useful to analyze the (1D) Ising model.
- will be able to cooperate with computer simulation.

# Future work

- Solving the state explosion problem.
  - abstraction, symmetry reduction, etc.
- Analyzing the 2D Ising model.
  - more practical problems such as phase transition.
- Analyzing other probabilistic systems.
  - genetic algorithm, etc.

(end of slides)

This presentation is based on:

T. Sekizawa, T. Tsuchiya, T. Kikuno, and K. Takahashi,
***"Analyzing the One Dimensional Ising Model by Probabilistic Model Checking"***,
Proceedings of the IASTED Asian Conference on Modelling and Simulation, pp.199-204, ACTA Press, October 2007.