A note on the weak topology for the constructive completion of the space $\mathcal{D}(\mathbb{R})$

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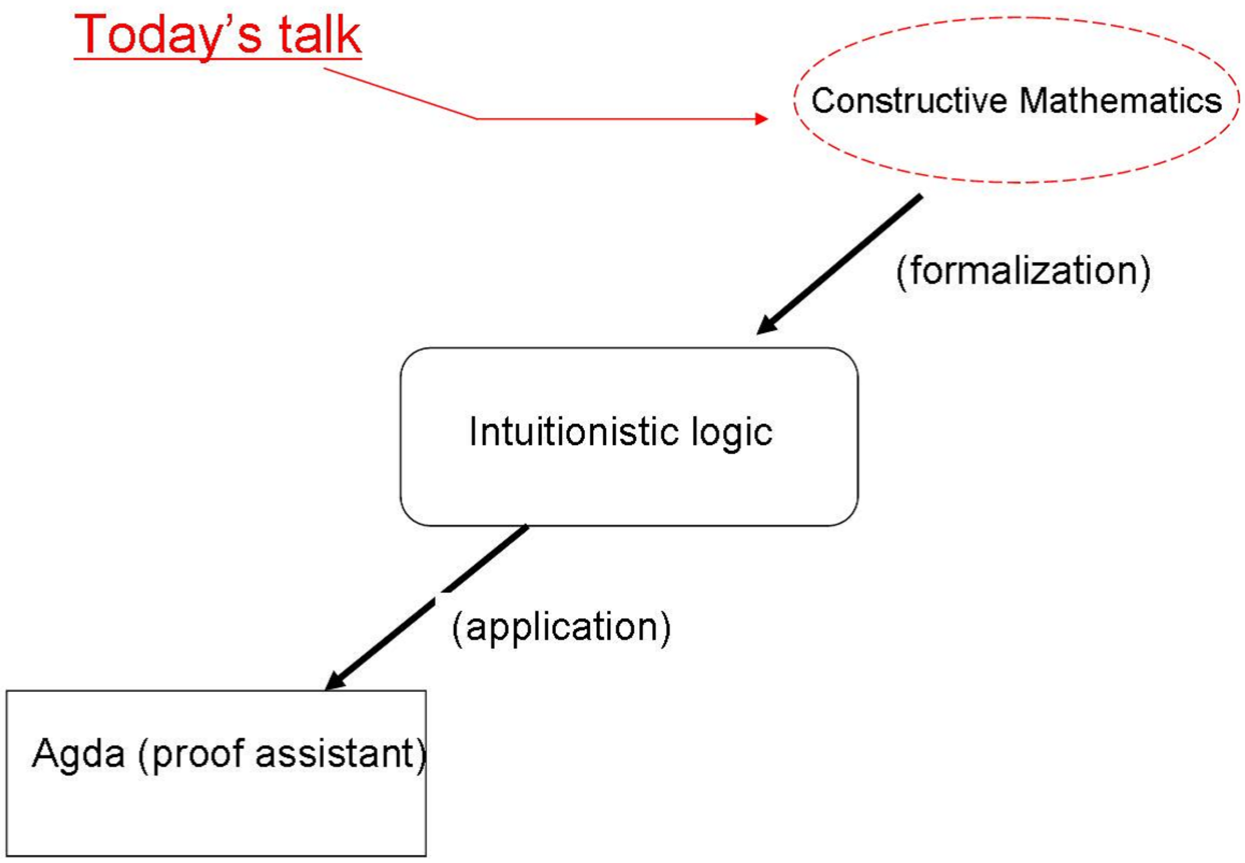

Today's talk
Constructive Mathematics
(formalization)
Intuitionistic logic
(application)
Agda (proof assistant)

Constructive mathematics is a framework of study for computability in mathematics.

Today we consider computability in generalized function theory, via constructive mathematics.

§1 Introduction

The space $\mathcal{D}(\mathbb{R})$ is an important example of a non-metrizable locally convex space.

A *distribution* (or *generalized function*) is a sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$ (a sequentially continuous linear function from $\mathcal{D}(\mathbb{R})$ into $\mathbb{R}$).

We had not had the start of constructive theory of generalized functions until the following problems, given by the 1960's monograph of Bishop, were solved:

1. How constructively difficult is the completeness properties of $\mathcal{D}(\mathbb{R})$ and its dual space $\mathcal{D}^*(\mathbb{R})$ (i.e. the space of distributions)?

2. Can we take the natural forms of the constructive completions of $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}^*(\mathbb{R})$ respectively ?

These problems are solved by the following results:

(1a) *$\mathcal{D}(\mathbb{R})$ is complete if and only if the principle BD-$\mathbb{N}$ is can be proved* [Ishihara and Y,'02].

(2) *The constructive completion $\widetilde{\mathcal{D}}(\mathbb{R})$ of $\mathcal{D}(\mathbb{R})$ is obtained by generalizing every element (test functions) of $\mathcal{D}(\mathbb{R})$* [Y, '05].

(1b) *The dual space $\widetilde{\mathcal{D}}^*(\mathbb{R})$ of $\widetilde{\mathcal{D}}(\mathbb{R})$ is equal to the dual space $\mathcal{D}^*(\mathbb{R})$ as sets, and the weak completeness of $\mathcal{D}^*(\mathbb{R})$ can be proved* [Y, '03].

$$
\begin{array}{ccccc}
\underline{\text{weakly complete}} \rightarrow & \widetilde{\mathcal{D}}^*(\mathbb{R}) & \underset{(1b)}{=} & \mathcal{D}^*(\mathbb{R}) & (*\text{: dual spaces}) \\
& \uparrow & & \uparrow & \\
(\sim\text{: completion}) & \underset{(2)}{\widetilde{\mathcal{D}}(\mathbb{R})} & \underset{(1a)}{\neq} & \mathcal{D}(\mathbb{R}) &
\end{array}
$$

Fig. The spaces $\mathcal{D}(\mathbb{R})$ and $\tilde{\mathcal{D}}(\mathbb{R})$ in Bishop's constructive mathematics.

Here

1a $\mathcal{D}(\mathbb{R})$ *is complete if and only if BD-*$\mathbb{N}$ *can be proved*.

Notes:

A subset A of $\mathbb{N}$ is *pseudobounded* if

$\forall\{a_n\} \in A^{\mathbb{N}} \exists N \in \mathbb{N}[a_n < n] \quad (n \geq N)$.

A bounded subset of $\mathbb{N}$ is pseudobounded.

BD-$\mathbb{N}$: every countable *pseudobounded* subset of $\mathbb{N}$ is bounded.

BD-$\mathbb{N}$ can be proved in

classical mathematics,

Brouwer's intuitionistic mathematics and

constructive recursive mathematics of Markov's school,

but not in Bishop's constructive mathematics.

2 The constructive completion $\widetilde{\mathcal{D}}(\mathbb{R})$ is obtained by generalizing every element (test functions) of $\mathcal{D}(\mathbb{R})$

Note:

- A *test function* is an infinitely differentiable function on $\mathbb{R}$ with *compact support*.

- A element of $\widetilde{\mathcal{D}}(\mathbb{R})$ is an infinitely differentiable function on $\mathbb{R}$ with *pseudobounded support*.

1b The weak completeness of the dual space $\mathcal{D}^*(\mathbb{R})$ holds.

This matter follows from the Banach-Steinhaus theorem for $\mathcal{D}(\mathbb{R})$: for a sequence $\{u_k\}$ of distributions (sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$), if the sequence $\{u_k(\phi)\}$ converges in $\mathbb{R}$ for any ϕ in $\mathcal{D}(\mathbb{R})$, then the limit u exists and is a distribution.

The completeness of $\mathcal{D}(\mathbb{R})$ is not necessary for proving the weak completeness of $\mathcal{D}^*(\mathbb{R})$, although many classical proofs require it.

Now this version of Banach-Steinhaus theorem was proved by showing the following properties.

- The Banach-Steinhaus theorem for the completion $\widetilde{\mathcal{D}}(\mathbb{R})$.

- every distribution is uniquely extended to $\widetilde{\mathcal{D}}(\mathbb{R})$.

In particular, the second implies that $\widetilde{\mathcal{D}}^*(\mathbb{R}) = \mathcal{D}^*(\mathbb{R})$ as <u>sets</u>.

We then have a question:
<u>is $\widetilde{\mathcal{D}}^*(\mathbb{R})$ topologically equivalent to $\mathcal{D}^*(\mathbb{R})$?</u>

We discuss this problem in this talk.

$$\widetilde{\mathcal{D}}^*(\mathbb{R}) \stackrel{?}{\simeq} \mathcal{D}^*(\mathbb{R}) \quad \text{(*: dual spaces)}$$

$$\uparrow \qquad\qquad \uparrow$$

$$\text{($\sim$: completion)} \quad \widetilde{\mathcal{D}}(\mathbb{R}) \neq \mathcal{D}(\mathbb{R})$$

Fig. The spaces $\mathcal{D}(\mathbb{R})$ and $\tilde{\mathcal{D}}(\mathbb{R})$ in Bishop's constructive mathematics.

Contents

§2 Preliminary

(2a) The principle BD-$\mathbb{N}$ in constructive mathematics

BHK(Brouwer-Heyting-Kolmogorov)-interpretation

	constructive math.	classical math.
$P \vee Q$	we can judge P or Q by finite procedures	$\neg P$ and $\neg Q$ imply a contradiction
$\exists x P(x)$	we can construct c such that $P(c)$ by finite procedures	"$\neg P(x)$ for all x" implies a contradiction

Under this interpretation, **Principle of Excluded Middle (PEM)**

for all proposition P, $P \vee \neg P$

and its negation cannot be proved.

Bishop's constructive mathematics is mathematics with BHK-interpretation.

The three frameworks of constructive mathematics:

- Brouwer's **intuitionistic mathematics**
- **Constructive recursive mathematics** of Markov's school
- Bishop's **Constructive mathematics**

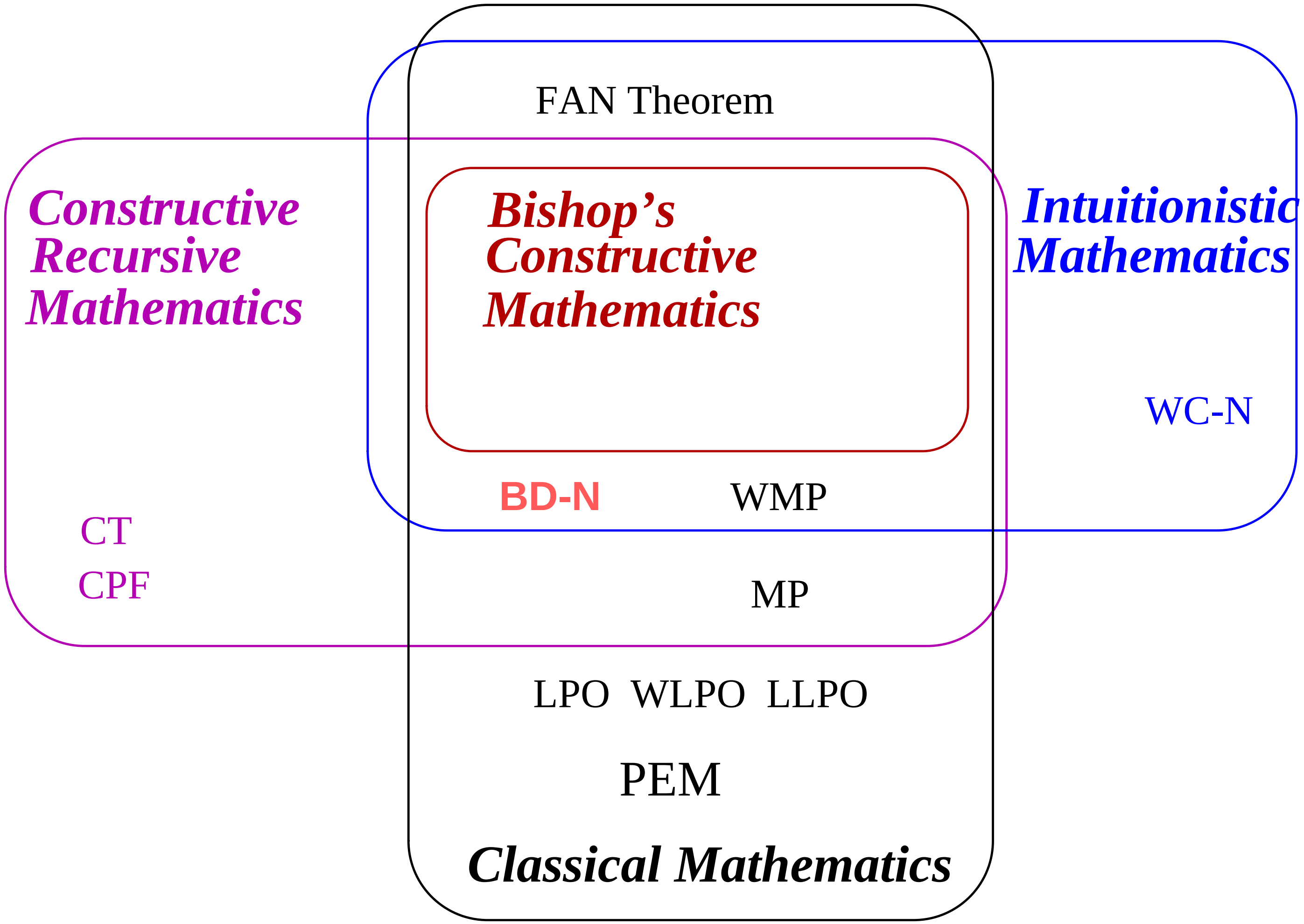


Fig.2.1 The sets of theorems in the four frameworks

(2b) Locally convex spaces

Let X be a vector space over $\mathbb{R}$.

A mapping $p : X \to \mathbb{R}^{0+}$ is a **seminorm** on X
if it satisfies that for $x, y \in X$ and $\lambda \in \mathbb{R}$, (1) $p(x+y) \leq p(x)+p(y)$
and (2) $p(\lambda x) = |\lambda| p(x)$.

A pair $(X, \{p_i\})$ is **locally convex space** over $\mathbb{R}$
if for all index i and x in X, whenever $p_i(x) = 0$ then $x = 0$.

$\{x_n\}$ **converges** to x in $X \overset{\text{def}}{\Leftrightarrow}$
$\forall k \in \mathbb{N} \forall i \in I \exists N \in \mathbb{N} \left[n \geq N \implies p_i(x - x_n) < 2^{-k} \right]$.

Let u be a linear functional on X.

u is **sequentially continuous** on $X \overset{\text{def}}{\Leftrightarrow}$
for each sequence $\{x_n\}$ in X and $x \in X$
$\{x_n\}$ converges to x in X
$\implies$ the sequence $\{u(x_n)\}$ converges to $u(x)$ in $\mathbb{R}$.

The **dual space** X^* **with weak topology** of a locally convex space $(X, \{p_i\})$ is a locally convex space of sequentially continuous linear functionals on X, with the seminorms $\{\|\cdot\|_x\}$ defined by $\|u\|_x := |u(x)| \quad (x \in X)$.

§3 The space $\mathcal{D}(\mathbb{R})$ and its completion $\widetilde{\mathcal{D}}(\mathbb{R})$

The set supp$f$ denotes the closure of the set $\{x \in \mathbb{R} : |f(x)| > 0\}$ in Euclid space $\mathbb{R}$.

Set $\quad \text{supp}_{\mathbb{N}} f := \{0\} \cup \{n \in \mathbb{N} : \exists q \in \mathbb{Q}[|q| \geq n \wedge |f(q)| > 0]\}$.

Notes that for a continuous function f, supp$f$ is bounded $\Longleftrightarrow$ $\text{supp}_{\mathbb{N}} f$ is bounded.

Example 1 (The bump function)**.**

$$\phi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then $\text{supp}\phi = [-1, 1]$ and $\text{supp}_{\mathbb{N}}\phi = \{0\}$.

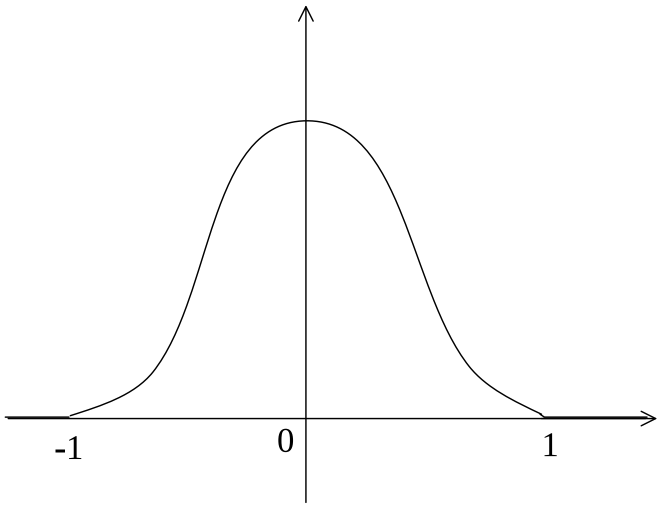


Fig.3.2 An example of a test function: the bump function

f *has compact support* $\overset{\text{def}}{\Leftrightarrow}$ supp$f$ is bounded.

f *has pseudobounded support* $\overset{\text{def}}{\Leftrightarrow}$ $\text{supp}_{\mathbb{N}} f$ is pseudobounded.

A *test function* is an infinitely differentiable functions from $\mathbb{R}$ into it itself with compact support.

$\mathcal{D}(\mathbb{R})$ denotes the locally convex space of test functions with the seminorms

$$p_{\alpha,\beta}(\phi) := \sup_{n} \max_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} \left| \phi^{(l)}(x) \right| \quad (\phi \in \mathcal{D}(\mathbb{R}), \alpha, \beta \in \mathbb{N} \to \mathbb{N}).$$

$\widetilde{\mathcal{D}}(\mathbb{R})$ denotes the locally convex space of infinitely differentiable functions on $\mathbb{R}$ with pseudobounded support, taken with the seminorms $\{p_{\alpha,\beta}\}$.

§4 he dual spaces $\mathcal{D}^*(\mathbb{R})$ and $\widetilde{\mathcal{D}}^*(\mathbb{R})$

Theorem 2 (Y, '03)**.** *Every distribution is uniquely extended to* $\widetilde{\mathcal{D}}(\mathbb{R})$*.*

That is, $\underline{\mathcal{D}^*(\mathbb{R}) = \widetilde{\mathcal{D}}^*(\mathbb{R}) \text{ as sets}}$.

Theorem 3. *A sequence* $\{u_n\}$ *of distributions converges to* 0 *in* $\mathcal{D}^*(\mathbb{R})$ *if and only if it does in* $\widetilde{\mathcal{D}}^*(\mathbb{R})$*.*

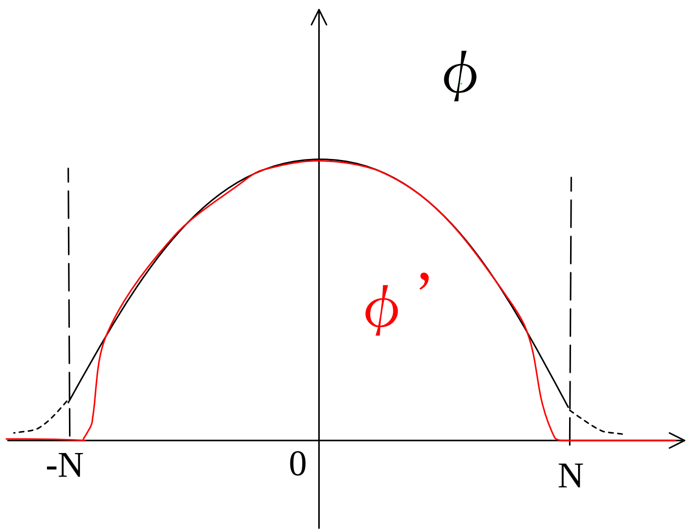
(Proof) We consider the part "only if".

It is sufficient to show that if $\{u_n\} \to 0$ in $\mathcal{D}^*(\mathbb{R})$, then $\{u_n\} \to 0$ in $\widetilde{\mathcal{D}}^*(\mathbb{R})$.

That is, we show that $\{u_n(\phi)\} \to 0$ for all $\underline{\phi \in \mathcal{D}^*(\mathbb{R})}$, then $\{u_n\} \to 0$ for all $\underline{\phi' \in \widetilde{\mathcal{D}}^*(\mathbb{R})}$.

Let $\{u_n\}$ be a given sequence of distributions, ϕ be in $\widetilde{\mathcal{D}}(\mathbb{R})$, and any k in $\mathbb{N}$.

We can then construct some ϕ' in $\mathcal{D}^*(\mathbb{R})$ such that for some N in $\mathbb{N}$, $|u_n(\phi')| < 2^{-(k+1)} \to |u_n(\phi)| < 2^{-k} \quad (n \geq N)$.



(q.e.d)

We again note that

> Given a sequence $\{u_n\}$ of distributions and an element ϕ in $\widetilde{\mathcal{D}}(\mathbb{R})$, we can construct a test function ϕ' such that for some N in $\mathbb{N}$,
>
> $$\|u_n\|_{\phi'} < 2^{-(k+1)} \rightarrow \|u_n\|_{\phi} < 2^{-k} \quad (n \geq N)$$

That is, $\widetilde{\mathcal{D}}^*(\mathbb{R})$ is equal to $\mathcal{D}^*(\mathbb{R})$ w.r.t. convergence.

$$\widetilde{\mathcal{D}}^*(\mathbb{R}) \simeq \mathcal{D}^*(\mathbb{R}) \quad \text{(*: dual spaces)}$$

$$\uparrow \qquad\qquad \uparrow$$

$$\text{($\sim$: completion)} \quad \widetilde{\mathcal{D}}(\mathbb{R}) \neq \mathcal{D}(\mathbb{R})$$

Open question:

Is $\widetilde{\mathcal{D}}^*(\mathbb{R})$ equal to $\mathcal{D}^*(\mathbb{R})$ w.r.t. neighbourhoods?

Reference

Satoru Yoshida,
A note on weak topology for the constructive completion of the space $\mathcal{D}(\mathbb{R})$,
http://unit.aist.go.jp/cvs/tr-data/ps07-001.pdf