A note on the weak topology for the constructive completion of the space $\mathcal{D}(\mathbb{R})$

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Today's talk

Constructive Mathematics

(formalization)

Intuitionistic logic

(application)

Agda (proof assistant)
Constructive mathematics is a framework of study for computability in mathematics.

Today we consider computability in generalized function theory, via constructive mathematics.
§1 Introduction

The space $\mathcal{D}(\mathbb{R})$ is an important example of a non-metrizable locally convex space.

A distribution (or generalized function) is a sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$ (a sequentially continuous linear function from $\mathcal{D}(\mathbb{R})$ into $\mathbb{R}$).

We had not had the start of constructive theory of generalized functions until the following problems, given by the 1960’s monograph of Bishop, were solved:
1. How constructively difficult is the completeness properties of $\mathcal{D}(\mathbb{R})$ and its dual space $\mathcal{D}^*(\mathbb{R})$ (i.e. the space of distributions)?

2. Can we take the natural forms of the constructive completions of $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}^*(\mathbb{R})$ respectively?

These problems are solved by the following results:

(1a) $\mathcal{D}(\mathbb{R})$ is complete if and only if the principle BD-$\mathbb{N}$ is can be proved [Ishihara and Y,'02].

(2) The constructive completion $\tilde{\mathcal{D}}(\mathbb{R})$ of $\mathcal{D}(\mathbb{R})$ is obtained by generalizing every element (test functions) of $\mathcal{D}(\mathbb{R})$ [Y, '05].

(1b) The dual space $\tilde{\mathcal{D}}^*(\mathbb{R})$ of $\tilde{\mathcal{D}}(\mathbb{R})$ is equal to the dual space $\mathcal{D}^*(\mathbb{R})$ as sets, and the weak completeness of $\mathcal{D}^*(\mathbb{R})$ can be proved [Y, '03].
weakly complete \rightarrow \tilde{\mathcal{D}}^* (\mathbb{R}) = \mathcal{D}^* (\mathbb{R}) \quad (\ast: \text{dual spaces})

(1b)

\uparrow \quad \uparrow

(\sim: \text{completion}) \quad \tilde{\mathcal{D}} (\mathbb{R}) \neq \mathcal{D} (\mathbb{R})

(2) (1a)

Fig. The spaces \mathcal{D}(\mathbb{R}) and \tilde{\mathcal{D}}(\mathbb{R}) in Bishop's constructive mathematics.
Here

1a \( D(\mathbb{R}) \) is complete if and only if \( BD-\mathbb{N} \) can be proved.

Notes:

A subset \( A \) of \( \mathbb{N} \) is \textit{pseudobounded} if
\[
\forall \{a_n\} \in \mathbb{N}^A \exists N \in \mathbb{N}[a_n < n] \quad (n \geq N).
\]

A bounded subset of \( \mathbb{N} \) is pseudobounded.

\( BD-\mathbb{N} \): every countable \textit{pseudobounded} subset of \( \mathbb{N} \) is bounded.

\( BD-\mathbb{N} \) can be proved in classical mathematics,
Brouwer's intuitionistic mathematics and constructive recursive mathematics of Markov's school,
but not in Bishop's constructive mathematics.
The constructive completion $\tilde{\mathcal{D}}(\mathbb{R})$ is obtained by generalizing every element (test functions) of $\mathcal{D}(\mathbb{R})$

Note:

- A *test function* is an infinitely differentiable function on $\mathbb{R}$ with *compact support*.

- A element of $\tilde{\mathcal{D}}(\mathbb{R})$ is an infinitely differentiable function on $\mathbb{R}$ with *pseudobounded support*.
The weak completeness of the dual space $\mathcal{D}^*(\mathbb{R})$ holds.

This matter follows from the Banach-Steinhaus theorem for $\mathcal{D}(\mathbb{R})$: for a sequence $\{u_k\}$ of distributions (sequentially continuous linear functionals on $\mathcal{D}(\mathbb{R})$), if the sequence $\{u_k(\phi)\}$ converges in $\mathbb{R}$ for any $\phi$ in $\mathcal{D}(\mathbb{R})$, then the limit $u$ exists and is a distribution.

The completeness of $\mathcal{D}(\mathbb{R})$ is not necessary for proving the weak completeness of $\mathcal{D}^*(\mathbb{R})$, although many classical proofs require it.
Now this version of Banach-Steinhaus theorem was proved by showing the following properties.

- The Banach-Steinhaus theorem for the completion $\tilde{D}(\mathbb{R})$.

- every distribution is uniquely extended to $\tilde{D}(\mathbb{R})$.

In particular, the second implies that $\tilde{D}^*(\mathbb{R}) = D^*(\mathbb{R})$ as sets.

We then have a question:

is $\tilde{D}^*(\mathbb{R})$ topologically equivalent to $D^*(\mathbb{R})$?

We discuss this problem in this talk.
\[
\tilde{\mathcal{D}}^*(\mathbb{R}) \overset{?}{\sim} \mathcal{D}^*(\mathbb{R}) \quad (\ast: \text{dual spaces})
\]

\[
\uparrow \quad \uparrow
\]

(\sim: \text{completion}) \quad \tilde{\mathcal{D}}(\mathbb{R}) \neq \mathcal{D}(\mathbb{R})

Fig. The spaces \(\mathcal{D}(\mathbb{R})\) and \(\tilde{\mathcal{D}}(\mathbb{R})\) in Bishop's constructive mathematics.
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§2 Preliminary

(2a) The principle BD-$\mathbb{N}$ in constructive mathematics

BHK(Brouwer-Heyting-Kolmogorov)-interpretation

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<td>$P \lor Q$</td>
<td>we can judge $P$ or $Q$ by finite procedures</td>
<td>$\neg P$ and $\neg Q$ imply a contradiction</td>
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<tr>
<td>$\exists x P(x)$</td>
<td>we can construct $c$ such that $P(c)$ by finite procedures</td>
<td>“$\neg P(x)$ for all $x$” implies a contradiction</td>
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Under this interpretation, Principle of Excluded Middle (PEM) for all proposition $P$, $P \lor \neg P$ and its negation cannot be proved.

Bishop’s constructive mathematics is mathematics with BHK-interpretation.
The three frameworks of constructive mathematics:

- Brouwer’s **intuitionistic mathematics**
- **Constructive recursive mathematics** of Markov’s school
- Bishop’s **Constructive mathematics**
Fig. 2.1 The sets of theorems in the four frameworks
(2b) Locally convex spaces

Let $X$ be a vector space over $\mathbb{R}$.

A mapping $p : X \to \mathbb{R}^{0+}$ is a seminorm on $X$ if it satisfies that for $x, y \in X$ and $\lambda \in \mathbb{R}$, (1) $p(x + y) \leq p(x) + p(y)$ and (2) $p(\lambda x) = |\lambda|p(x)$.

A pair $(X, \{p_i\})$ is locally convex space over $\mathbb{R}$ if for all index $i$ and $x$ in $X$, whenever $p_i(x) = 0$ then $x = 0$. 
\{x_n\} \textbf{converges} to \(x\) in \(X\) \(\iff\) 
\(\forall k \in \mathbb{N} \forall i \in I \exists N \in \mathbb{N} \left[n \geq N \implies p_i(x - x_n) < 2^{-k}\right].\)

Let \(u\) be a linear functional on \(X\).

\(u\) is \textbf{sequentially continuous} on \(X\) \(\iff\) for each sequence \(\{x_n\}\) in \(X\) and \(x \in X\)
\(\{x_n\}\) converges to \(x\) in \(X\)
\(\implies\) the sequence \(\{u(x_n)\}\) converges to \(u(x)\) in \(\mathbb{R}\).

The \textbf{dual space} \(X^*\) \textbf{with weak topology} of a locally convex space \((X, \{p_i\})\) is a locally convex space of sequentially continuous linear functionals on \(X\), with the seminorms \(\{\| \cdot \|_x\}\) defined by \(\|u\|_x := |u(x)|\) \((x \in X)\).
§3 The space $\mathcal{D}(\mathbb{R})$ and its completion $\tilde{\mathcal{D}}(\mathbb{R})$

The set $\text{supp} f$ denotes the closure of the set $\{x \in \mathbb{R} : |f(x)| > 0\}$ in Euclid space $\mathbb{R}$.

Set $\text{supp}_\mathbb{N} f := \{0\} \cup \{n \in \mathbb{N} : \exists q \in \mathbb{Q}[|q| \geq n \land |f(q)| > 0]\}$.

Notes that for a continuous function $f$, $\text{supp} f$ is bounded $\iff$ $\text{supp}_\mathbb{N} f$ is bounded.
Example 1 (The bump function).

\[
\phi(x) = \begin{cases} 
\exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\
0 & \text{if } |x| \geq 1.
\end{cases}
\]

Then \(\text{supp}\phi = [-1, 1]\) and \(\text{supp}_N\phi = \{0\}\).

Fig. 3.2 An example of a test function: the bump function
$f$ has compact support $\iff \text{supp} f$ is bounded.

$f$ has pseudobounded support $\iff \text{supp}_\mathbb{N} f$ is pseudobounded.

A test function is an infinitely differentiable functions from $\mathbb{R}$ into it itself with compact support.

$\mathcal{D}(\mathbb{R})$ denotes the locally convex space of test functions with the seminorms

$$p_{\alpha,\beta}(\phi) := \sup_n \max_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} |\phi^{(l)}(x)| \quad (\phi \in \mathcal{D}(\mathbb{R}), \alpha, \beta \in \mathbb{N} \to \mathbb{N}).$$

$\tilde{\mathcal{D}}(\mathbb{R})$ denotes the locally convex space of infinitely differentiable functions on $\mathbb{R}$ with pseudobounded support, taken with the seminorms $\{p_{\alpha,\beta}\}$. 
§4 The dual spaces $\mathcal{D}^*(\mathbb{R})$ and $\mathcal{\tilde{D}}^*(\mathbb{R})$

**Theorem 2** (Y, ’03). *Every distribution is uniquely extended to $\mathcal{\tilde{D}}(\mathbb{R})$.  
That is, $\mathcal{D}^*(\mathbb{R}) = \mathcal{\tilde{D}}^*(\mathbb{R})$ as sets.*
Theorem 3. A sequence \( \{u_n\} \) of distributions converges to 0 in \( \mathcal{D}^*(\mathbb{R}) \) if and only if it does in \( \tilde{\mathcal{D}}^*(\mathbb{R}) \).

(Proof) We consider the part “only if”.

It is sufficient to show that if \( \{u_n\} \rightarrow 0 \) in \( \mathcal{D}^*(\mathbb{R}) \), then \( \{u_n\} \rightarrow 0 \) in \( \tilde{\mathcal{D}}^*(\mathbb{R}) \).

That is, we show that \( \{u_n(\phi)\} \rightarrow 0 \) for all \( \phi \in \mathcal{D}^*(\mathbb{R}) \), then \( \{u_n\} \rightarrow 0 \) for all \( \phi' \in \tilde{\mathcal{D}}^*(\mathbb{R}) \).
Let \( \{u_n\} \) be a given sequence of distributions, \( \phi \) be in \( \mathcal{D}(\mathbb{R}) \), and any \( k \) in \( \mathbb{N} \).

We can then construct some \( \phi' \) in \( \mathcal{D}^*(\mathbb{R}) \) such that for some \( N \) in \( \mathbb{N} \),
\[
|u_n(\phi')| < 2^{-(k+1)} \quad \Rightarrow \quad |u_n(\phi)| < 2^{-k} \quad (n \geq N).
\]

(q.e.d)
We again note that

Given a sequence \( \{u_n\} \) of distributions and an element \( \phi \) in \( \tilde{\mathcal{D}}(\mathbb{R}) \), we can construct a test function \( \phi' \) such that for some \( N \) in \( \mathbb{N} \),

\[
\|u_n\|_{\phi'} < 2^{-(k+1)} \Rightarrow \|u_n\|_{\phi} < 2^{-k} \quad (n \geq N)
\]

That is, \( \tilde{\mathcal{D}}^*(\mathbb{R}) \) is equal to \( \mathcal{D}^*(\mathbb{R}) \) w.r.t. convergence.
\[ \tilde{\mathcal{D}}^*(\mathbb{R}) \simeq \mathcal{D}^*(\mathbb{R}) \quad (\ast: \text{dual spaces}) \]

\[ \uparrow \quad \uparrow \]

(\sim: \text{completion}) \quad \tilde{\mathcal{D}}(\mathbb{R}) \neq \mathcal{D}(\mathbb{R})
Open question:

Is $\tilde{\mathcal{D}}^*(\mathbb{R})$ equal to $\mathcal{D}^*(\mathbb{R})$ w.r.t. neighbourhoods?

Reference

Satoru Yoshida,
A note on weak topology for the constructive completion of the space $\mathcal{D}(\mathbb{R})$,