First-Order Substructural Logics: An Algebraic Approach

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Substructural logics have been traditionally characterized as those logics such that, when presented by means of a (sequent) proof system, lack one or more of the usual structural rules: weakening, exchange and contraction. Nonetheless, it is well known that they can also be, very usefully, roughly described as the *logics of residuated lattices*. Indeed, from this point of view we have seen, specially in the last decade, a florescence of works on propositional substructural logics, mainly capitalizing on the fact that they can be given an algebraic semantics based on some class of (expansions or (sub)reducts of) residuated lattices, and hence by using the tools and techniques from (Abstract) Algebraic Logic (see e.g. [8]). The same applies, to a lesser extent, to first-order formalisms for substructural logics, inasmuch they can given a semantics which, though not purely algebraic as in the propositional case, it contains an essential algebraic part together with a domain of individuals to interpret first-order variables and terms. Prominent examples of this approach are the following:

- Rasiowa-Sikorski style Intuitionistic first-order predicate logic (see [12])
- Rasiowa implicative first-order predicate logics (see [11])
- Gödel-Dummett first-order predicate logic (see [6])
- First-order fuzzy logics (Hájek, Esteva, Godo, ...) (see e.g. [9, 7, 10, 2])

In all these cases one starts from a propositional logic, which enjoys an equivalent algebraic semantics in the sense of [1], and an implication connective defining an order relation on the algebras. Then, a first-order language is defined in the traditional way (with the usual notions of variables, functionals, predicates, existential and universal quantifiers, bounded and free occurrences of variables, notion of substitutability, atomic formulae, formulae etc.) by using the connectives of the underlying propositional logic to combine first-order atomic formulae and quantifications thereof. Finally, a semantics for this syntax is introduced in such a way that: (1) predicate symbols are interpreted as mappings sending (tuples of) individuals in the domain to truth-values in a member of the algebraic semantics, (2) connectives are interpreted as their corresponding operations in the algebra, (3) the existential (resp. universal) quantification of a formula is interpreted as the supremum (resp. infimum) of the values of its instances w.r.t. the order given by the implication.

In this talk we will generalize this approach to first-order logics based on any propositional substructural logic[3] and obtain the following results:

1. A general uniform proof of completeness theorem for these logics.
2. In particular, we will have (1) a notion of *minimal* first-order logic over a given propositional logic L, which is complete w.r.t. the widest algebraic semantics based on L-algebras and its axiomatization, and (2) a notion of *first-order logic over* L which are complete w.r.t. subclasses of that semantics.
3. By means of a generalized notion of disjunction,[4] we obtain a characterization and axiomatization of the first-order logic over a given L which is complete w.r.t. the semantics based on RFSI L-algebras (in particular, we will have the axiomatization of the extension given by linearly ordered algebras).
4. A form of Skolemization for these logics.

The technical details of our results will be available in the forthcoming paper [5].

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[3]We assume that substructural logics are algebraizable finitary expansions of the implicational fragment of the logic of non-associative residuated lattices.

[4]The necessary background will be introduced in the previous talk CINTULA, NOGUERA: *Lattice disjunction is not a disjunction (in many substructural logics)* and it is based on the papers [3, 4].

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