Lattice disjunction is not a disjunction (in many substructural logics)

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Disjunction is a crucial connective in both classical and intuitionistic logics. One of its most natural properties is the *proof by cases property* (PCP for short):

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}$$

In substructural logics[3] the situation is quite different. We start by showing that their usual lattice connective $\vee$ is not actually a good disjunction (because it does not satisfy PCP) in neither FL nor FL_e nor FL_w (see [8] for definitions of these logics), while it works in all extensions of FL_{ew}. This, of course, entails several relevant questions:

1. Can we characterize those substructural logics where $\vee$ satisfies PCP?

2. Can we, for a given substructural logic, axiomatize its weakest extension where $\vee$ satisfies PCP?

3. Is it possible to find a *different* connective (say ∇) which would satisfy PCP?

4. Why do we care? Is disjunction (and PCP for that matter) really that important?

The goal of this talk is to answer these questions. Let us now show some hints to answer them (for full details see [3, 4]). We will see that the third question is formulated "too optimistically": it will not be enough to consider a different connective, but we will be forced to work with a *generalized* connective, similarly to the generalized equivalences well-known from abstract algebraic logic. However our useful notation will hide the generalized nature of our new disjunctions; indeed, given a parameterized set of formulae $\nabla(p, q, \overrightarrow{r})$ we define:

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}^{\leq \omega}\}.$$

We say that (parameterized) set of formulae ∇ is a *(p-)disjunction* in a substructural logic L, whenever it satisfies (a form of) PCP:

$$\Gamma, \varphi \vdash_{\mathrm{L}} \chi \text{ and } \Gamma, \psi \vdash_{\mathrm{L}} \chi \text{ if and only if, } \Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi.$$

This level of generality is actually needed as shown by the following examples:

- The connective $\varphi \nabla \psi = (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$ is a disjunction in FL_e.
- The set $\varphi \nabla \psi = \{(p \to q) \to q, (q \to p) \to p\}$ is a disjunction in the purely implicational fragment of the Gödel-Dummett logic (see [6]) and no single formula can define a disjunction in this logic.
- The infinite set with parameters

  $$\varphi \nabla \psi = \{\gamma_1(\varphi \wedge \overline{1}) \vee \gamma_2(\psi \wedge \overline{1}) \mid \text{where } \gamma_1, \gamma_2 \text{ are iterated conjugates}\}$$

  is a p-disjunction in FL and no finite set of formulae can define a disjunction in this logic.

Now we can start answering the questions:

1. We can show that ∇ is a disjunction in a given logic iff it satisfies certain simple conditions and for each rule $\varphi_1, \ldots, \varphi_n \vdash \psi$ of some of its axiomatic systems the logic proves $\varphi_1 \nabla \chi, \ldots, \varphi_n \nabla \chi \vdash \psi \nabla \chi$. This, in particular, entails that the property of being a disjunction is preserved in axiomatic extensions of a given logic. We can also provide other characterizations (e.g. based on a generalized notion of prime filter).

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[3]In this talk we assume that substructural logics are algebraizable finitary expansions of the implicational fragment of the logic of non-associative residuated lattices.

2. All we have to do is to add the necessary rules required in point **1.** In particular the rule $\varphi_1 \vee \chi, \varphi_2 \vee \chi \vdash (\varphi_1 \vee \varphi_2) \vee \chi$ axiomatizes the weakest logic extending FL_e, where $\vee$ is disjunction.

3. It is a known result [5] that in each filter-distributive substructural logic some p-disjunction has to exist. Answer **1.** tells us how to recognize one. Other methods to construct a (p-)disjunction use the description of filters generated by a set or the (parameterized) local deduction theorem.

4. Why are disjunctions and PCP so important? Well, in logics with a (p-)disjunction ∇ we can use it to:

- axiomatize any of its extensions given by a universal class of its algebras, where ∇ is still a p-disjunction.
- axiomatize any of its extensions given by a positive-universal class of its algebras (a class given by positive universal sentences, i.e., disjuncions of identities), extending a known result by Galatos [7].
- axiomatize the extension given by its *linearly ordered algebras* (i.e. the minimum *semilinear*, or *fuzzy*, logic above it).
- characterize which of its extensions are complete w.r.t. *densely-linearly ordered algebras*, using the so-called *density rule* [1].
- axiomatize the intersections of its axiomatic extensions (or equivalently, the joints of (relative) subvarieties of the corresponding algebras).
- given a class $\mathbb{K}$ of its algebras, we can characterize strong completeness and finite strong completeness of our logic w.r.t. $\mathbb{K}$ in terms of embedding properties, generalizing results in [2].
- construct its first-order variant which remains complete w.r.t. finitely relatively-subdirectly irreducible algebras (i.e., the class of its linearly ordered algebras in the case of semilinear logics).[4]

Furthermore we can solve several open problems (e.g. necessity of using $\triangle$ in axiomatizing $\mathrm{MTL}_{\sim}$) and give alternative proof of theorems (e.g. showing that the logic $\mathrm{PŁ}'$ is complete w.r.t. chains) from the fuzzy logic literature.

References

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[4] More on this topic will be explained in the talk CINTULA, NOGUERA: *First-Order Substructural Logics: an algebraic approach.*