Quasi-subtractive varieties

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Algebras in classical varieties have the property that their lattice of congruences is isomorphic to the lattice of some "special" subsets: e.g. normal subgroups of groups, two-sided ideals of rings, filters (or ideals) of Boolean algebras. Why do such pleasant theorems occur? A rather satisfactory explanation is usually provided along the following lines:

- In every member $\mathbf{A}$ of a τ-regular variety $\mathbb{V}$ - namely, a variety that arises as the equivalent algebraic semantics of an algebraisable deductive system - the lattice of congruences is isomorphic to the lattice of deductive filters on $\mathbf{A}$ of the τ-assertional logic of $\mathbb{V}$ ([1]; [2]; [3]).
- In the pointed case, if $\mathbb{V}$ is 1-subtractive [7], then the deductive filters on $\mathbf{A} \in \mathbb{V}$ of the 1-assertional logic of $\mathbb{V}$ coincide with the $\mathbb{V}$-ideals of $\mathbf{A}$ in the sense of [5], which is even better due to the availability of a manageable concept of *ideal generation*.

However, ideal-congruence isomorphism theorems abound in the literature that are not subsumed by this picture: for example, the correspondence between open filters and congruences in pseudointerior algebras, or between deductive filters and congruences in residuated lattices, or between ideals and MV congruences in quasi-MV algebras. The aim of the present talk is to appropriately generalise the concepts of subtractivity and τ-regularity in such a way as to obtain a general framework encompassing all the previously mentioned results.

A variety $\mathbb{V}$ whose type ν includes a nullary term 1 and a unary term $\Box$ is called *quasi-subtractive* w.r.t. 1 and $\Box$ iff there is a binary term $\rightarrow$ of type ν s.t. $\mathbb{V}$ satisfies the equations

(i) $\Box x \rightarrow x \approx 1$ (iii) $\Box(x \rightarrow y) \approx x \rightarrow y$
(ii) $1 \rightarrow x \approx \Box x$ (iv) $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1$

If $\Box$ is the identity, this definition collapses onto the standard definition of subtractive variety. If it is not, we call $\mathbb{V}$ *properly* quasi-subtractive.

Examples include: subtractive varieties (and their nilpotent shifts), pseudointerior algebras, Boolean algebras with operators, residuated lattices, Nelson algebras, subresiduated lattices, basic algebras, quasi-MV algebras. Some of these varieties are not subtractive; some are but can be viewed as properly quasi-subtractive with a different choice of witness terms.

The most important tool for the investigation of quasi-subtractive varieties is the notion of *open filter*, which is to quasi-subtractive varieties as the concept of Gumm-Ursini ideal is to subtractive varieties. A $\mathbb{V}$-*open filter term* in the variables $\overrightarrow{x}$ is is an $n+m$-ary term $p(\overrightarrow{x}, \overrightarrow{y})$ of type ν s.t.

$$\{\Box x_i \approx 1 : i \leq n\} \vdash_{Eq(\mathbb{V})} \Box p(\overrightarrow{x}, \overrightarrow{y}) \approx 1.$$

A $\mathbb{V}$-*open filter* of $\mathbf{A} \in \mathbb{V}$ is a subset $F \subseteq A$ which is closed w.r.t. all $\mathbb{V}$-open filter terms p (i.e. whenever $a_1, ..., a_n \in F, b_1, ..., b_m \in A$, $p\left(\overrightarrow{a}, \overrightarrow{b}\right) \in F$) and such that for every $a \in A$, we have that $a \in F$ iff $\Box a \in F$.

Finally, a variety $\mathbb{V}$ is called *weakly* $(\Box x, 1)$*-regular* iff the $\{\Box x \approx 1\}$-assertional logic $\mathcal{S}(\mathbb{V})$ of $\mathbb{V}$ is strongly and finitely algebraisable.

We have achieved several results, including:

1) A proof that every quasi-subtractive variety $\mathbb{V}$ is such that for all $\mathbf{A} \in \mathbb{V}$, every $\mathbb{V}$-open filter of $\mathbf{A}$ is a τ-class of some $\theta \in \mathrm{Con}(\mathbf{A})$ (for $\tau = \{\Box x \approx 1\}$);

2) A proof that, if $\mathbb{V}$ is quasi-subtractive and $\mathbf{A} \in \mathbb{V}$, $\mathbb{V}$-open filters of $\mathbf{A}$ coincide with deductive filters on $\mathbf{A}$ of $\mathcal{S}(\mathbb{V})$;

3) A manageable description of generated $\mathbb{V}$-open filters;

4) A decomposition of a quasi-subtractive variety as a subdirect product of a subtractive and a "flat" variety (together with conditions under which the decomposition is direct), having as corollaries the Galatos-Tsinakis decomposition theorem for GMV algebras [4] and the Jónsson-Tsinakis decomposition theorem for the join $\mathbb{LG} \vee \mathbb{IRL}$ in the lattice of subvarieties of residuated lattices [6];

5) An investigation of constructions generalising kernel contractions in residuated lattices;

6) A proof that, if $\mathbb{V}$ is quasi-subtractive and weakly $(\Box x, 1)$-regular and $\mathbb{V}'$ is the equivalent algebraic semantics of $\mathcal{S}(\mathbb{V})$, then in any $\mathbf{A} \in \mathbb{V}$ there is a lattice isomorphism between the lattice of $\mathbb{V}$-$\mathbb{V}'$ congruences on $\mathbf{A}$ and the lattice of $\mathbb{V}$-open filters on $\mathbf{A}$.

In view of the examples mentioned above, we believe that the notion of quasi-subtractive variety could provide a common umbrella for the algebraic investigation of several families of logics, including substructural logics, modal logics, quantum logics, logics of constructive mathematics.

References

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