Pre-ideals of basic algebras

Jan Kühr and Jan Krňávek

Department of Algebra and Geometry, Palacký University at Olomouc
Tř. 17. listopadu 12, 77146 Olomouc, Czech Republic
e-mail: kuhr@inf.upol.cz

A *basic algebra* [3] is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ that satisfies the identities

$$
\begin{aligned}
&x \oplus 0 = x, \\
&\neg\neg x = x, \\
&\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x, \\
&\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0.
\end{aligned}
$$

These algebras have been introduced in [3] as a counterpart of bounded lattices with sectional antitone involutions and can be regarded as a common generalization of MV-algebras and orthomodular lattices, also including lattice effect algebras. More precisely, for every basic algebra $(A, \oplus, \neg, 0)$, the underlying order defined by $x \leq y$ iff $\neg x \oplus y = \neg 0$ makes A into a bounded lattice where, for each $a \in A$, the map $x \mapsto \neg x \oplus a$ is an antitone involution on the principal filter $[a)$. On the other hand, if we are given a bounded lattice with sectional antitone involutions $x \mapsto x^a$ $(a \in A)$, then the rule $x \oplus y := (x^0 \vee y)^y$ and $\neg x := x^0$ defines a basic algebra. Concerning the interconnection between basic algebras, MV-algebras and orthomodular lattices, MV-algebras are precisely the associative basic algebras (commutativity is then derivable from the other axioms), and orthomodular lattices may be characterized as basic algebras satisfying the identity $x \oplus (x \wedge y) = x$.

We should emphasize that our basic algebras are not much connected with Hájek's basic logic and BL-algebras; in fact, the intersection with BL-algebras are MV-algebras. We used the name 'basic algebra' just to express the fact that these algebras capture some basic common features of all the structures considered in [3].

In the present paper we study what we call *pre-ideals*, i.e., non-empty subsets that are closed under $\oplus$ and downwards closed. Since the variety of basic algebras is ideal determined, the term ‘ideal’ is reserved for the congruence kernels. It is not hard to show that the ideal lattice forms a complete sublattice of the lattice of pre-ideals.

We restrict ourselves to basic algebras satisfying the identity

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z) \tag{1}$$

or, more generally, the identity

$$\neg x \oplus (x \wedge y) = \neg x \oplus y. \tag{2}$$

Some results are also proved for commutative basic algebras. These algebras are much closer to MV-algebras; for instance, the induced lattice is distributive, and we prove that finite basic algebras satisfying (2) are automatically MV-algebras.

The first important observation is that basic algebras satisfying (2) have the following *Riesz decomposition property*: If $a \leq \tau(b_1, \dots, b_n)$, where τ is an additive term, then there exist $c_i \leq b_i$ such that $a = \tau(c_1, \dots, c_n)$.

Further, for every $a \in A$ and integer $n \geq 0$ we define $n \otimes a$ inductively: $0 \otimes a := 0$ and $n \otimes a := a \oplus ((n-1) \otimes a)$ for $n \geq 1$.

If A is finite and satisfies (2), then for every atom $a \in A$, the set $N(a) = \{n \otimes a \mid n \geq 0\}$ is a finite chain $0 < a < \cdots < \hat{a}$. The Riesz decomposition property entails $N(a) = [0, \hat{a}]$. Since intervals in basic algebras are basic algebras, it follows that $N(a)$ is a finite MV-chain.

Theorem 1. *Let A be a finite basic algebra satisfying* (2). *Then A is isomorphic to the direct product $\prod_{a \in M} N(a)$ where M denotes the set of atoms of A. Consequently, A is an MV-algebra.*

Given a basic algebra $(A, \oplus, \neg, 0)$ satisfying (2) and its pre-ideal I, we can define an equivalence relation θ_I on A as follows: $(x, y) \in \theta_I$ iff $x = a_1 \oplus (\cdots \oplus (a_m \oplus y') \dots)$ and $y = b_1 \oplus (\cdots \oplus (b_n \oplus x') \dots)$ for some $a_i, b_j \in I$ and $x', y' \in A$ with $x' \leq x$ and $y' \leq y$. Then θ_I is compatible with the meet-operation; the underlying order of A/θ_I is given by: $[x]_{\theta_I} \leq [y]_{\theta_I}$ iff $x = a_1 \oplus (\cdots \oplus (a_m \oplus y') \dots)$ for some $a_i \in I$ and $y' \in A$ with $y' \leq y$. Moreover, if I is an ideal, i.e., $I = [0]_\phi$ for some congruence ϕ, then $\theta_I = \phi$.

Next, we focus on basic algebras satisfying (1). It is easy to show that the pre-ideal generated by $\emptyset \neq X \subseteq A$ consists of those $a \in A$ which are less than or equal to some finite sum of elements of X. The pre-ideal lattice is an algebraic distributive lattice, and we can characterize its meet-prime elements that we call *prime pre-ideals* of A.

Theorem 2. *Let* $(A, \oplus, \neg, 0)$ *be a basic algebra that satisfies* (1). *Then for every pre-ideal* P*, the following are equivalent:*

(i) P *is prime;*

(ii) for all $x, y \in A$*, if* $x \wedge y \in P$*, then* $x \in P$ *or* $y \in P$*;*

(iii) for all $x, y \in A$*, if* $x \wedge y = 0$*, then* $x \in P$ *or* $y \in P$*;*

(iv) for all $x, y \in A$*,* $\neg(\neg x \oplus y) \in P$ *or* $\neg(\neg y \oplus x) \in P$*;*

(v) $(A/\theta_P, \leq)$ *is a chain;*

(vi) the set of all pre-ideals containing P *is a chain under inclusion.*

In what follows, let $(A, \oplus, \neg, 0)$ be a commutative basic algebra. We say that a pre-ideal is *closed* if it is closed under all existing suprema. By a *value* of $0 \neq a \in A$ we mean a pre-ideal which is maximal with respect to not containing a. If a pre-ideal is the only value of some non-zero element, then we call it a *special value*.

Theorem 3. *Every special value is closed.*

We denote by $D(A)$ the intersection of all closed prime pre-ideals of A and call it the *distributive radical* of A. It easily follows that $D(A)$ equals the intersection of the closed values and contains those elements having no closed values.

A basic algebra is *completely distributive* if

$$\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f \colon I \to J} \bigwedge_{i \in I} a_{if(i)}$$

for all $\{a_{ij} \mid i \in I, j \in J\}$ for which the indicated suprema and infima exist.

Theorem 4. *A commutative basic algebra* $(A, \oplus, \neg, 0)$ *is completely distributive if and only if* $D(A) = \{0\}$.

References

[1] M. Botur: *An example of a commutative basic algebra which is not an MV-algebra*, Math. Slovaca, to appear.

[2] M. Botur and R. Halaš: *Finite commutative basic algebras are MV-algebras*, J. Mult.-Val. Log. Soft Comput. **14** (2008), 69–80.

[3] I. Chajda, R. Halaš and J. Kühr: *Many valued quantum algebras*, Algebra Univers. **60** (2009), 63–90.