FINITENESS PROPERTIES FOR IDEMPOTENT RESIDUATED STRUCTURES

JAMES RAFTERY AND AI-NI HSIEH

(to be presented by Raftery)

A class K of similar algebras is said to have the *finite embeddability property* (briefly, the *FEP*) if every finite subset of an algebra in K can be extended to a finite algebra in K, with preservation of all partial operations. If a finitely axiomatized variety or quasivariety of finite type has the FEP, then its universal first order theory is decidable, hence its equational and quasi-equational theories are decidable as well. Where the algebras are residuated ordered groupoids, these theories are often interchangeable with logical systems of independent interest. Partly for this reason, there has been much recent investigation of finiteness properties such as the FEP in varieties of residuated structures.

A residuated partially ordered monoid is said to be *idempotent* if its monoid operation is idempotent. In this case, the partial order is equationally definable, so the structures can be treated as pure algebras. Such an algebra is said to be *conic* if each of its elements lies above or below the monoid identity $\mathtt{t}$; it is *semiconic* if it is a subdirect product of conic algebras. We prove that

> *the class* SCIP *of all semiconic idempotent commutative residuated po-monoids is locally finite,*

i.e., every finitely generated member of this class is a finite algebra. It turns out that SCIP is a quasivariety; it is not a variety.

The lattice-ordered members of SCIP form a variety SCIL, provided that we add the lattice operations $\wedge, \vee$ to the similarity type. This variety is not locally finite, but the local finiteness of SCIP facilitates a proof that SCIL has the FEP. In fact, we show that

> *for every relative subvariety* K *of* SCIP, *the lattice-ordered members of* K *form a variety with the FEP.*

(A *relative subvariety* of SCIP is a subclass axiomatized, relative to SCIP, by some set of equations.) It is also shown that

> SCIL *has a continuum of semisimple subvarieties.*

Note that SCIL contains all Brouwerian lattices, i.e., the algebraic models of positive intuitionistic logic. It also includes all positive Sugihara monoids (cf. [3]); these algebras model the positive fragment of the system **R**–mingle. The results here give a unified explanation of the strong finite model property for many extensions of these and other systems. They generalize Diego's Theorem, as well as the main theorem of [5], which showed that the variety generated by all idempotent commutative residuated *chains* is locally finite. Another generalization of the latter result, in a different direction, has been obtained in [4]. Further, we show that

> *the involutive algebras in* SCIL *are subdirect products of chains.*

Although SCIL is finitely axiomatized, it is not clear whether SCIP has a finite basis. Motivated in part by this question, we consider the larger quasivariety IP of *all* idempotent commutative residuated po-monoids. It is proved that

> *a relative subvariety of* IP *consists of semiconic algebras if and only if it satisfies* $x \approx (x \to \mathsf{t}) \to x$.

It follows that SCIP is not itself a relative subvariety of IP. The result also has corollaries for the logical system $\mathbf{RMO}^*$, which adds fusion and the Ackermann truth constant to Anderson and Belnap's $\mathbf{RMO}_{\to}$ in a natural (and conservative) manner. Because the axiomatic extensions of $\mathbf{RMO}^*$ are in one-to-one correspondence with the relative subvarieties of IP, we can infer the following:

> *If an axiomatic extension of* $\mathbf{RMO}^*$ *has* $((p \to \mathsf{t}) \to p) \to p$ *among its theorems, then it is locally tabular*

(i.e., it has only finitely many inequivalent n–variable formulas, for every finite n). In particular, such an extension is strongly decidable, provided that it is finitely axiomatized.

Most of the results reported here have been written up in [1, 2].

References

[1] A. Hsieh, *Some locally tabular logics with contraction and mingle*, Rep. Math. Logic **45** (2010), 143–159.

[2] A. Hsieh, J.G. Raftery, *Semiconic idempotent residuated structures*, Algebra Universalis **61** (2009), 413–430.

[3] J.S. Olson, J.G. Raftery, *Positive Sugihara monoids*, Algebra Universalis **57** (2007), 75–99.

[4] J.S. Olson, J.G. Raftery, *Residuated structures, concentric sums and finiteness conditions*, Communications in Algebra **36** (2008), 3632–3670.

[5] J.G. Raftery, *Representable idempotent commutative residuated lattices*, Trans. Amer. Math. Soc. **359** (2007), 4405–4427.

School of Mathematical Sciences, University of KwaZulu-Natal, Westville Campus, Private Bag X54001, Durban 4000, South Africa

E-mail address: raftery@ukzn.ac.za

School of Mathematical Sciences, University of KwaZulu-Natal, Westville Campus, Private Bag X54001, Durban 4000, South Africa

E-mail address: akongrung@gmail.com