LEIBNIZ INTERPOLATION PROPERTIES

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In the framework of Abstract Algebraic Logic (see [2] for a general survey) a number of different interpolation properties have been considered: Maehara Interpolation Property, Robinson Interpolation Property, and Deductive Interpolation Property among others (see [1] as well as the references therein).

In [3], Kihara and Ono introduce some new notions of interpolation for substructural logics. For instance, one of the interpolation properties that they study is the following: for any arbitrary pair of formulas φ, ψ, if $\vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$, then there exists a formula δ with variables $\mathrm{var}(\delta) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ such that

$$\vdash (\varphi\backslash\delta) \wedge (\delta\backslash\varphi) \quad \text{and} \quad \vdash (\delta\backslash\psi) \wedge (\psi\backslash\delta).$$

In the same paper they present algebraic characterizations for all these notions and investigate their relation with the usual interpolation properties. The authors also remark: "only a few properties specific to FL-algebras or residuated lattices are used in our discussion," and they propose the study of these interpolation properties in a more general setting. The objective of the present paper is to give an answer to that remark by exhibiting a general framework in which these interpolation properties can be expressed and characterized by means of algebraic tools.

First, we note that $\nabla(x, y) = \{(x\backslash y) \wedge (y\backslash x)\}$ is a set of equivalence formulas (in the sense of Abstract Algebraic Logic) for the subestructural logics, that is, it defines the Leibniz congruences of their models. In particular, for every theory Γ and every pair of formulas φ, ψ,

$$\Gamma \vdash \nabla(\varphi, \psi) \quad \Leftrightarrow \quad \langle \varphi, \psi \rangle \in \boldsymbol{\Omega}\Gamma.$$

This motivates the choice of the equivalential logics, i.e., logics having a set of equivalence formulas, as our general setting.

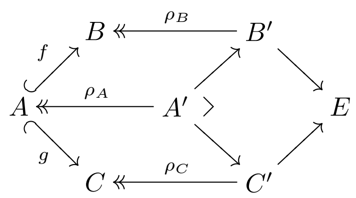
We introduce a family of what we call 'Leibniz' interpolation properties, in which we are mainly interpolating pairs of elements in the Leibniz congruence of a particular free algebra. Applied to the case of substructural logics, some of them coincide with those introduced in [3].

The first notion of interpolation that we introduce is the *Leibniz Interpolation Property* (LIP). A sentential logic $\mathcal{S}$ has the LIP if for every pair of formulas φ, ψ, if $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega}_{\mathcal{S}}$, then there exists a formula δ with variables $\mathrm{var}(\delta) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ such that $\langle \varphi, \delta \rangle, \langle \delta, \psi \rangle \in \boldsymbol{\Omega}_{\mathcal{S}}$. Here, $\boldsymbol{\Omega}_{\mathcal{S}}$ stands for the Leibniz congruence of the set of theorems of $\mathcal{S}$.

By adding new conditions we can strengthen this property in several different ways. For instance, we say that a sentential logic $\mathcal{S}$ has the *Leibniz Interpolation Property for Theories* (LIPT) if for every set of formulas Γ and every pair of formulas φ, ψ, if $\langle \varphi, \psi \rangle \in \boldsymbol{\Omega}\overline{\Gamma}$, then there exists a formula δ with variables $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma, \varphi) \cap \mathrm{var}(\Gamma, \psi)$ such that $\langle \varphi, \delta \rangle, \langle \delta, \psi \rangle \in \boldsymbol{\Omega}\overline{\Gamma}$, where $\overline{\Gamma} = Cn_{\mathcal{S}}(\Gamma)$ is the theory of $\mathcal{S}$ generated by Γ.

In the context of equivalential logics, we find characterizations for these interpolations properties in terms of categorical properties of their classes of reduced models, reduced algebras, or both. We say that a concrete category **C** has *covering pullbacks for (pairs of) monos* if whenever $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ is a pair of monos in **C**, there exists a pullback and onto morphisms ρ_A, ρ_B, ρ_C rendering commutative the following diagram:

(1)



We obtaing the following result:

Theorem 1. *For an equivalential logic $\mathcal{S}$, the following statements are equivalent:*

(i) *$\mathcal{S}$ has the LIP,*
(ii) *the category $\mathbf{Mod}^*\mathcal{S}$ of its reduced models has covering pullbacks for monos,*
(iii) *the category $\mathbf{Alg}^*\mathcal{S}$ of its reduced algebras has covering pullbacks for monos.*

For the others interpolations properties we find similar characterizacions, by strengthening the conditions in Diagram (1). For instance, the LIPT for an equivalential logic $\mathcal{S}$ is equivalent to the existence in $\mathbf{Mod}^*\mathcal{S}$ of covering pullbacks for monos of the form of (1) where, moreover, ρ_A is an isomorphism.

Three of the notions that we introduce and deserve a special attention are the *Relative Leibniz*, *Robinson-Leibniz*, and *Maehara-Leibniz Interpolation Properties*. They are the natural generalizations of three of the interpolation properties considered in [3] for substructural logics. We investigate the relation of these properties with those studied by Czelakowsky and Pigozzi in [1] and obtain the following result for the equivalential logics satisfying the G-rule, that is, the regularly algebraizable logics.

Theorem 2. *For every regularly equivalential logic $\mathcal{S}$, we have the following implications:*

(i) *The Relative Interpolation Property implies Interpolation Property.*
(ii) *The Robinson-Leibniz Interpolation Property implies the Robinson Interpolation Property.*
(iii) *The Maehara-Leibniz Interpolation Property implies the Maehara Inerpolation Property.*

References

[1] J. Czelakowski, D. Pigozzi. Amalgamation and interpolation in abstract algebraic logic. In *Models, Algebras, and Proofs*, X. Caicedo and C. H. Montenegro (eds.) Vol. 203 of Lecture Notes in Pure and Applied Mathematics, 187265 (1998).
[2] J. M. Font, R. Jansana, and D. Pigozzi. A survey of abstract algebraic logic. *Studia Logica*, 74(1-2):13–97, 2003. Abstract algebraic logic, Part II (Barcelona, 1997).
[3] H. Kihara, H. Ono, Interpolation Properties, Beth Definability Properties and Amalgamation Properties for Substructural Logics *Journal of Logic and Computation* DOI 10.1093/logcom/exn084.

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