Lattice disjunction is not a disjunction
(in many substructural logics)

Petr Cintula[1] Carles Noguera[2]

[1]Institute of Computer Science, Czech Academy of Sciences
Prague, Czech Republic

[2]Artificial Intelligence Research Institute (IIIA - CSIC)
Bellaterra, Catalonia

Disjunction in Classical Logic

(PD)	$\varphi \vdash_{\mathrm{CPC}} \varphi \vee \psi$ and $\psi \vdash_{\mathrm{CPC}} \varphi \vee \psi$
(C)	$\varphi \vee \psi \vdash_{\mathrm{CPC}} \psi \vee \varphi$
(I)	$\varphi \vee \varphi \vdash_{\mathrm{CPC}} \varphi$
(A)	$\varphi \vee (\psi \vee \chi) \dashv\vdash_{\mathrm{CPC}} (\varphi \vee \psi) \vee \chi$
(PCP)	If $\Gamma, \varphi \vdash_{\mathrm{CPC}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CPC}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CPC}} \chi$.

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Notice that properties (C), (I), (A) are redundant

The properties (PD) and (PCP) could be equivalently formulated as:

$\Gamma, \varphi \vdash_{\mathrm{CPC}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{CPC}} \chi$, if and only if, $\Gamma, \varphi \vee \psi \vdash_{\mathrm{CPC}} \chi$.

A problem

Theorem

In FL_e, the lattice connective $\vee$ does not satisfy PCP.

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Proof.

From $\varphi \vdash \varphi \wedge \mathbf{1}$ we easily get $\varphi \vdash (\varphi \wedge \mathbf{1}) \vee \psi$. As also $\psi \vdash (\varphi \wedge \mathbf{1}) \vee \psi$ we would by (PCP) get:

$$\varphi \vee \psi \vdash (\varphi \wedge \mathbf{1}) \vee \psi$$

Consider the non-distributive lattice 'diamond' $\{\bot, a, b, \mathbf{1}, \top\}$, this is the lattice reduct an FL_e-algebra but $a \vee b = \top$ and $(a \wedge \mathbf{1}) \vee b = \bot \vee b = b$ □

Theorem

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A possible solution of this problem:

Theorem

The connective $\vee'$ defined as $\varphi \vee' \psi = (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})$ satisfies

$$\text{(PD)} \qquad \varphi \vdash (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1}) \quad \textit{and} \quad \psi \vdash (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})$$

$$\text{(PCP)} \qquad \textit{If } \Gamma, \varphi \vdash \chi \textit{ and } \Gamma, \psi \vdash \chi, \textit{ then } \Gamma, (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1}) \vdash \chi.$$

A bigger problem

Theorem

In the implication fragment of Gödel-Dummet logic we cannot define any connective $\vee$ *satisfying* (PD) *and* (PCP)*.*

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Proof.

Consider connective $\vee'$ with those properties. Then in the full logic:

$$\varphi \vee \psi \dashv\vdash \varphi \vee' \psi$$

Thus using the deduction theorem it would also prove:

$$\varphi \vee \psi \leftrightarrow \varphi \vee' \psi$$

But it can be shown that $\vee$ is not definable in the implication fragment of Gödel-Dummet logic. □

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Theorem

In the implication fragment of Gödel-Dummet logic we cannot define any connective $\vee$ *satisfying* (PD) *and* (PCP).

A possible solution of this problem:

Theorem

The 'connective' $\{(\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi\}$ *satisfies*

$$
\begin{array}{lllll}
(\mathrm{PD})_\varphi & \varphi \vdash (\varphi \to \psi) \to \psi & \textit{and} & \varphi \vdash (\psi \to \varphi) \to \varphi \\
(\mathrm{PD})_\psi & \psi \vdash (\varphi \to \psi) \to \psi & \textit{and} & \psi \vdash (\psi \to \varphi) \to \varphi \\
(\mathrm{PCP}) & \multicolumn{3}{l}{\textit{If } \Gamma, \varphi \vdash \chi \textit{ and } \Gamma, \psi \vdash \chi\textit{, then}} \\
 & \multicolumn{3}{r}{\Gamma, (\varphi \to \psi) \to \psi, (\psi \to \varphi) \to \varphi \vdash \chi.}
\end{array}
$$

An even bigger problem

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In FL no set of formulas of two variables defines 'connective' satisfying (PD) *and and* (PCP)*.*

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Proof.

Nick Galatos. *Personal communication*, 2009. □

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A possible solution of this problem:

Theorem

The following "connective" satisfies both (PD) *and* (PCP)

$\{\gamma_1(\varphi \wedge \mathbf{t}) \vee \gamma_2(\psi \wedge \mathbf{t}) \mid$ *where* γ_1, γ_2 *are iterated conjugates*$\}$

An *iterated conjugate* of φ is a formula $\gamma_{\alpha_1}(\gamma_{\alpha_2}(\ldots\gamma_{\alpha_n}(\varphi)\ldots))$ where $\gamma_{\alpha_i} = \lambda_{\alpha_i}(\varphi) = (\alpha_i \backslash \varphi \& \alpha_i) \wedge \mathbf{t}$ or $\gamma_{\alpha_i} = \rho_{\alpha_i}(\varphi) = (\alpha_i \& \varphi / \alpha_i) \wedge \mathbf{t}$ for some formulas α_i.

Useful conventions

Let $\nabla(p, q, \overrightarrow{r})$ be a set of formulas. We write

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{F}m^{\leq \omega}\}.$$

$$\Sigma_1 \nabla \Sigma_2 = \bigcup \{\varphi \nabla \psi \mid \varphi \in \Sigma_1, \psi \in \Sigma_2\}$$

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By $\rightarrow$ we denote any of the two implications, $/$ or $\backslash$, of FL

By $\leq_{\mathcal{A}}$ we denote the order in an FL-algebra $\mathcal{A}$.

Fixing the logical framework

Definition (Substructural logics by Ono)

A substructural logic as an axiomatic extension of the logic of pointed residuated lattices.

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Definition (Substructural logics—for this talk)

A substructural logic as an finitary extension of the logic of pointed residuated lattices.

Logics and algebras

Substrutural logics are in one-one correspondence with the subquasivarieties of FL-algebras:

Definition

For a logic L, we define $\mathbf{ALG}^*(\mathrm{L})$

$$\mathcal{A} \in \mathbf{ALG}^*(\mathrm{L}) \text{ iff } \models_{\mathcal{A}} \bigwedge_{\varphi \in T} \psi \wedge \mathbf{t} \approx \mathbf{t} \ \Rightarrow \ \psi \wedge \mathbf{t} \approx \mathbf{t} \text{ whenever } T \vdash_{\mathrm{L}} \varphi$$

Definition

For each quasivariety of FL-algebras $\mathcal{Q}$ we define $\mathrm{L}_{\mathcal{Q}}$:

$$\{\alpha \leftrightarrow \beta \mid \alpha \approx \beta \in \Gamma\} \vdash \varphi \leftrightarrow \psi \text{ whenever } \models_{\mathcal{Q}} \bigwedge \Gamma \ \Rightarrow \ \varphi \approx \psi$$

Definition

Given a logic L we say that (parameterized) set of formulas ∇ is a (p-)disjunction if it has the properties:

(PD) $\varphi \vdash_{\mathrm{L}} \varphi\nabla\psi$ and $\psi \vdash_{\mathrm{L}} \varphi\nabla\psi$

(PCP) If $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \varphi\nabla\psi \vdash_{\mathrm{L}} \chi$.

It is a (p-)protodisjunction if it only satisfies (PD).

Generalized disjunction

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To avoid confusion we will call the connective $\vee$

the lattice protodisjunction

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Lemma

If ∇ is a (p-)disjunction, then it also satisfies:

(C)	$\varphi \nabla \psi \vdash_{\mathrm{L}} \psi \nabla \varphi$
(I)	$\varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi$
(A)	$\varphi \nabla (\psi \nabla \chi) \dashv\vdash_{\mathrm{L}} (\varphi \nabla \psi) \nabla \chi$

A parameterized set ∇' is (p-)disjunction iff $\varphi \nabla \psi \dashv\vdash_{\mathrm{L}} \varphi \nabla' \psi$.

Syntactical characterization

Definition

Let $R = \Gamma \rhd \varphi$ be a consecution. Then

$$R^{\nabla} = \{\Gamma\nabla\chi \rhd \delta \mid \chi \in \mathrm{Fm}_{\mathcal{L}} \text{ and } \delta \in \varphi\nabla\chi\}.$$

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Examples

- $(\varphi \rhd \Box\varphi)^{\vee} = \{\varphi \vee \chi \rhd \Box\varphi \vee \chi\}$
- $(\varphi \rhd \varphi \wedge \mathbf{t})^{\vee} = \{\varphi \vee \chi \rhd (\varphi \wedge \mathbf{t}) \vee \chi\}$
- $(\varphi \rhd \varphi \wedge \mathbf{t})^{\vee'} = \{(\varphi \wedge \mathbf{t}) \vee (\chi \wedge \mathbf{t}) \rhd ((\varphi \wedge \mathbf{t}) \wedge \mathbf{t}) \vee (\chi \wedge \mathbf{t})\}$

Syntactical characterization

Theorem

Let L be a substructural logic with a presentation $\mathcal{AS}$ and ∇ a commutative and idempotent p-protodisjunction. Then the following are equivalent:

1. ∇ *is a p-disjunction,*
2. $R^{\nabla} \subseteq \mathrm{L}$ *for each consecution* $R \in \mathrm{L}$*,*
3. $R^{\nabla} \subseteq \mathrm{L}$ *for each* $R \in \mathcal{AS}$*.*

Applications

$\varphi \vee \chi, (\varphi \to \psi) \vee \chi \vdash_{FL_{ew}} \psi \vee \chi$ AND SO $\vee$ is disjunction in FL_{ew}

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$(\varphi \wedge \mathbf{t}) \vee (\chi \wedge \mathbf{t}) \vdash_{FL_e} ((\varphi \wedge \mathbf{t}) \wedge \mathbf{t}) \vee (\chi \wedge \mathbf{t})$ AND SO

$(\varphi \wedge \mathbf{t}) \vee (\psi \wedge \mathbf{t})$ is disjunction FL_e

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Recall

$\varphi \nabla \psi = \{\gamma_1(\varphi \wedge \mathbf{t}) \vee \gamma_2(\psi \wedge \mathbf{t}) \mid$ where γ_1, γ_2 are iterated conjugates$\}$

Then for each iterated conjugates γ_1, γ_2:

$$\varphi \nabla \chi \vdash_{FL} \gamma_1(\lambda_\alpha(\varphi) \wedge \mathbf{t}) \vee \gamma_2(\chi \wedge \mathbf{t})$$

$$\varphi \nabla \chi \vdash_{FL} \gamma_1(\rho_\alpha(\varphi) \wedge \mathbf{t}) \vee \gamma_2(\chi \wedge \mathbf{t})$$

P-disjunction preservation

Corollary

Let ∇ be a p-disjunction in a logic L_1 and L_2 is a logic axiomatized by adding a set of consecutions $\mathcal{C}$ to L_1. Then ∇ is a p-disjunction in L_2 iff $R^{\nabla} \subseteq \mathrm{L}_2$ for each $R \in \mathcal{C}$.

Corollary

Let ∇ a be p-disjunction in a logic L. Then it is a p-disjunction in all its axiomatic extensions.

Axiomatization of ‘disjunctional companion’

Definition

L^{∇}: the least logic extending L where ∇ is a p-disjunction.

Theorem

Let L be a logic with a presentation $\mathcal{AS}$ and ∇ a commutative idempotent associative p-protodisjunction. Then L^{∇} is axiomatized by $\mathcal{AS} \cup \bigcup\{R^{\nabla} \mid R \in \mathcal{AS}\}$.

Disjunction and filter distributivity

Theorem

Let ∇ be a p-disjunction in L and $\mathcal{A}$ an L-algebra. Then

- $\mathrm{Fi}(X, x) \cap \mathrm{Fi}(X, y) = \mathrm{Fi}(X, x \nabla^{\mathcal{A}} y)$

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Theorem

Let ∇ *be a p-disjunction in* L *and* $\mathcal{A}$ *an* L*-algebra. Then*

- $\mathrm{Fi}(X, x) \cap \mathrm{Fi}(X, y) = \mathrm{Fi}(X, x \nabla^{\mathcal{A}} y)$
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- $\mathrm{Fi}(X) \cap \mathrm{Fi}(Y) = \mathrm{Fi}(X \nabla^{\mathcal{A}} Y)$

Theorem

Let L *be a substructural logic. Then the following are equivalent:*

1. *there is p-disjunction* ∇ *in* L*.*
2. *the lattice of all theories (deductively closed sets) is distributive.*
3. *for each* L*-algebra the lattice of its filters (relative congruences) is distributive.*

Prime filters

A filter F is ∇-*prime* if: $a\nabla^{\mathcal{A}}b \subseteq F$ iff $a \in F$ or $b \in F$.

Theorem

Let L be a substructural logic and ∇ a p-protodisjunction. Then ∇ is p-disjunction iff for each L-algebra its ∇-prime filters form a base of the closure system of all its filters.

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Corollary

Let ∇ be a p-disjunction in a substructural logic L and $\mathcal{A}$ a relatively finitely subdirectly irreducible L-algebra. Then $\mathrm{Fi}(\mathbf{1})$ is a ∇-prime filter.

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A filter F is ∇*-prime* if: $a\nabla^{\mathcal{A}}b \subseteq F$ iff $a \in F$ or $b \in F$.

Theorem

Let L be a substructural logic and ∇ a p-protodisjunction. Then ∇ is p-disjunction iff for each L-algebra its ∇-prime filters form a base of the closure system of all its filters.

Corollary

Let ∇ be a p-disjunction in a substructural logic L and $\mathcal{A}$ a relatively finitely subdirectly irreducible L-algebra. Then $\mathrm{Fi}(\mathbf{1})$ is a ∇-prime filter.

Proof.

$x\nabla^{\mathcal{A}}y \subseteq \mathrm{Fi}(\mathbf{1})$ iff $\mathrm{Fi}(x\nabla^{\mathcal{A}}y) = \mathrm{Fi}(\mathbf{1})$ iff $\mathrm{Fi}(x) \cap \mathrm{Fi}(y) = \mathrm{Fi}(\mathbf{1})$ iff $\mathrm{Fi}(x) = \mathrm{Fi}(\mathbf{1})$ or $\mathrm{Fi}(y) = \mathrm{Fi}(\mathbf{1})$ iff $x \in \mathrm{Fi}(\mathbf{1})$ or $y \in \mathrm{Fi}(\mathbf{1})$ □

Axiomatization of positive universal classes

A positive universal formula has the form
$\Psi = (\forall \overrightarrow{x})(\varphi_1(\overrightarrow{x}) \approx \psi_1(\overrightarrow{x}) \vee \ldots \vee \varphi_n(\overrightarrow{x}) \approx \psi_n(\overrightarrow{x}))$

Theorem

Let L *be a substructural logic with a p-disjunction* ∇ *and* Ψ *a positive universal formula. Then*

$$\models_{\{\mathcal{A} \mid \mathcal{A} \models \Psi\}} = \mathrm{L} + (\varphi_1 \leftrightarrow \psi_1)\nabla \ldots \nabla(\varphi_n \leftrightarrow \psi_n).$$

This generalizes the work by Nick Galatos.

Axiomatization of intersection of logics

Corollary

Let L *be a substructural logic with a p-disjunction* ∇. *Let* L_1 *and* L_2 *be axiomatic extensions, by finite sets of axioms* Γ_1 *and* Γ_2, *of the logic* L. *Then*

$$\mathrm{L}_1 \cap \mathrm{L}_2 = \mathrm{L} + \Gamma_1 \nabla \Gamma_2.$$

Corollary

Let L be a substructural logic with a p-disjunction ∇. Then:

$$\begin{aligned}\models_{\{\mathcal{A}\,|\,\mathcal{A}\text{ is }\mathrm{L}\text{-chain}\}} &= \mathrm{L} + (\varphi \wedge \psi \leftrightarrow \psi)\nabla(\varphi \wedge \psi \leftrightarrow \varphi)\\ &= \mathrm{L} + (\varphi \to \psi)\nabla(\psi \to \varphi)\end{aligned}$$

Definition

A L be a substructural logic is semilinear logic if

$$\mathrm{L} = \models_{\{\mathcal{A} \in \mathbf{ALG}^*(\mathrm{L}) \mid \mathcal{A} \text{ is chain}\}}$$

Semilinearity Property (SLP):

$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$

Theorem

The following are equivalent:

1. L *is semilinear logic,*
3. L *has the* (SLP)*,*
6. $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}} = \{\mathcal{A} \mid \mathcal{A} \text{ is chain}\}$.

Interplay of implications and disjunctions

Proposition

Let L be a logic. If ∇ is a p-disjunction, then:

$$(\mathrm{MP}_\nabla) \quad \varphi \to \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \psi \qquad \textit{and} \qquad \varphi \to \psi, \psi \nabla \varphi \vdash_{\mathrm{L}} \psi$$

If L is semilinear then for any p-protodisjunction ∇ it holds:

$$(\mathrm{P}) \quad \vdash_{\mathrm{L}} (\varphi \to \psi) \nabla (\psi \to \varphi)$$

Interplay of implications and disjunctions

Theorem

Let L *be a semilinear logic and* ∇ *a p-protodisjunction. Then,*

∇ *is p-disjunction iff* L *satisfies* (MP_{∇}).

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Theorem

Let L *be a logic and* ∇ *a p-disjunction. Then,*

L *is semilinear iff* L *satisfies* (P).

We can prove in FL:

$\varphi \to \psi \vdash \varphi \to \psi$
$\varphi \to \psi \vdash \psi \to \psi$
$\varphi \to \psi \vdash (\varphi \to \psi) \wedge (\psi \to \psi)$
$\varphi \to \psi \vdash \varphi \vee \psi \to \psi$
$\varphi \to \psi, \varphi \vee \psi \vdash \psi$

Thus FL proves $(\mathrm{MP}_\vee)$ and so:

Corollary

$\vee$ *is disjunction in all semilinear logics.*

Semilinear companion

Definition

L^{ℓ}: the least semilinear logic extending L.

Theorem

Let L be a logic, ∇ a commutative, idempotent and associative p-protodisjunction, satisfying (MP_{∇}). Then

$$\mathrm{L}^{\ell} \textit{ is the extension of } \mathrm{L}^{\nabla} \textit{ by } (\mathrm{P}).$$

Corollary

Let L be a finitary logic, ∇ a p-disjunction. Then

$$\mathrm{L}^{\ell} \textit{ is the extension of } \mathrm{L} \textit{ by } (\mathrm{P}).$$

Semilinear companion: an example

Theorem

Let L *be a finitary logic,* ∇ *a commutative, idempotent and associative p-protodisjunction, satisfying* (MP_∇)*. Then*

$$\mathrm{L}^\ell \text{ is the extension of } \mathrm{L}^\nabla \text{ by } (\mathrm{P}).$$

$$\mathsf{FL}_e^\ell = (\mathsf{FL}_e)^\vee + (\varphi \to \psi) \vee (\psi \to \varphi) =$$
$$= \mathsf{FL}_e + \varphi \vee \chi \vdash (\varphi \wedge \mathbf{t}) \vee \chi + (\varphi \to \psi) \vee (\psi \to \varphi)$$

Corollary

Let L *be a finitary logic,* ∇ *a p-disjunction. Then*

$$\mathrm{L}^\ell \text{ is the extension of } \mathrm{L} \text{ by } (\mathrm{P}).$$

$$\mathsf{FL}_e^\ell = \mathsf{FL} + ((\varphi \to \psi) \wedge \mathbf{t}) \vee ((\psi \to \varphi) \wedge \mathbf{t})$$

Dense linear orders

$\mathbf{ALG}^{\delta}(\mathrm{L})$ is the set of linear densely ordered L-algebras

Theorem

Let L *be a semilinear logic. Then the following are equivalent:*

1. $\mathrm{L} = \models_{\mathbf{ALG}^{\delta}(\mathrm{L})}$
2. L *has the Density property* (DP)*:*

   if $\Gamma \vdash_{\mathrm{L}} (\varphi \to p) \vee (p \to \psi) \vee \chi$
   for some variable p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$*,*
   then $\Gamma \vdash_{\mathrm{L}} (\varphi \to \psi) \vee \chi$*.*

Characterization of strong completeness

Let us restrict ourselves to the substructural logics L where $\vee$ is disjunction. Let us also fix a set $\mathbb{K} \subseteq \mathbf{ALG}^*(\mathrm{L})$.

Theorem

The following are equivalent:

(i) $\mathrm{L} = \models_{\mathbb{K}}$.

(ii) *Every countable member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is embeddable into some member of* $\mathbb{K}$.

(iii) *Every countable member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$ *is embeddable into some member of* $\mathbb{K}$.

Characterization of finite strong completeness

Theorem

The following are equivalent:

(i) L *is the finitary companion of* $\models_{\mathbb{K}}$.

(ii) *Every member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is* *partially* *embeddable into* $\mathbb{K}$.

(iii) *Every countable member of* $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RSI}}$ *is* *partially* *embeddable into* $\mathbb{K}$.

(iv) $\mathbf{ALG}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{ISP}_{\mathrm{U}}(\mathbb{K})$.

Conclusions

(Generalized) Disjunction connectives are present in many logics and bring several important properties:

- Description of filter intersection
- Axiomatization of positive universal classes of algebras
- Axiomatization of intersections of logics (joins of relative subvarieties)
- Characterization of semilinearity (fuzziness).
- Axiomatization of the semilinear companion.
- Characterization of completeness properties w.r.t. dense chains and other distinguished semantics.

It is also crucial in the first-order case.

... the next talk

Thank you
for your attention