Modules over quantaloids and the Isomorphism Problem

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JAIST

Joint work with Nikolaos Galatos

June, 2010

Two notions of equivalence among logics

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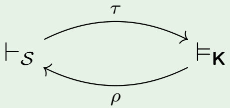
Syntactic

- existence of structural translations:



- respecting and reflecting the conseq. relations $\vdash_{\mathcal{S}}$ and $\models_{\mathbf{K}}$
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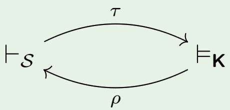

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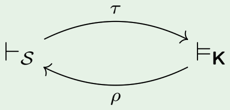

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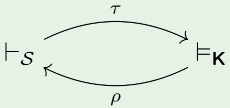

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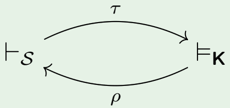

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Def: A structural closure operator on an M-set $\langle A, \cdot \rangle$ is a closure operator C on A satisfying:

$$\forall \sigma \in M, \ \sigma C \leqslant C\sigma \qquad \text{(Str)}$$

The lifting

$M = \langle M, \circ, 1\rangle$ monoid $\rightsquigarrow \mathcal{A}_M = \langle \mathcal{P}M, \cdot, \{1\}\rangle$ complete residuated lattice

$$a \cdot b = \{\sigma \circ \tau : \sigma \in a,\ \tau \in b\}$$

$\cdot : M \times X \to X$ action $\rightsquigarrow$ $* : \mathcal{P}M \times \mathcal{P}X \to \mathcal{P}X$ biresiduated map

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Module over a quantale

Fix a quantale $\mathcal{A} = \langle \mathbf{A}, \cdot, 1\rangle$, which is a tuple where $\mathbf{A}$ is a complete lattice and $\cdot : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ is a biresiduated map rendering $\langle A, \cdot, 1\rangle$ a monoid.

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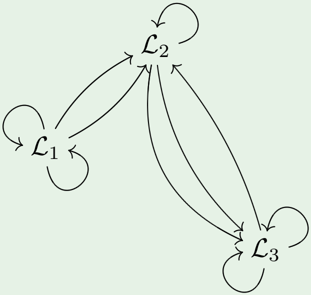
(i) $a * (b * x) = (a \cdot b) * x$,

(ii) $1 * x = x$.



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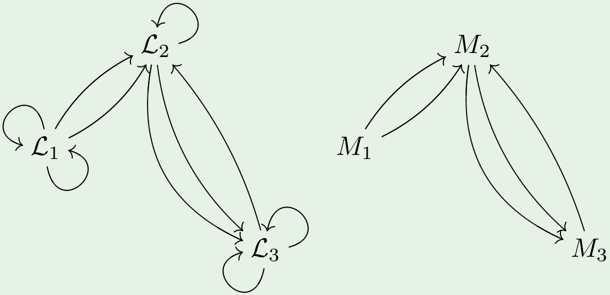
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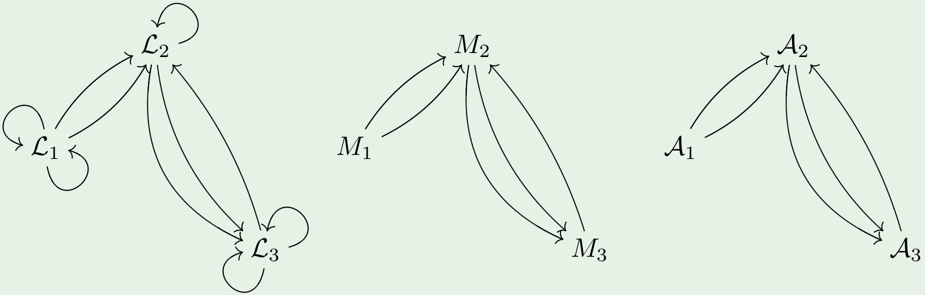
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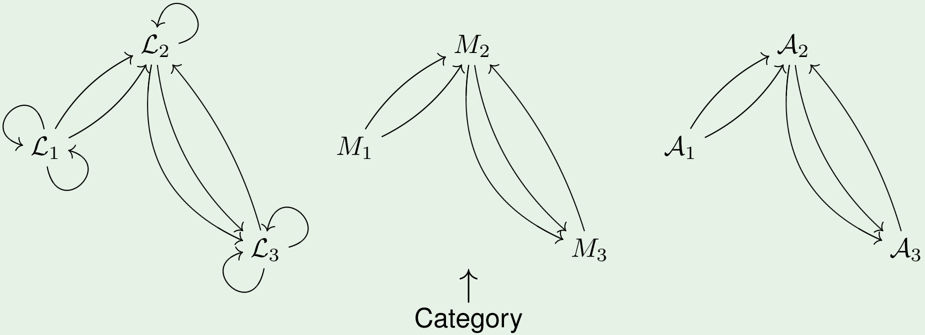

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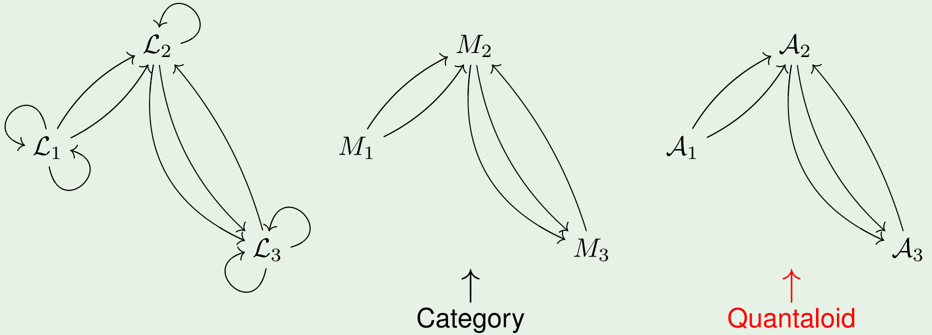
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Quantaloid

Def: A *quantaloid* is an enriched category $\mathcal{Q}$ over the category $\mathcal{S}\ell$ of $\bigvee$-complete lattices:

- for every $A, B \in \mathcal{Q}$, the hom-set $\mathcal{Q}(A, B)$ is a $\bigvee$-complete lattice;
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- for every $A \in \mathcal{Q}$, TA is a complete lattice;
- for every $A, B \in \mathcal{Q}$, $*_T : \mathcal{Q}(A, B) \times TA \to TB$ is biresiduated;
- for every $A \in \mathcal{Q}$, $1_A *_T x = x$;
- for every $a : A \to B$, $b : B \to C$ in $\mathcal{Q}$, and $x \in TA$,

$$(b \circ a) *_T x = b *_T (a *_T x).$$

Def: A closure operator on a $\mathcal{Q}$-module is a family $\gamma = \{\gamma_A : TA \to TA\}_{A \in \mathcal{Q}}$ satisfying

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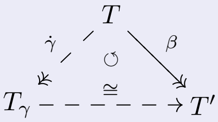

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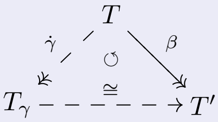

$(\gamma = \beta^\partial \beta)$

Interpretations and Representations

Def: Given a quantaloid $\mathcal{Q}$, two modules T and T', and closure operators γ and δ in T and T' respectively,

- an interpretation of γ into δ is a morphism $\tau : T \to T'$, satisfying

$$\forall A, \forall x, x' \in TA, \quad x \leqslant \gamma_A(x') \iff \tau x \leqslant \gamma_A(\tau x').$$

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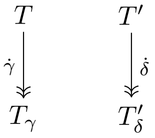

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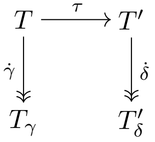

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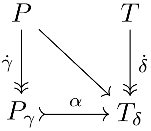

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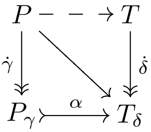

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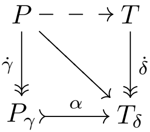

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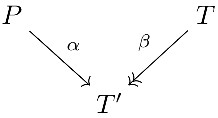

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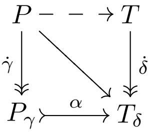

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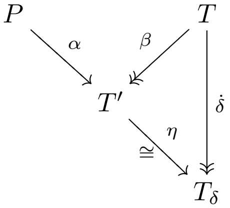

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A $\mathcal{Q}$-module P is projective if and only if every representation of a closure operator on P into another closure operator is induced.

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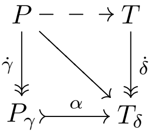

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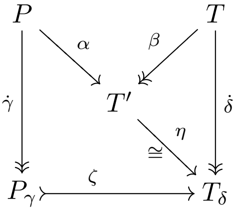

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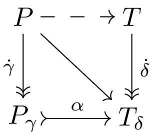

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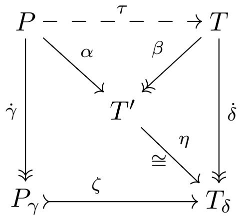

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Isomorphism Theorem (Modules over quantaloids)

The Isom. Theorem hods for a $\mathcal{Q}$-module T if and only if T is projective.

Def: A generalized variable for an M-set $\langle X, \cdot \rangle$ is an element $p \in X$ such that there exists $u \subseteq M$, satisfying:

- for every $\varphi \in X$, there exists $v_\varphi \in M$, such that $v_\varphi \cdot p = \varphi$,
- $u * \{p\} = \{p\}$,
- for every $\pi, \sigma \in M$, $\pi \cdot p = \sigma \cdot p \Longrightarrow \{\pi\}u = \{\sigma\}u$.

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Theorem

Let $\langle X, \cdot \rangle$ be an M-set and $\mathbb{R}$ the associated $\mathcal{A}_M$-module. Then, $\mathbb{R}$ is cyclic and projective if and only if $\langle X, \cdot \rangle$ has a g-variable.

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This theorem can be lifted to the setting of modules over quantaloids.

$\mathcal{Q}$

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$\mathcal{Q}\text{-}Mod \qquad \mathcal{Q}^{\mathrm{op}}\text{-}Mod$

$$(\_)^{\partial} : \mathcal{Q}\text{-}\mathit{Mod} \to \mathcal{Q}^{\mathrm{op}}\text{-}\mathit{Mod}.$$

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Proposition

Monos in the categories of modules over quantaloids are those morphisms such that every component is injective. Dually, epis in the categories of modules over quantaloids are those morphisms such that every component is onto.

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- Products $\prod_I T^i$ are conputed componentwise;
- projections $\pi^j : \prod_I T^i \to T^j$ are the componentwise projections;
- insertions $e^j : T^j \to \prod_I T^i$ are determined by

$$\left(e^j_A(x)\right)_i = \begin{cases} \bot_{T^i A} & \text{if } i \neq j, \\ x & \text{if } i = j. \end{cases}$$

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Corollary

For every quantaloid $\mathcal{Q}$, $\mathcal{Q}$-$\mathcal{M}od$ is complete and cocomplete.

Every $\mathcal{Q}$-morphism $\alpha : T \to T'$ induces a closure operator $\gamma = \alpha^{\partial}\alpha$ on T.

(This is similar to the fact that the composition of a residuated map with its residuum is a closure operator.)

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Lemma

If $\alpha : T \to T'$ is a $\mathcal{Q}$-morphism, (η^1, η^2) is the congruence of α, and $\gamma = \alpha^\partial \alpha$ is the closure operator induced by α, then $\dot{\gamma} : T \twoheadrightarrow T_\gamma$ is the coequalizer of $\eta^1, \eta^2 : H \to T$.

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Corollary

All epis in $\mathcal{Q}$-Mod are regular (i.e., coequalizers).

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Corollary

All epis in $\mathcal{Q}\text{-}\mathcal{M}od$ are regular (i.e., coequalizers).

Proof:

- By the Lemma, epis induced by closure operators are regular.
- By a previous theorem, all epis are induced by closure operators, up to isomorphism.

Lemma

Epis are stable by pullbacks: if $\varepsilon : T_1 \twoheadrightarrow R$ is an epi in $\mathcal{Q}\text{-}Mod$ and the following diagram is a pullback

$$\begin{array}{ccc} H & \longrightarrow & T_1 \\ {\scriptstyle \overline{\varepsilon}}\downarrow & \lrcorner & \downarrow{\scriptstyle \varepsilon} \\ T_2 & \longrightarrow & R \end{array}$$

then $\overline{\varepsilon}$ is also an epi.

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Corollary

For every $\mathcal{Q}$, *the category* $\mathcal{Q}\text{-}\mathcal{M}od$ *has the* ***strong amalgamation property****: for every pair of monos* $i : T \rightarrowtail T'$ *and* $j : T \rightarrowtail T''$, *there exist two monos* $j' : T' \rightarrowtail T'''$ *and* $i' : T'' \rightarrowtail T'''$ *such that the following diagram is a pullback and a pushout:*

$$\begin{array}{ccc} T & \overset{i}{\rightarrowtail} & T' \\ {\scriptstyle j}\downarrow & \lrcorner \quad \ulcorner & \downarrow{\scriptstyle j'} \\ T'' & \underset{i'}{\rightarrowtail} & T''' \end{array}$$

Theorem

Let $\mathcal{Q}$ be a small quantaloid,

- *then $\mathcal{Q}$-$\mathcal{M}od$ has a separating set.*
- *Therefore $\mathcal{Q}$-$\mathcal{M}od$ is wellpowered (and cowellpowered by duality),*
- *and since it is complete it is strogly complete (strongly cocomplete by duality).*

Proposition

If $\mathcal{Q}$ is small, then $\mathcal{Q}$-$\mathcal{M}od$ is $(\mathrm{Epi}, \mathrm{Mono})$-structured. That is to say, every morphism in $\mathcal{Q}$-$\mathcal{M}od$ admits an $(\mathrm{Epi}, \mathrm{Mono})$-decomposition, unique up to isomorphism.

Proposition

If $\mathcal{Q}$ is small, then $\mathcal{Q}$-$\mathcal{M}od$ has enough projectives and enough injectives.

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