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From Games to Truth Tables
— A Generalization of Giles's Game

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Motivation

Robin Giles's characterization of Łukasiewicz logic by a combination of a Lorenzen style dialogue game with bets on uncertain atomic statements is one of the most important attempts to derive a logic from first principles about reasoning in a non-classical setting.

We plan to

1. re-visit the original source
2. analyze and disect it into different parts
3. isolate underlying principles from a GT point of view
4. re-assemble the game in a more general setting

Result:
A theorem stating quite general, but sufficient conditions for extracting truth functional semantics from a Giles-style game.

Giles about reasoning within theories of physics

Robin Giles 1974/77: A non-classical logics for physics
Studia Logica 33 / *Sel. Papers on Łukasiewicz Sentential Calculi*

Key words for Giles's analysis of reasoning:

- each atomic assertion p is to be tested with respect to a concrete experiment E_p that may either fail or succeed
- experiments may show dispersion: different instances of the same experiment may yield different results
- to provide a tangible meaning to sentences one imagines a dialogue between me and you, where we are willing to pay 1€ to the opponent for each false atomic assertion, i.e., one where the corresponding instance of the experiment fails
- since experiments are dispersive, assertions are risky

Important observations:

- A tenet collects all assertions of a player (me or you): The expected loss for my tenet $\{q_1, \ldots, q_n\}$ of atomic assertions gets quantified by assigning a subjective failure probability $\langle q_i \rangle$ to the experiment E_{q_i}.

- Events are independent instances of elementary experiments. In other words: experiments are event types who's instances share the same failure probability.

- An elementary (or: atomic, final) state of the game is denoted by $[p_1, \ldots, p_n \,\|\, q_1, \ldots, q_m]$, where $\{p_1, \ldots, p_n\}$ is your tenet and $\{q_1, \ldots, q_m\}$ is my tenet of assertions.
My corresponding risk, i.e., my expected loss of money is

$$\sum_{1 \leq i \leq m} \langle q_i \rangle € - \sum_{1 \leq j \leq n} \langle p_j \rangle €$$

What about logically complex statements?

NB: So far, no logic has been involved!

For the reduction of logically complex assertions to atomic states Giles refers to the logical rules introduced by Paul Lorenzen in his dialogue game for constructive reasoning.

Giles states the rules in the following (old fashioned) way:

- *He who asserts $A \supset B$ agrees to assert B if his opponent will assert A.*
- *He who asserts $A \vee B$ undertakes to assert either A or B at his own choice.*
- *He who asserts $A \wedge B$ undertakes to assert either A or B at his opponent's choice.*

Defining $\neg A = A \supset \bot$ leads to

- *He who asserts $\neg A$ agrees to pay 1€ to his opponent if he will assert A.*

Observations about the dialogue part of Giles's game

(1) Each assertion can be attacked at most once: i.e. we respect Mundici's principle *repetita juvant*. (Here turned into: *repetitions are risky*.)

The players may also choose not to attack an assertion.

(2) In contrast to Lorenzen:

- no regulations on the succession of moves
- no restrictions on what can be attacked when

(3) Giles defends the $\wedge$-rule by reference to a principle of limited liability: each assertion carries a maximal risk of 1€. There is no rule for strict conjunction (&).

(4) While Lorenzen seeks to characterize (intuitionistic) validity by reducing to *ipse dixisti states*, Giles reduces to a given many valued interpretation of atomic statements.

In this respect Giles's game is more like a Hintikka style evaluation game than a Lorenzen style dialogue game.

Adequateness of Giles's game for Ł

Theorem (coarse version):

I have a strategy for avoiding expected loss for precisely those initial statements that are valid in Łukasiewicz logic.

Theorem (refined version):

Suppose we play the game starting with my assertion of F with respect to given assignment $\langle \cdot \rangle$ of risk values to atomic assertions.

The following are equivalent:

- F evaluates to $1-x$ in Łukasiewicz logic under the interpretation that assigns $1 - \langle p \rangle$ to each atom p.
- My best strategy guarantees that the play ends in an elementary state, where my risk is not higher than x€, but you have a strategy enforcing an elementary state, where my risk is not less than x€.

Remarks on the proof of Giles's Theorem

We have to show that my risk $\langle \cdot \rangle$ can be extended from elementary to arbitrary states in such a way that

$$\langle \Gamma \parallel A \supset B, \Delta \rangle = \max(\langle \Gamma \parallel \Delta \rangle, \langle \Gamma, A \parallel B, \Delta \rangle) \quad (1)$$
$$\langle \Gamma, A \supset B \parallel \Delta \rangle = \min(\langle \Gamma \parallel \Delta \rangle, \langle \Gamma, B \parallel A, \Delta \rangle) \quad (2)$$

(analogous conditions have to hold for other connectives)

This can be achieved by defining

$$\langle \Gamma \parallel \Delta \rangle^v =_{def} |\Delta| - |\Gamma| + \sum_{G \in \Gamma} v(G) - \sum_{F \in \Delta} v(F).$$

for the valuation v assigning $1 - \langle p \rangle$ to each atom p.

The fact that no regulations are needed in Giles's game falls out from the proof. From a game theoretic point of view it is more natural to assume regulations, and prove that they don't affect the players' respective 'power'. (See [FM, StudLog09])

Adding strong conjunction

Remember Giles's rule for conjunction:

- *He who asserts $A \wedge B$ undertakes to assert either A or B at his opponent's choice.*

Why don't we have to assert both conjuncts?
Limited liability principle: one is never forced to risk more than 1€.

However there is an even more direct way to respect that principle:
[formulated Giles-style:]

- *He who asserts $A \& B$ undertakes to assert either both, A and B, or else pay 1€ to the opponent, at his own choice.*

[formulated Chris-style:]

- *Asserting $A \& B$ obliges one to assert either A and B, or else to pay 1€.*

Even better: forget about the 'or else'-part of the rule and add the following general principle of limited liability:

- To any attack on a (logically complex) assertion one has either to reply according to the appropriate rule or else pay to 1€.

Beyond Łukasiewicz logic Ł

Note: all n-valued Łukasiewicz logics $\textbf{Ł}_n$ are characterized if possible risk values are restricted to $\{\frac{i}{n-1} : 0 \leq i < n\}$.

CHL is characterized by removing experiments that always fail.

There are also game characterizations of Gödel logic **G** and of Product logic **P**. However there is a high(?) price to pay:

- the refined version of the adequateness theorem fails: only validity, but not (graded) truth is captured
- An additional flag has to be introduced. (Alternatively: two types of states are needed.)
- The implication rule has to be extended in a somewhat problematic manner.

NB: these latter game variants are still well worth investigating. They directly correspond to a kind of uniform hypersequent calculus for **Ł**,**G**, and **P**.

Liberating from risk: the case of Abelian logic A

NB: The interpretation of intermediary truth values as risk values and the corresponding story about dispersive experiments is completely independent from the dialogue game.

From a game theoretic perspective, one does not need to talk about probabilities or risk at all. These numbers are nothing but inverted payoff values.

From now on, we will reverse the inversion and assume that players are maximizing payoff, instead of minimizing expected payments according to a particular betting scheme.

Three simple modifications turn Giles's game for **Ł** into one for Abelian logic **A**:

(1) allow arbitrary real numbers as pay off values

(2) drop the option not to attack an implication at all

(3) drop the ('or else pay 1€') principle of limited liability

Characterizing degrees of truth as payoff values

Our aim here is more ambitious than just to characterize various many valued logics by variations of the original game:

We want to capture the general principle underlying the refined version of Giles-style game semantics.

We have to consider two types of general conditions:

(1) principles regarding payoff functions

(2) principles regarding the form of dialogue rules

Payoff principles

We are (still) interested in games ending in states where my atomic assertions faces your atomic assertions.

While the order of assertions is irrelevant, repetitions are not. In other words:

- final states take the form of pairs of multisets of atomic statements (tenets), denoted $[\Gamma \parallel \Delta]$
- the corresponding payoff value ($\in \mathbb{R}$) is denoted by $\langle \Gamma \mid \Delta \rangle$

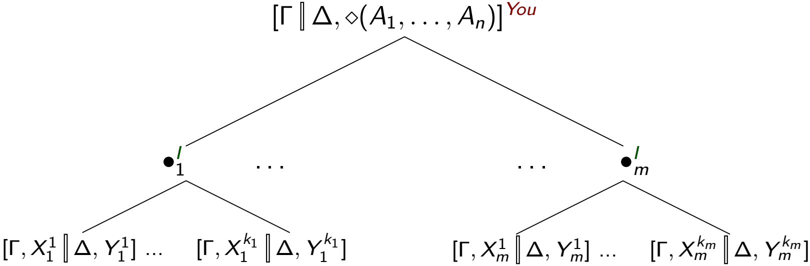
A payoff function $\langle \cdot \mid \cdot \rangle$ is called

- context independent if
  $\langle \Gamma' \mid \Delta' \rangle = \langle \Gamma'' \mid \Delta'' \rangle$ implies $\langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle = \langle \Gamma, \Gamma'' \mid \Delta'', \Delta \rangle$
- monotone if
  $\langle \Gamma' \mid \Delta' \rangle \leq \langle \Gamma'' \mid \Delta'' \rangle$ implies $\langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle \leq \langle \Gamma, \Gamma'' \mid \Delta'', \Delta \rangle$
- symmetric if $\langle \Gamma \mid \Delta \rangle = - \langle \Delta \mid \Gamma \rangle$

We call payoff functions that are context independent, monotone, and symmetric discriminating.

General format of (decomposing) dialogue rules

You may attack my assertion of $\diamond(A_1, \ldots, A_n)$ in different ways:



where X_i^j, Y_i^j are multisets of the form $\{A_1^{\ell_1}, \ldots, A_n^{\ell_n}\} \cup C$ for some multiset C of truth constants.

Rules for my attacks on your assertions of $\diamond(A_1, \ldots, A_n)$ are dual!

The presence of a node $[\Gamma \parallel \Delta]$ amounts to granting (no attack).

Main result

Theorem

Let ∂ be a game with discriminating payoff function $\langle \cdot \mid \cdot \rangle$ and decomposing dialogue rules respecting duality. Then one can extract from $\langle \cdot \mid \cdot \rangle$ and the rules a set truth functions $\mathcal{F}_{\partial}$ over $\mathbb{R}$ such that the following two values are equivalent for every formula A:

- the optimal payoff guaranteed by my best strategy for a ∂-play starting in $[\,\|\, A]$,
- the truth value of A according to $\mathcal{F}_{\partial}$ under the interpretation that assigns $\langle \mid p \rangle$ to p for all atomic formulas p.

In other words: discriminating payoff and dual decomposing rules are sufficient for a game to characterize a many valued logic!

Some comments

- The proof of the theorem relies on the fact that context independent payoff is sufficient to guarantee the existence of an AC-function $\circ$ s.t. $\langle \Gamma, \Gamma' \mid \Delta, \Delta' \rangle = \langle \Gamma \mid \Delta \rangle \circ \langle \Gamma' \mid \Delta' \rangle$. The decomposing and dual rules induce truth functions via the min-max principle.
- Restricted truth value sets (like $[0, 1]$ for Ł) are obtained by examining which payoffs for single formulas result from corresponding restrictions of $\langle \mid p \rangle$ fror atomic p.
- A lot of interesting many-valued logics, like **G** and **P** (provably) do not admit a characterization in terms of Giles-style dialogue games.
- Among the games covered are: $\mathbf{Ł}_n$ for all $n \geq 2$, **Ł**, **CHL**, **A**; but also extensions of such logics by by arbitrary truth constants and new connectives.

Summary

- We have isolated the essential principles underlying Giles's characterization of Łukasiewicz logic and synthesized a corresponding general 'toolkit' for assembling games that are adequate for a certain type of truth functional logics.

Topics for further research:

- Concise characterization of the class of logics where truth functions can be extracted from a Giles style game. NB: this should allow to prove also negative results.
- Connection to proof theory: do winning strategies always correspond to analytic proofs in a hypersequent system?
- Generalizing further in different directions, e.g.:
  - only care about winning conditions / designated values
  - allow for non-decomposing rules
  - consider other types of states
  - ...

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