Quasi-Subtractive Varieties, Part I

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Isomorphism Theorems

Algebras in classical varieties have pleasant properties:

Groups

Congruences ⟷ Normal Subgroups

Rings

Congruences ⟷ Two-Sided Ideals

Boolean algebras

Congruences ⟷ Filters (Ideals)

Some definitions

Let $\mathbb{K}$ be a class of algebras, and let τ be a formula-equation transformer. Define

$$\Gamma \vdash_{\mathrm{K}_\tau} \alpha \text{ iff } \tau(\Gamma) \vdash_{\mathbb{K}} \tau(\alpha).$$

The logic $\mathbf{K}_\tau = (\mathbf{Fm}, \vdash_{\mathrm{K}_\tau})$ is called the τ-*assertional logic* of $\mathbb{K}$.

A variety $\mathbb{V}$ is τ-*protoregular* if its τ-assertional logic is protoalgebraic

A variety $\mathbb{V}$ is τ-*regular* if its τ-assertional logic is strongly and finitely algebraisable with $\mathbb{V}$ as equivalent algebraic semantics

Equivalently: if for any algebra $\mathbf{A} \in \mathbb{V}$ and for any congruences θ, φ on $\mathbf{A}$, $\tau^{\mathbf{A}}/\theta = \tau^{\mathbf{A}}/\varphi$ implies $\theta = \varphi$, where:

$$\tau^{\mathbf{A}}/\theta = \left\{a \in A : \tau^{\mathbf{A}}(a) \subseteq \theta\right\}$$

1-*protoregular*, 1-*regular*: the same with $\tau = \{x \approx 1\}$

Some definitions

A pointed variety $\mathbb{V}$ of type ν is 1-*subtractive* iff there exists a binary term $x \to y$ of type ν s.t. $\mathbb{V}$ satisfies the following equations:

- $x \to x \approx 1$;
- $1 \to x \approx x$.

A variety $\mathbb{V}$ is 1-*subtractive* iff it is 1-*permutable:* for every $\mathbf{A} \in \mathbb{V}$ and for every $\theta, \psi \in \mathrm{Con}\,(\mathbf{A})$

$$1^{\mathbf{A}}/\theta \circ \psi = 1^{\mathbf{A}}/\psi \circ \theta$$

The Received View (1)

In every member $\mathbf{A}$ of a τ-regular variety $\mathbb{V}$ - namely, a variety that arises as the equivalent algebraic semantics of an algebraisable deductive system - there is a lattice isomorphism between the lattice of congruences of $\mathbf{A}$ and the lattice of deductive filters on $\mathbf{A}$ of the τ-assertional logic of $\mathbb{V}$ ([2]; [1]; [5, Section 4.4]; [4]).

The Received View (2)

If, moreover, $\mathbb{V}$ is also 1-subtractive, then the deductive filters on $\mathbf{A} \in \mathbb{V}$ of the 1-assertional logic of $\mathbb{V}$ coincide with the $\mathbb{V}$-ideals of $\mathbf{A}$ in the sense of Gumm and Ursini [27], which is even better in that deductive filters might be very hard to describe in general, while for $\mathbb{V}$-ideals this is less likely thanks to the availability of a manageable concept of *ideal generation* [9].

All Well and Good. However…

- There are several isomorphism theorems that are not subsumed by this nice picture

- HOW CAN WE EXPLAIN THEM?

?

Example 1: Pseudointerior Algebras

The variety $\mathbb{PI}$ of *pseudointerior algebras* [8], introduced by Blok and Pigozzi as a tool for the investigation of varieties with a commutative regular TD term, is not 1-subtractive; however, it allows for a very manageable concept of *open filter* (provably distinct from the concept of $\mathbb{PI}$-ideal). It can be shown that in every pseudointerior algebra there is an isomorphisms between the lattices of congruences and of open filters.

Example 2: Residuated Lattices

The variety $\mathbb{RL}$ of *residuated lattices* [23] is 1-ideal determined and, in fact, it is well-known that in every residuated lattice the lattice of congruences is isomorphic to the lattice of $\mathbb{RL}$-ideals. It is likewise known that $\mathbb{RL}$-ideals on any residuated lattice coincide with convex normal subalgebras of such. There is a further isomorphism theorem, however (namely, between congruences and *deductive filters* in the sense of [23]), which cannot be explained by recourse to the general results we mentioned earlier.

Example 3: Quasi-MV Algebras

The variety $q\mathbb{MV}$ of *quasi-MV algebras* [35], generalisations of MV algebras introduced in the context of an investigation into the foundations of quantum computing, is neither 1-subtractive nor 1-regular; still, in every quasi-MV algebra $\mathbf{A}$ the lattice of $q\mathbb{MV}$-$\mathbb{MV}$ congruences (where $\theta \in \mathrm{Con}(\mathbf{A})$ is said to be $q\mathbb{MV}$-$\mathbb{MV}$ iff $\mathbf{A}/\theta$ is an MV algebra) is isomorphic to the lattice of the deductive filters on $\mathbf{A}$ of infinite-valued Łukasiewicz logic.

Subtractivity Generalised to an Arbitrary Translation

a variety $\mathbb{V}$ is τ-*permutable* iff for any congruences θ, φ on any $\mathbf{A} \in \mathbb{V}$, $\tau^{\mathbf{A}}/\theta \circ \varphi = \tau^{\mathbf{A}}/\varphi \circ \theta$.

Theorem 1 *Let* $\tau = \{\delta_i(x) \approx \epsilon_i(x) : i \leq n\}$, *where* $\epsilon_i^{\mathbf{A}}$ *is a constant operation on every* $\mathbf{A} \in \mathbb{V}$. *Then* $\mathbb{V}$ *is a* τ-*permutable variety iff there exist* n *binary terms, denoted by* $\rightarrow_1, \ldots, \rightarrow_n$ *and written in infix notation, such that* $\mathbb{V}$ *satisfies the following equations for any* $i \leq n$:

$$\begin{aligned} \delta_i(x) \rightarrow_i x &\approx \epsilon_i(x) \\ \epsilon_i(x) \rightarrow_i x &\approx \delta_i(x) \end{aligned}$$

Quasi-Subtractive Variety

Definition 2 A variety $\mathbb{V}$ whose type ν includes a nullary term 1 and a unary term $\Box$ is called *quasi-subtractive* w.r.t. 1 and $\Box$ iff there is a binary term $\rightarrow(x, y)$ (hereafter written in infix notation) of type ν s.t. $\mathbb{V}$ satisfies the equations

Q1 $\Box x \rightarrow x \approx 1$

Q2 $1 \rightarrow x \approx \Box x$

Q3 $\Box(x \rightarrow y) \approx x \rightarrow y$

Q4 $\Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1$

Examples (1)

Example 4 *(Subtractive varieties).* Every 1-subtractive variety $\mathbb{V}$ ([42], [2], [3], [4]) is quasi-subtractive: it suffices to take as arrow the term witnessing 1-subtractivity for $\mathbb{V}$, and as box the identity term.

Example 5 *(Pointed varieties).* Let $\mathbb{V}$ be any pointed variety, i.e., a variety whose type includes a constant 1. Defining $\Box x = 1 = x \to y$ it is immediately verified that $\mathbb{V}$ is quasi-subtractive with the above witness terms.

Examples (2)

Example 7 *(Residuated lattices).* Residuated lattices (see e.g. [23]) are the equivalent algebraic semantics of the 0-free fragment RL of the substructural logic FL, and arise in several different areas of mathematics. The variety $\mathbb{RL}$ of residuated lattices is both 1-subtractive and 1-regular; however: (i) 1-subtractivity and 1-regularity are witnessed by terms bearing no connection whatsoever to each other; (ii) the $\mathbb{RL}$ ideals of a residuated lattice $\mathbf{A}$ are its convex normal subalgebras, which are not the deductive filters on $\mathbf{A}$ of the substructural logic RL. The latter coincide with the upward closures (w.r.t. the residuated lattice order) of convex normal subalgebras.

The variety $\mathbb{RL}$ is quasi-subtractive, witness the terms $(x \backslash y) \wedge 1$, $x \wedge 1$ and 1. Remark that the symmetrisation of the term $(x \backslash y) \wedge 1$ also witnesses point regularity for $\mathbb{RL}$.

Examples (4)

Let $\mathbb{V}$ be a variety with a commutative TD term $p(x, y, z)$ and let 1 be a constant term of $\mathbb{V}$. If we define:

$$\begin{aligned} \Box x &= p(x, 1, 1) \\ x \to y &= p(x, p(x, y, x), 1) \end{aligned}$$

then V is quasi-subtractive with the above witness terms. The following examples are noteworthy applications of this observation.

Example 12 *(Pseudointerior algebras).* The variety of pseudointerior algebras is not 1-subtractive; however, it is quasi-subtractive with open left residuation as arrow and the pseudointerior operation as box.

Examples (5)

Example 14 *(Interior algebras).* Interior algebras are the equivalent algebraic semantics of the modal deductive system S4. Blok and Pigozzi [8] show that every interior algebra is a pseudointerior algebra with open left residuation defined by

$$x \to y = \Box\left(\neg x \vee \Box\left(\left(\neg x \vee y\right) \wedge \left(\neg y \vee x\right)\right)\right).$$

Therefore, any interior algebra is quasi-subtractive according to Example 12. However, interior algebras have a Boolean algebra reduct, whence material implication $\neg x \vee y$ clearly witnesses 1-subtractivity (and then, a fortiori, quasi-subtractivity w.r.t. 1 and the identity term, according to Example 4). Finally, it should be observed that according to Example 13 quasi-subtractivity of interior algebras is also witnessed by strict implication $\Box\left(\neg x \vee y\right)$, together with $\Box$ and 1.

Examples (6)

Example 15 *(Integral k-potent residuated lattices and Nelson algebras).* A residuated lattice is k-potent if it satisfies the identity $x^k \approx x^{k+1}$ for some $k \in \omega$. In integral RLs, k-potency implies that $x^n \approx x^k$ holds for any $n \geq k$. It is well-known that k-potent integral commutative residuated lattices have EDPC, and it can be proved that all semisimple varieties of integral commutative residuated lattices are k-potent, and hence are discriminator varieties [31].

Let $\mathbb{V}$ be a variety of k-potent integral RLs. Define $\Box x = x^k$ and $x \to y = (x \setminus y)^k$. These terms witness quasi-subtractivity of $\mathbb{V}$. In particular, the variety $\mathbb{P}_k\mathbb{IRL}$ of all k-potent integral RLs is quasi-subtractive with the above witness terms, and it again emphasises the fact that quasi-subtractivity is a relative notion. Namely, the terms $x \wedge 1$ and $x \setminus y \wedge 1$, witnessing quasi-subtractivity of $\mathbb{RL}$, reduce to the identity and $x \setminus y$ respectively, so $\mathbb{P}_k\mathbb{IRL}$ is subtractive with these witness terms.

A particular case of the above is given by Nelson algebras, the equivalent algebraic semantics of constructive logic with strong negation (see e.g. [43]). In [40], [41], in fact, it is shown that the variety of Nelson algebras is term equivalent to a particular variety $\mathbb{NRL}$ of 3-potent FL_{ew}-algebras (commutative, integral and double-pointed residuated lattices).

Examples (3)

Example 9 *(Quasi-MV algebras).* Quasi-MV algebras are a generalisation of Chang's MV algebras motivated by an investigation into quantum computational logics (see e.g. [35], [38], [11]). The variety of quasi-MV algebras is neither 1-subtractive nor 1-regular, and it differs from the nilpotent shift of the variety of MV algebras [16]. However, it is quasi-subtractive, witness the terms $x' \oplus y$, $x \oplus 0$ and 1.

Example (Modular congruence lattices with CEP)

Let $\mathcal{V}$ be a congruence modular variety. The congruence lattice $\mathrm{Con}\,\mathbf{A}$ of any $\mathbf{A} \in \mathcal{V}$ can be expanded to a bounded residuated ℓ-groupoid, with the commutator $[\theta, \varphi]$ as multiplication and the centraliser $(\theta : \varphi)$ as its residual. Suppose that the commutator identity $[x, y] = x \wedge y \wedge [1, 1]$ is satisfied. It holds trivially in Abelian and congruence distributive varieties, but also [Kiss] in any modular variety with CEP, and [Burris, McKenzie] in every variety with decidable first-order theory. There, every so expanded congruence lattice is a quasi-subtractive algebra, as witnessed by:

$$\begin{aligned} x \to y &= [(y : x), (y : x)] \\ \Box x &= [x, x] \\ 1 &= [1, 1] \end{aligned}$$

Open Filters

Definition 16 Let $\mathbb{V}$ be a variety whose type ν includes a nullary term 1 and a unary term $\Box$. A $\mathbb{V}$-*open filter term* in the variables $\overrightarrow{x}$ is is an $n+m$-ary term $p\left(\overrightarrow{x}, \overrightarrow{y}\right)$ of type ν s.t.

$$\{\Box x_i \approx 1 : i \leq n\} \vdash_{Eq(\mathbb{V})} \Box p\left(\overrightarrow{x}, \overrightarrow{y}\right) \approx 1.$$

Definition 18 Let $\mathbb{V}$ be as in Definition 16. A $\mathbb{V}$-*open filter* of $\mathbf{A} \in \mathbb{V}$ is a subset $F \subseteq A$ with the following properties:

i) it is closed w.r.t. all $\mathbb{V}$-open filter terms p: whenever $a_1, ..., a_n \in F, b_1, ..., b_m \in A$, $p\left(\overrightarrow{a}, \overrightarrow{b}\right) \in F$;

ii) for every $a \in A$, we have that $a \in F$ iff $\Box a \in F$.

Examples

Variety	Ursini ideals	Open filters
Pseudointerior algebras	?	Open filters
Residuated lattices	Convex normal subalgebras	Deductive filters
Quasi-MV algebras	Weak ideals	Ideals
Interior algebras	Congruence filters	Congruence filters

Normal Open Filters

Definition 20 Let $\mathbb{V}$ be as in Definition 16. $\mathbb{V}$ has *normal open filters* iff for all $\mathbf{A} \in \mathbb{V}$, every $\mathbb{V}$-open filter of A is a $(\Box x, 1)$-class of some $\theta \in \mathrm{Con}(\mathbf{A})$.

Theorem 25 *Every quasi-subtractive variety $\mathbb{V}$ has normal open filters.*

According to established notational practice (see e.g. [10]), the τ-assertional logic of a variety $\mathbb{V}$ is referred to as $\mathcal{S}(\mathbb{V}, \tau)$. Here, however, we only work with the particular translation $\tau = \{\Box x \approx 1\}$, a circumstance which gives us the freedom of simplifying our notation to $\mathcal{S}(\mathbb{V})$.

Lemma 26 *If $\mathbb{V}$ is a variety with normal open filters and $\mathbf{A} \in \mathbb{V}$, $\mathbb{V}$-open filters of $\mathbf{A}$ coincide with deductive filters on $\mathbf{A}$ of $\mathcal{S}(\mathbb{V})$.*

Theorem 27 *If $\mathbb{V}$ is a quasi-subtractive variety and $\mathbf{A} \in \mathbb{V}$, $\mathbb{V}$-open filters of $\mathbf{A}$ coincide with deductive filters on $\mathbf{A}$ of $\mathcal{S}(\mathbb{V})$.*

Weakly τ-Regular Varieties

Definition 58 A variety $\mathbb{V}$ is called *weakly τ-regular* iff the τ-assertional logic $S(\mathbb{V}, \tau)$ of $\mathbb{V}$ is strongly and finitely algebraisable.

Lemma 59 *$\mathbb{V}$ is weakly τ-regular iff, given $\mathbf{A}$ in $\mathbb{V}$ and given any two $\mathbb{V}$-$\mathbb{V}'$ congruences θ, φ on $\mathbf{A}$ (where $\mathbb{V}'$ is the equivalent algebraic semantics of $\mathcal{S}(\mathbb{V}, \tau)$), we have that $\tau^{\mathbf{A}}/\theta = \tau^{\mathbf{A}}/\varphi$ implies $\theta = \varphi$.*

Open Filter Determinacy

We now leave again the general scenario of an arbitrary translation τ to focus once more on translations of the form $\{\Box x \approx 1\}$. Recall that $\mathcal{S}(\mathbb{V})$ is short for $\mathcal{S}(\mathbb{V}, \{\Box x \approx 1\})$.

Theorem 60 *If $\mathbb{V}$ is weakly $(\Box x, 1)$-regular and $\mathbb{V}'$ is the equivalent algebraic semantics of $\mathcal{S}(\mathbb{V})$, then in any $\mathbf{A} \in \mathbb{V}$ there is a lattice isomorphism between the lattice of $\mathbb{V}$-$\mathbb{V}'$ congruences on $\mathbf{A}$ and the lattice of deductive filters on $\mathbf{A}$ of $\mathcal{S}(\mathbb{V})$.*

Corollary 61 *If $\mathbb{V}$ is quasi-subtractive and weakly $(\Box x, 1)$-regular and $\mathbb{V}'$ is the equivalent algebraic semantics of $\mathcal{S}(\mathbb{V})$, then in any $\mathbf{A} \in \mathbb{V}$ there is a lattice isomorphism between the lattice of $\mathbb{V}$-$\mathbb{V}'$ congruences on $\mathbf{A}$ and the lattice of $\mathbb{V}$-open filters on $\mathbf{A}$.*

Applications (1)

If we let $\Box$ be the identity term therein, then $\mathbb{V}$ is 1-subtractive and weakly 1-regular. However, for a variety $\mathbb{V}$, being weakly 1-regular means nothing else than that the 1-assertional logic of $\mathbb{V}$ is finitely and regularly algebraisable *with* $\mathbb{V}$ *as equivalent algebraic semantics*. So $\mathbb{V} = \mathbb{V}'$ and $\mathbb{V}$ is after all 1-subtractive and 1-regular, i.e. 1-ideal determined. In this particular case, therefore, we get as an instance of Corollary 61 the isomorphism between the lattice of congruences and the lattice of $\mathbb{V}$-ideals in members of ideal determined varieties.

Applications (2)

The variety $\mathbb{RL}$ of residuated lattices is quasi-subtractive and $(x \wedge 1, 1)$-regular: as aleady mentioned, its $(x \wedge 1, 1)$-assertional logic is nothing but the 0-free fragment RL of the substructural logic FL (see [23]), which is algebraisable with $\mathbb{RL}$ as equivalent algebraic semantics. According to Corollary 61, therefore, in any residuated lattice $\mathbf{A}$ there is an isomorphism between the lattice of congruences on $\mathbf{A}$ and the lattice of $\mathbb{RL}$-open filters on $\mathbf{A}$, which are its deductive filters in the sense of [23]. This isomorphism theorem is exactly the content of Theorem 3.47 in [23].

Applications (3)

The variety $\mathbb{PI}$ of pseudointerior algebras is quasi-subtractive and $(x^\circ, 1)$-regular: its $(x^\circ, 1)$-assertional logic is algebraisable with $\mathbb{PI}$ as equivalent algebraic semantics. According to Corollary 61, therefore, in any pseudointerior algebra $\mathbf{A}$ there is an isomorphism between the lattice of congruences on $\mathbf{A}$ and the lattice of $\mathbb{PI}$-open filters on $\mathbf{A}$, which are its open filters in the sense of [7]. This isomorphism theorem is exactly the content of Theorem 2.16 in [7].

Applications (4)

The variety $q\mathbb{MV}$ of quasi-MV algebras is quasi-subtractive and $(x \oplus 0, 1)$-regular: its $(x \oplus 0, 1)$-assertional logic is nothing but infinite-valued Łukasiewicz logic (see [33]), which is algebraisable with the variety $\mathbb{MV}$ of MV algebras as equivalent algebraic semantics. According to Corollary 61, therefore, in any quasi-MV algebra $\mathbf{A}$ there is an isomorphism between the lattice of $q\mathbb{MV} - \mathbb{MV}$ congruences on $\mathbf{A}$ and the lattice of $q\mathbb{MV}$-open filters on $\mathbf{A}$, which are (the dualisation of) ideals in the sense of [35]. This isomorphism theorem is exactly the content of Theorem 45 in [35].