Quasi-Subtractive Varieties, Part I

T. Kowalski, F. Paoli, M. Spinks

AsubL Take 4 – JAIST, June 9, 2010
Isomorphism Theorems

Algebras in classical varieties have pleasant properties:

- **Groups**
  - Congruences \leftrightarrow Normal Subgroups

- **Rings**
  - Congruences \leftrightarrow Two-Sided Ideals

- **Boolean algebras**
  - Congruences \leftrightarrow Filters (Ideals)
Some definitions

Let $\mathcal{K}$ be a class of algebras, and let $\tau$ be a formula-equation transformer. Define

$$\Gamma \vdash_{K_\tau} \alpha \text{ iff } \tau(\Gamma) \vdash_{K} \tau(\alpha).$$

The logic $K_\tau = (\text{Fm}, \vdash_{K_\tau})$ is called the $\tau$-assertional logic of $\mathcal{K}$.

A variety $\mathcal{V}$ is $\tau$-protoregular if its $\tau$-assertional logic is protoalgebraic.

A variety $\mathcal{V}$ is $\tau$-regular if its $\tau$-assertional logic is strongly and finitely algebrisable with $\mathcal{V}$ as equivalent algebraic semantics.

Equivalently: if for any algebra $A \in \mathcal{V}$ and for any congruences $\theta, \varphi$ on $A$, $\tau^A / \theta = \tau^A / \varphi$ implies $\theta = \varphi$, where:

$$\tau^A / \theta = \{ a \in A : \tau^A(a) \subseteq \theta \}$$

1-protoregular, 1-regular: the same with $\tau = \{ x \approx 1 \}$
Some definitions

A pointed variety $\mathcal{V}$ of type $\nu$ is 1-subtractive iff there exists a binary term $x \to y$ of type $\nu$ s.t. $\mathcal{V}$ satisfies the following equations:

- $x \to x \approx 1$;
- $1 \to x \approx x$.

A variety $\mathcal{V}$ is 1-subtractive iff it is 1-permutable: for every $A \in \mathcal{V}$ and for every $\theta, \psi \in \text{Con}(A)$

$$1^A / \theta \circ \psi = 1^A / \psi \circ \theta$$
In every member $A$ of a $\tau$-regular variety $\mathcal{V}$ - namely, a variety that arises as the equivalent algebraic semantics of an algebraizable deductive system - there is a lattice isomorphism between the lattice of congruences of $A$ and the lattice of deductive filters on $A$ of the $\tau$-assertional logic of $\mathcal{V}$ ([2]; [1]; [5, Section 4.4]; [4]).
The Received View (2)

If, moreover, \( \mathcal{V} \) is also 1-subtractive, then the deductive filters on \( A \in \mathcal{V} \) of the 1-assertional logic of \( \mathcal{V} \) coincide with the \( \mathcal{V} \)-ideals of \( A \) in the sense of Gumm and Ursini [27], which is even better in that deductive filters might be very hard to describe in general, while for \( \mathcal{V} \)-ideals this is less likely thanks to the availability of a manageable concept of ideal generation [9].
All Well and Good. However...

- There are several isomorphism theorems that are not subsumed by this nice picture

- HOW CAN WE EXPLAIN THEM?
Example 1: Pseudointerior Algebras

The variety $\mathfrak{P\Pi}$ of \textit{pseudointerior algebras} [8], introduced by Blok and Pigozzi as a tool for the investigation of varieties with a commutative regular TD term, is not 1-subtractive; however, it allows for a very manageable concept of \textit{open filter} (provably distinct from the concept of $\mathfrak{P\Pi}$-ideal). It can be shown that in every pseudointerior algebra there is an isomorphisms between the lattices of congruences and of open filters.
Example 2: Residuated Lattices

The variety $\mathbb{RL}$ of residuated lattices [23] is 1-ideal determined and, in fact, it is well-known that in every residuated lattice the lattice of congruences is isomorphic to the lattice of $\mathbb{RL}$-ideals. It is likewise known that $\mathbb{RL}$-ideals on any residuated lattice coincide with convex normal subalgebras of such. There is a further isomorphism theorem, however (namely, between congruences and deducive filters in the sense of [23]), which cannot be explained by recourse to the general results we mentioned earlier.
Example 3: Quasi-MV Algebras

The variety $q\text{MV}$ of quasi-$MV$ algebras [35], generalisations of MV algebras introduced in the context of an investigation into the foundations of quantum computing, is neither 1-subtractive nor 1-regular; still, in every quasi-MV algebra $A$ the lattice of $q\text{MV}$-$MV$ congruences (where $\theta \in \text{Con}(A)$ is said to be $q\text{MV}$-$MV$ iff $A/\theta$ is an MV algebra) is isomorphic to the lattice of the deductive filters on $A$ of infinite-valued Łukasiewicz logic.
Subtractivity Generalised to an Arbitrary Translation

A variety $\mathbb{V}$ is $\tau$-permutable iff for any congruences $\theta, \varphi$ on any $A \in \mathbb{V}$, $\tau^A / \theta \circ \varphi = \tau^A / \varphi \circ \theta$.

**Theorem 1** Let $\tau = \{\delta_i(x) \approx \epsilon_i(x) : i \leq n\}$, where $\epsilon^A_i$ is a constant operation on every $A \in \mathbb{V}$. Then $\mathbb{V}$ is a $\tau$-permutable variety iff there exist $n$ binary terms, denoted by $\rightarrow_1, \ldots, \rightarrow_n$ and written in infix notation, such that $\mathbb{V}$ satisfies the following equations for any $i \leq n$:

\[
\delta_i(x) \rightarrow_i x \ \approx \ \epsilon_i(x)
\]

\[
\epsilon_i(x) \rightarrow_i x \ \approx \ \delta_i(x)
\]
**Definition 2** A variety $\mathcal{V}$ whose type $\nu$ includes a nullary term $1$ and a unary term $\square$ is called *quasi-subtractive* w.r.t. $1$ and $\square$ iff there is a binary term $\rightarrow (x, y)$ (hereafter written in infix notation) of type $\nu$ s.t. $\mathcal{V}$ satisfies the equations

1. $\square x \rightarrow x \approx 1$
2. $1 \rightarrow x \approx \square x$
3. $\square (x \rightarrow y) \approx x \rightarrow y$
4. $\square (x \rightarrow y) \rightarrow (\square x \rightarrow \square y) \approx 1$
Examples (1)

Example 4 (Subtractive varieties). Every 1-subtractive variety \( \mathbb{V} \) ([42], [2], [3], [4]) is quasi-subtractive: it suffices to take as arrow the term witnessing 1-subtractivity for \( \mathbb{V} \), and as box the identity term.

Example 5 (Pointed varieties). Let \( \mathbb{V} \) be any pointed variety, i.e., a variety whose type includes a constant 1. Defining \( \Box x = 1 = x \to y \) it is immediately verified that \( \mathbb{V} \) is quasi-subtractive with the above witness terms.
Examples (2)

Example 7 \textit{(Residuated lattices).} Residuated lattices (see e.g. [23]) are the equivalent algebraic semantics of the 0-free fragment \( \mathbb{RL} \) of the substructural logic \( \mathbb{FL} \), and arise in several different areas of mathematics. The variety \( \mathbb{RL} \) of residuated lattices is both 1-subtractive and 1-regular; however: (i) 1-subtractivity and 1-regularity are witnessed by terms bearing no connection whatsoever to each other; (ii) the \( \mathbb{RL} \) ideals of a residuated lattice \( \mathbf{A} \) are its convex normal subalgebras, which are not the deductive filters on \( \mathbf{A} \) of the substructural logic \( \mathbb{RL} \). The latter coincide with the upward closures (w.r.t. the residuated lattice order) of convex normal subalgebras.

The variety \( \mathbb{RL} \) is quasi-subtractive, witness the terms \((x \backslash y) \wedge 1\), \(x \wedge 1\) and \(1\). Remark that the symmetrisation of the term \((x \backslash y) \wedge 1\) also witnesses point regularity for \( \mathbb{RL} \).
Examples (4)

Let $V$ be a variety with a commutative TD term $p(x, y, z)$ and let $1$ be a constant term of $V$. If we define:

$$
\Box x = p(x, 1, 1) \\
x \to y = p(x, p(x, y, x), 1)
$$

then $V$ is quasi-subtractive with the above witness terms. The following examples are noteworthy applications of this observation.

**Example 12** (Pseudointerior algebras). The variety of pseudointerior algebras is not 1-subtractive; however, it is quasi-subtractive with open left residuation as arrow and the pseudointerior operation as box.
Examples (5)

Example 14 (Interior algebras). Interior algebras are the equivalent algebraic semantics of the modal deductive system S4. Blok and Pigozzi [8] show that every interior algebra is a pseudointerior algebra with open left residuation defined by

$$x \rightarrow y = \Box (\neg x \lor \Box ((\neg x \lor y) \land (\neg y \lor x))).$$

Therefore, any interior algebra is quasi-subtractive according to Example 12. However, interior algebras have a Boolean algebra reduct, whence material implication $\neg x \lor y$ clearly witnesses 1-subtractivity (and then, a fortiori, quasi-subtractivity w.r.t. 1 and the identity term, according to Example 4). Finally, it should be observed that according to Example 13 quasi-subtractivity of interior algebras is also witnessed by strict implication $\Box (\neg x \lor y)$, together with $\Box$ and 1.
Examples (6)

Example 15 (Integral k-potent residuated lattices and Nelson algebras). A residuated lattice is $k$-potent if it satisfies the identity $x^k \approx x^{k+1}$ for some $k \in \omega$. In integral RLSs, $k$-potency implies that $x^n \approx x^k$ holds for any $n \geq k$. It is well-known that $k$-potent integral commutative residuated lattices have EDPC, and it can be proved that all semisimple varieties of integral commutative residuated lattices are $k$-potent, and hence are discriminator varieties [31].

Let $\mathbb{V}$ be a variety of $k$-potent integral RLSs. Define $\square x = x^k$ and $x \rightarrow y = (x \setminus y)^k$. These terms witness quasi-subtractivity of $\mathbb{V}$. In particular, the variety $\mathbb{P}_k\mathbb{IRL}$ of all $k$-potent integral RLSs is quasi-subtractive with the above witness terms, and it again emphasises the fact that quasi-subtractivity is a relative notion. Namely, the terms $x \land 1$ and $x \setminus y \land 1$, witnessing quasi-subtractivity of $\mathbb{RL}$, reduce to the identity and $x \setminus y$ respectively, so $\mathbb{P}_k\mathbb{IRL}$ is subtractive with these witness terms.

A particular case of the above is given by Nelson algebras, the equivalent algebraic semantics of constructive logic with strong negation (see e.g. [43]). In [40], [41], in fact, it is shown that the variety of Nelson algebras is term equivalent to a particular variety $\mathbb{NRL}$ of 3-potent $\mathbb{FL}_{e\omega}$-algebras (commutative, integral and double-pointed residuated lattices).
Example 9 \((Quasi-MV \text{ algebras})\). Quasi-MV algebras are a generalisation of Chang’s MV algebras motivated by an investigation into quantum computational logics (see e.g. [35], [38], [11]). The variety of quasi-MV algebras is neither 1-subtractive nor 1-regular, and it differs from the nilpotent shift of the variety of MV algebras [16]. However, it is quasi-subtractive, witness the terms \(x' \oplus y\), \(x \oplus 0\) and \(1\).
Example (Modular congruence lattices with CEP)

Let $\mathcal{V}$ be a congruence modular variety. The congruence lattice $\text{Con} \ A$ of any $A \in \mathcal{V}$ can be expanded to a bounded residuated $\ell$-groupoid, with the commutator $[\theta, \varphi]$ as multiplication and the centraliser $(\theta : \varphi)$ as its residual. Suppose that the commutator identity $[x, y] = x \land y \land [1, 1]$ is satisfied. It holds trivially in Abelian and congruence distributive varieties, but also [Kiss] in any modular variety with CEP, and [Burris, McKenzie] in every variety with decidable first-order theory. There, every so expanded congruence lattice is a quasi-subtractive algebra, as witnessed by:

\[
x \rightarrow y = [(y : x), (y : x)] \\
\square x = [x, x] \\
1 = [1, 1]
\]
Open Filters

**Definition 16** Let $\mathbb{V}$ be a variety whose type $\nu$ includes a nullary term 1 and a unary term $\Box$. A $\mathbb{V}$-open filter term in the variables $\overrightarrow{x}$ is is an $n + m$-ary term $p(\overrightarrow{x}, \overrightarrow{y})$ of type $\nu$ s.t.

$$\{ \Box x_i \approx 1 : i \leq n \} \vdash_{\text{Eq}(\mathbb{V})} \Box p(\overrightarrow{x}, \overrightarrow{y}) \approx 1.$$ 

**Definition 18** Let $\mathbb{V}$ be as in Definition 16. A $\mathbb{V}$-open filter of $A \in \mathbb{V}$ is a subset $F \subseteq A$ with the following properties:

i) it is closed w.r.t. all $\mathbb{V}$-open filter terms $p$: whenever $a_1, \ldots, a_n \in F, b_1, \ldots, b_m \in A$, $p(\overrightarrow{a}, \overrightarrow{b}) \in F$;

ii) for every $a \in A$, we have that $a \in F$ iff $\Box a \in F$. 
## Examples

<table>
<thead>
<tr>
<th>Variety</th>
<th>Ursini ideals</th>
<th>Open filters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pseudointerior algebras</td>
<td>?</td>
<td>Open filters</td>
</tr>
<tr>
<td>Residuated lattices</td>
<td>Convex normal subalgebras</td>
<td>Deductive filters</td>
</tr>
<tr>
<td>Quasi-MV algebras</td>
<td>Weak ideals</td>
<td>Ideals</td>
</tr>
<tr>
<td>Interior algebras</td>
<td>Congruence filters</td>
<td>Congruence filters</td>
</tr>
</tbody>
</table>
Normal Open Filters

Definition 20 Let $\mathbb{V}$ be as in Definition 16. $\mathbb{V}$ has normal open filters iff for all $A \in \mathbb{V}$, every $\mathbb{V}$-open filter of $A$ is a $(\Box x, 1)$-class of some $\theta \in \text{Con}(A)$.

Theorem 25 Every quasi-subtractive variety $\mathbb{V}$ has normal open filters.

According to established notational practice (see e.g. [10]), the $\tau$-assertional logic of a variety $\mathbb{V}$ is referred to as $S(\mathbb{V}, \tau)$. Here, however, we only work with the particular translation $\tau = \{\Box x \approx 1\}$, a circumstance which gives us the freedom of simplifying our notation to $S(\mathbb{V})$.

Lemma 26 If $\mathbb{V}$ is a variety with normal open filters and $A \in \mathbb{V}$, $\mathbb{V}$-open filters of $A$ coincide with deductive filters on $A$ of $S(\mathbb{V})$.

Theorem 27 If $\mathbb{V}$ is a quasi-subtractive variety and $A \in \mathbb{V}$, $\mathbb{V}$-open filters of $A$ coincide with deductive filters on $A$ of $S(\mathbb{V})$. 
Weakly $\tau$-Regular Varieties

**Definition 58** A variety $\mathbb{V}$ is called weakly $\tau$-regular iff the $\tau$-assertional logic $S(\mathbb{V}, \tau)$ of $\mathbb{V}$ is strongly and finitely algebraisable.

**Lemma 59** $\mathbb{V}$ is weakly $\tau$-regular iff, given $A$ in $\mathbb{V}$ and given any two $\mathbb{V}$-$\mathbb{V}'$ congruences $\theta, \varphi$ on $A$ (where $\mathbb{V}'$ is the equivalent algebraic semantics of $S(\mathbb{V}, \tau)$), we have that $\tau^A / \theta = \tau^A / \varphi$ implies $\theta = \varphi$. 
Open Filter Determinacy

We now leave again the general scenario of an arbitrary translation \( \tau \) to focus once more on translations of the form \( \{ \Box x \approx 1 \} \). Recall that \( S(\forall) \) is short for \( S(\forall, \{ \Box x \approx 1 \}) \).

**Theorem 60** If \( \forall \) is weakly \( (\Box x, 1) \)-regular and \( \forall' \) is the equivalent algebraic semantics of \( S(\forall) \), then in any \( A \in \forall \) there is a lattice isomorphism between the lattice of \( \forall-\forall' \) congruences on \( A \) and the lattice of deductive filters on \( A \) of \( S(\forall) \).

**Corollary 61** If \( \forall \) is quasi-subtractive and weakly \( (\Box x, 1) \)-regular and \( \forall' \) is the equivalent algebraic semantics of \( S(\forall) \), then in any \( A \in \forall \) there is a lattice isomorphism between the lattice of \( \forall-\forall' \) congruences on \( A \) and the lattice of \( \forall \)-open filters on \( A \).
Applications (1)

If we let $\square$ be the identity term therein, then $\mathcal{V}$ is 1-subtractive and weakly 1-regular. However, for a variety $\mathcal{V}$, being weakly 1-regular means nothing else than that the 1-assertional logic of $\mathcal{V}$ is finitely and regularly algebraisable with $\mathcal{V}$ as equivalent algebraic semantics. So $\mathcal{V} = \mathcal{V}'$ and $\mathcal{V}$ is after all 1-subtractive and 1-regular, i.e. 1-ideal determined. In this particular case, therefore, we get as an instance of Corollary 61 the isomorphism between the lattice of congruences and the lattice of $\mathcal{V}$-ideals in members of ideal determined varieties.
The variety $\mathbb{RL}$ of residuated lattices is quasi-subtractive and $(x \wedge 1, 1)$-regular: as already mentioned, its $(x \wedge 1, 1)$-assertional logic is nothing but the 0-free fragment $\mathbb{RL}$ of the substructural logic $\mathbb{FL}$ (see [23]), which is algebraisable with $\mathbb{RL}$ as equivalent algebraic semantics. According to Corollary 61, therefore, in any residuated lattice $A$ there is an isomorphism between the lattice of congruences on $A$ and the lattice of $\mathbb{RL}$-open filters on $A$, which are its deductive filters in the sense of [23]. This isomorphism theorem is exactly the content of Theorem 3.47 in [23].
Applications (3)

The variety $\mathbb{P\Pi}$ of pseudointerior algebras is quasi-subtractive and $(x^\circ, 1)$-regular: its $(x^\circ, 1)$-assertional logic is algebraisable with $\mathbb{P\Pi}$ as equivalent algebraic semantics. According to Corollary 61, therefore, in any pseudointerior algebra $A$ there is an isomorphism between the lattice of congruences on $A$ and the lattice of $\mathbb{P\Pi}$-open filters on $A$, which are its open filters in the sense of [7]. This isomorphism theorem is exactly the content of Theorem 2.16 in [7].
Applications (4)

The variety $q\text{MV}$ of quasi-MV algebras is quasi-subtractive and $(x \oplus 0, 1)$-regular: its $(x \oplus 0, 1)$-assertional logic is nothing but infinite-valued Łukasiewicz logic (see [33]), which is algebraisable with the variety $\text{MV}$ of MV algebras as equivalent algebraic semantics. According to Corollary 61, therefore, in any quasi-MV algebra $A$ there is an isomorphism between the lattice of $q\text{MV} – \text{MV}$ congruences on $A$ and the lattice of $q\text{MV}$-open filters on $A$, which are (the dualisation of) ideals in the sense of [35]. This isomorphism theorem is exactly the content of Theorem 45 in [35].