Quasi-subtractive varieties (part II)

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- Bad: such that (i) holds but not (ii).
- Ugly: all the rest.

Open filters

Let $\mathcal{V}$ be quasi-subtractive, and $\mathbf{A} \in \mathcal{V}$.

Definition

An open filter term in the variables $\mathbf{x}$ is is an $n + m$-ary term $p(\mathbf{x}, \mathbf{y})$ such that $\Box\mathbf{x} \approx 1$ implies $\Box p(\mathbf{x}, \mathbf{y}) \approx 1$

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Definition

An open filter of $\mathbf{A}$ is a subset $F \subseteq A$ such that:

1. if p is an open filter term, and $\mathbf{a} \in F, \mathbf{b} \in A$, then $p(\mathbf{a}, \mathbf{b}) \in F$
2. $a \in F$ iff $\Box a \in F$

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For a set $X \subseteq A$, we put $\Gamma(X)$ to be the closure of X under open filter terms, and $\uparrow X$ to be $\Box^{-1}(X)$.

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Let $\mathcal{V}$ be a quasi-subtractive variety, and $\mathbf{A} \in \mathcal{V}$. The set of open filters of $\mathbf{A}$ under the operations of $\_ \cap \_$ and $\uparrow\Gamma(\_ \cup \_)$, forms an algebraic modular lattice.

Open and flat subvarieties

Let $\mathcal{V}$ be quasi-subtractive. The subvariety $\mathcal{V}_O$ of $\mathcal{V}$ defined by $\Box x \approx x$ we call open. Any subvariety whose intersection with $\mathcal{V}_O$ is trivial, we call flat.

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Lemma

Let $\mathcal{V}_F \subseteq \mathcal{V}$ be flat. Then, there exists a unary term $\boxtimes x$ such that $\mathcal{V}_O \models \boxtimes x \approx x$ and $\mathcal{V}_F \models \boxtimes x \approx 1$.

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Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be varieties. We write $\mathcal{V}_1 \times_s \mathcal{V}_2$ for the class $\{\mathbf{A} \hookrightarrow_s \mathbf{B}_1 \times \mathbf{B}_2 \colon \mathbf{B}_1 \in \mathcal{V}_1 \text{ and } \mathbf{B}_2 \in \mathcal{V}_2\}$.

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Theorem

If $\boxtimes$ commutes with all operations not preserving $\{1\}$ on all algebras in $\mathcal{V}_O \cup \mathcal{V}_F$, then $\mathcal{V}_O \vee \mathcal{V}_F = \mathcal{V}_O \times_s \mathcal{V}_F$.

Known results as corollaries

Let $\mathcal{V}$ be a quasi-subtractive variety with open and flat subvarieties $\mathcal{V}_O$ and $\mathcal{V}_F$ such that, for some binary term $x \circ y$ and unary term $\boxdot$, the following hold:

1. $\mathcal{V}_O \models \boxdot x \approx 1, \quad x \circ 1 \approx x$
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Let $\mathcal{V}$, $\mathcal{V}_O$ and $\mathcal{V}_F$ be as above. Then $\mathcal{V}_O \vee \mathcal{V}_F = \mathcal{V}_O \times \mathcal{V}_F$.

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Some known direct decomposition theorems become corollaries. Such are the decomposition theorems for certain varieties of residuated lattices, due to Jónsson-Tsinakis and Galatos-Tsinakis, or for *sircomonoids* due to Raftery-Van Alten.

Unknown results as corollaries

Varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ are independent if there is a binary term $x \star y$ such that $\mathcal{V}_1 \models x \star y = x$ and $\mathcal{V}_1 \models x \star y = y$.

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Theorem

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be varieties of groups. Then, $\mathcal{V}_1$ and $\mathcal{V}_2$ are independent if and only if there is a positive integer k such that $\mathcal{V}_1 \models x^k = e$ and $\mathcal{V}_2 \models x^{k-1} = e$, up to renumbering of $\mathcal{V}_1$ and $\mathcal{V}_2$. Moreover, if both $\mathcal{V}_1$ and $\mathcal{V}_2$ are nontrivial, then $k > 2$.

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Theorem

Let $\mathcal{V}$ be a variety of groups. The following are equivalent.

1. *$\mathcal{V}$ satisfies the identities $x^{k(k-1)} = e$ and $(xy)^{1-k}(zu)^k = x^{1-k}z^k y^{1-k}u^k$ for some $k > 1$.*
2. *$\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, for independent varieties $\mathcal{V}_1$ and $\mathcal{V}_2$.*

Transfer of CEP and AP

Theorem

Let $\mathcal{V}_O$ and $\mathcal{V}_F$ constitute an open-flat decomposition of a quasi-subtractive variety $\mathcal{V}$. Then, the variety $\mathcal{V}$ has CEP if and only if both $\mathcal{V}_O$ and $\mathcal{V}_F$ have CEP.

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Theorem

Let $\mathcal{V}_O$ and $\mathcal{V}_F$ constitute an open-flat decomposition of a quasi-subtractive variety $\mathcal{V}$. Then,

1. *$\mathcal{V}$ has AP iff $\mathcal{V}_O$ and $\mathcal{V}_F$ have AP,*
2. *$\mathcal{V}$ has SAP iff $\mathcal{V}_O$ and $\mathcal{V}_F$ have SAP.*

Open contractions

For any term $t(\mathbf{x})$, we define its open translation $t^{\Box}$ inductively:

- $x^{\Box} = x$, for a variable x,
- $o^{\Box}(t_1, \ldots, t_k) = \Box o(t_1^{\Box}, \ldots, t_k^{\Box})$, for a k-ary basic operation o and terms $t_1, \ldots, t_k$.

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On $A^{\Box} = \{a \in A \colon \Box a = a\}$, we define operations, putting $(o^{\Box})_{o \in O}$, where O is the set of all basic operations in the type. Then $\mathbf{A}^{\Box}$ is the algebra $\langle A^{\Box}, (o^{\Box})_{o \in O} \rangle$.

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Functoriality

Lemma

Let $h\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then, $h|_{A^\Box}\colon \mathbf{A}^\Box \to \mathbf{B}^\Box$ is a homomorphism, and the diagram

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\quad h \quad} & \mathbf{B} \\ \Box\big\downarrow & & \big\downarrow\Box \\ \mathbf{A}^\Box & \xrightarrow{\quad h|_{A^\Box} \quad} & \mathbf{B}^\Box \end{array}$$

commutes. In particular, if θ is a congruence on $\mathbf{A}$, then $\theta|_{\mathbf{A}^\Box}$ is a congruence on $\mathbf{A}^\Box$.

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Suppose $\mathbf{A} \in \mathcal{V}$. In general, $\mathbf{A}^\Box$ may not belong to $\mathcal{V}$. Things begin to improve if the open translation preserves some structure.

Divide . . .

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- smooth, if $\square^{\square}$ and $\to^{\square}$ coincide on $A^{\square}$ with $\square$ and $\to$.
- contractive, if $\mathbf{A} \in \mathcal{V}$ implies $\mathbf{A}^{\square} \in \mathcal{V}$.

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- invertible, if for every algebra $\mathbf{A} \in \mathcal{V}$ and every congruence φ on $\mathbf{A}^{\Box}$ there is a congruence θ on $\mathbf{A}$ such that $\varphi = \theta|_{\mathbf{A}^{\Box}}$.

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- The notions are pairwise independent (e.g., translation from ℓ-groups to their negative cones is invertible, but not contractive; translation from residuated lattices to negative cones is contractive but not invertible).
- A subvariety of a contractive variety may fail to be contractive.

. . . et impera

Theorem

Let $\mathbf{A}$ *be quasi-subtractive with witness terms* $\Box$ *and* $\rightarrow$*. If* $\mathbf{A}^{\Box}$ *is smooth, then* $\mathbf{A}^{\Box}$ *is subtractive with witness term* $\rightarrow^{\Box}$.

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Let $\mathcal{V}$ be smooth and contractive. Then, the class $\mathcal{V}^{\Box}$ is a variety and it coincides with $\mathcal{V}_O$, the open subvariety of $\mathcal{V}$.

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Theorem

Let $\mathcal{V}$ *be smooth and invertible and* $\mathbf{A} \in \mathcal{V}$*. Suppose* F *is a* $\mathcal{V}^{\Box}$*-open filter on* $\mathbf{A}^{\Box}$*. Then,* $\uparrow F$ *is a* $\mathcal{V}$*-open filter on* $\mathbf{A}$*.*

Bases for contractive varieties

An identity $t(\mathbf{x}) \approx s(\mathbf{x})$ will be called stable if it survives open translation, that is, if

- $\mathcal{V} \models t(\mathbf{x}) \approx s(\mathbf{x})$, and
- $\mathcal{V} \models t^{\Box}(\Box\mathbf{x}) \approx s^{\Box}(\Box\mathbf{x})$,

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Theorem

Let $\mathcal{V}$ be quasi-subtractive. Then the following are equivalent:

1. *$\mathcal{V}$ is contractive;*
2. *$\mathcal{V}$ has a basis of stable identities;*
3. *every basis of $\mathcal{V}$ consists of stable identities;*
4. *the equational theory of $\mathcal{V}$ consists of stable identities.*

Examples of open contractions

Example

Let $\mathcal{V}$ be the variety of quasi-MV algebras, and $\Box x = x \oplus 0$, $x \to y = \neg x \oplus y$. Then, $\mathcal{V}^{\Box} \subseteq \mathcal{V}$ is the variety of MV algebras.

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Let $\mathcal{V}$ be the variety of residuated lattices, and $\Box x = x \wedge 1$, $x \to y = x \backslash y$. Then $\mathcal{V}^{\Box} \subseteq \mathcal{V}$ is the variety of integral residuated lattices, and the translation is known as the negative cone contraction.

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Let $\mathcal{V}$ be the variety of ℓ-groups, and $\Box x = x \wedge 1$, $x \to y = x^{-1}y$. Then $\mathcal{V}^{\Box}$ is the variety of negative cones of ℓ-groups, and $\mathcal{V} \vee \mathcal{V}^{\Box} = \mathcal{V} \times \mathcal{V}^{\Box}$.

Open contractions as translations between logics

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Let $\mathcal{V}$ be the variety of interior algebras and $\Box x$ be the interior operator. Then $\mathcal{V}^{\Box}$ is the variety of Heyting algebras, and the open translation is the usual Gödel translation.

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Open contractions as translations between logics

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Example

Same for any varieties $\mathcal{V}$ and $\mathcal{W}$ of FL-algebras, such that $\mathcal{V}$ has the Glivenko property with respect to $\mathcal{W}$. Actually, left or right Glivenko property suffices.

Some open problems

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Develop a theory of open contractions. In particular, investigate the properties of smoothness, contractivity and invertibility.

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Problem

Develop a theory of translations. What properties do they preserve? Under what conditions?

Some open problems

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Sometimes (e.g., in RLs) Gumm-Ursini ideals and open filters do not coincide, yet there are only good congruences. What general feature is responsible for this property?

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Problem

Investigate the bad and the ugly congruences. Do they have something to do with presence of types 1 and 2 in the sense of Tame Congruence Theory?

Some open problems

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Prove direct and/or subdirect decomposition theorems for some interesting classes of groupoids (maybe loops?).

Problem

How does the super-amalgamation property (SupAP) in the sense of Maximova fare with respect to open-flat decompositions?