Craig Interpolation for Semilinear Logics

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Joint work with Enrico Marchioni, IIIA-CSIC, Barcelona

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Craig Interpolation: Success and Failure

Craig interpolation holds for **Classical logic** CL: whenever

$$\vdash_{\mathrm{CL}} \varphi(\vec{p}, \vec{q}) \to \psi(\vec{q}, \vec{r})$$

there exists a formula $\chi(\vec{q})$ such that

$$\vdash_{\mathrm{CL}} \varphi(\vec{p}, \vec{q}) \to \chi(\vec{q}) \quad \text{and} \quad \vdash_{\mathrm{CL}} \chi(\vec{q}) \to \psi(\vec{q}, \vec{r}).$$

However, in **Łukasiewicz logic** Ł,

$$\vdash_{Ł} (p \wedge \neg p) \to (q \vee \neg q)$$

but there is no variable-free formula φ such that

$$\vdash_{Ł} (p \wedge \neg p) \to \varphi \quad \text{and} \quad \vdash_{Ł} \varphi \to (q \vee \neg q)$$

since $\frac{1}{2}$ is not definable.

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This Talk

We address issues of Craig interpolation for "semilinear logics":

- When does Craig interpolation hold for these logics?
- When does it fail?
- Is there a general characterization?

  Conjecture: the logic must at least be *idempotent*; i.e., admit $\varphi \equiv (\varphi \cdot \varphi)$ as a theorem.

We follow various strategies described in:

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Craig Interpolation

Let L be a finitary structural consequence relation based on a language $\mathcal{L}$ with a distinguished implication connective $\rightarrow$.

We write φ, ψ, χ for typical members of the set of formulas $\mathrm{Fm}_{\mathcal{L}}$, denoting by $\varphi(\vec{p})$ that the variables of φ are among $\vec{p} = p_1 \ldots p_n$.

L is said to have the *Craig interpolation property* CIP if whenever

$$\vdash_{\mathsf{L}} \varphi(\vec{p}, \vec{q}) \rightarrow \psi(\vec{q}, \vec{r})$$

there exists a formula $\chi(\vec{q})$ such that

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UL-extensions

We use a language with binary connectives $\cdot, \to, \wedge, \vee$ and constants $\bot, \top, \mathrm{f}, \mathrm{e}$, and define $\neg\varphi = \varphi \to \mathrm{f}$ and $\varphi \equiv \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Uninorm logic UL extends an axiomatization of MAILL (or $\mathrm{FL}_e^{\bot}$) with

$$\begin{array}{rl} \text{(PRL)} & (\varphi \to \psi) \vee (\psi \to \varphi) \\ \text{(DIS)} & (\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)). \end{array}$$

Some significant UL-extensions are

$$\begin{array}{rcl@{\qquad}rcl} \text{MTL} & = & \text{UL} + \text{(W)} & \text{G} & = & \text{MTL} + \text{(ID)} \\ \text{IMTL} & = & \text{MTL} + \text{(INV)} & \text{BL} & = & \text{MTL} + \text{(DIV)} \\ \text{UML} & = & \text{UL} + \text{(ID)} & \text{Ł} & = & \text{BL} + \text{(INV)} \\ \text{IUML} & = & \text{UML} + \text{(INV)} + \text{(F)} & \text{P} & = & \text{BL} + \text{(P)} \end{array}$$

where the extra axioms are

$$\begin{array}{rl@{\qquad}rl} \text{(INV)} & \neg\neg\varphi \to \varphi & \text{(W)} & (\varphi \to \mathrm{e}) \wedge (\mathrm{f} \to \varphi) \\ \text{(ID)} & \varphi \equiv (\varphi \cdot \varphi) & \text{(DIV)} & (\varphi \wedge \psi) \to (\varphi \cdot (\varphi \to \psi)) \\ \text{(F)} & \mathrm{e} \equiv \mathrm{f} & \text{(P)} & \neg\varphi \vee ((\varphi \to (\varphi \cdot \psi)) \to \psi). \end{array}$$

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Algebraic Semantics

Definition

A *bounded pointed commutative residuated lattice (BPCRL)* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, \mathrm{e}, \mathrm{f}, \bot, \top \rangle$ such that:

1. $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice;
2. $\langle A, \cdot, \mathrm{e} \rangle$ is a commutative monoid;
3. $z \leq x \to y$ iff $x \cdot z \leq y$ for all $x, y, z \in A$.

A UL-*algebra* satisfies additionally:

$$\mathrm{e} \leq (x \to y) \vee (y \to x) \quad \text{and} \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

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A Useful Lemma

Lemma

Let L *be any* UL*-extension such that either* $Ł_n$ *for some* $n \geq 3$ *or* P *is an* L*-extension. Then* L *does not have the* CIP*.*

Proof.

Observe first that:

$$\vdash_L (q \wedge (q \to p)) \to (r \vee (r \to p)).$$

If L has the CIP, then for some formula $\psi(p)$:

$$\vdash_L (q \wedge (q \to p)) \to \psi(p) \quad \text{and} \quad \vdash_L \psi(p) \to (r \vee (r \to p)).$$

Now suppose that either $Ł_n$ for some $n \geq 3$ or P is an L-extension. Then some $c \in (0, 1)$ is definable in $Ł_n$ for some $n \geq 3$ or a square root connective is definable in P, neither of which is possible. □

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Extensions of BL

Theorem (Baaz and Veith 1999, Montagna 2005)

The only consistent BL*-extensions with the* CIP *are* G, G_3*, and* CL*.*

Proof.

Let L be a consistent BL-extension with the CIP. Then L must be a G-extension; i.e., $\vdash_L p \to (p \cdot p)$. Otherwise, there is an L-extension complete with respect to a non-Boolean MV-chain or product chain, and L has $Ł_n$ for some $n \geq 3$ or P as an extension; by the previous lemma, this is not possible. Moreover, the only G-extensions with the CIP (see Baaz and Veith or the next lemma) are G, G_3, and CL. □

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Theorem

The only consistent MTL*-extensions whose product-free-fragments have the* CIP *are* G, G_3, *and* CL.

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Let L be a consistent MTL-extension whose product-free-fragment has the CIP. Since $\vdash_L (q \wedge (q \to p)) \to (r \vee (r \to p))$, there exists a formula $\psi(p)$ in the $\{\to, \wedge, \bot\}$-fragment ($\vee$, e, f, $\top$ are definable) such that

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But $\psi(\bot)$ is equivalent to $\top$ or $\bot$. In the first case, $\vdash_L r \vee \neg r$ so L is CL. In the second, $\vdash_L \neg(q \wedge \neg q)$, and (inductively) $\psi(p)$ is equivalent to

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A careful case analysis shows that L is G, G_3, or CL. □

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Let L be a consistent MTL-extension whose product-free-fragment has the CIP. Since $\vdash_L (q \wedge (q \to p)) \to (r \vee (r \to p))$, there exists a formula $\psi(p)$ in the $\{\to, \wedge, \bot\}$-fragment ($\vee$, e, f, $\top$ are definable) such that

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But $\psi(\bot)$ is equivalent to $\top$ or $\bot$. In the first case, $\vdash_L r \vee \neg r$ so L is CL. In the second, $\vdash_L \neg(q \wedge \neg q)$, and (inductively) $\psi(p)$ is equivalent to

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A careful case analysis shows that L is G, G_3, or CL. □

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Extensions of IMTL

Since $\varphi \cdot \psi$ is definable as $\neg(\varphi \to \neg\psi)$ in IMTL, we may also conclude:

Corollary

The only IMTL*-extension with the* CIP *is* CL*.*

For UL-extensions without weakening / integrality, our results are much less general…

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Extensions of IUML

Let us consider the logic

$$\text{IUML} = \text{UL} + \text{(INV)}\ \neg\neg\varphi \to \varphi + \text{(ID)}\ \varphi \equiv (\varphi \cdot \varphi) + \text{(F)}\ \mathrm{e} \equiv \mathrm{f}.$$

which is complete with respect to the BPCRL

$$\mathbf{A}_{\text{IUML}} = \langle [0,1], \min, \max, *, \to_*, \frac{1}{2}, \frac{1}{2}, 0, 1 \rangle$$

$$\text{where} \quad x * y = \begin{cases} \min(x,y) & \text{if } x + y \leq 1; \\ \max(x,y) & \text{otherwise.} \end{cases}$$

The only consistent extensions of IUML are the finite-valued logics IUML_{2n+1} ($n \in \mathbb{N}$) complete with respect to the BPCRLs

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Interpolation for IUML

Notice that treating formulas as truth functions over $\mathbf{A}_{\mathrm{IUML}}$:

$$\vdash_{\mathrm{IUML}} \varphi(\vec{p},\vec{r}) \to \psi(\vec{q},\vec{r}) \quad \text{iff} \quad \varphi(\vec{p},\vec{r}) \leq \psi(\vec{q},\vec{r}) \text{ for all } \vec{p},\vec{q},\vec{r} \in [0,1].$$

Moreover, the truth-function

$$T(\vec{r}) = \sup\{\varphi(\vec{p},\vec{r}) \mid \vec{p} \in [0,1]\}$$

is an interpolant in the sense that $\varphi(\vec{p},\vec{r}) \leq T(\vec{r}) \leq \psi(\vec{q},\vec{r})$.

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$\prec$-Chains

Consider the following formulas:

$$\begin{array}{rcl} p \prec q & = & (\mathrm{e} \wedge (p \to q)) \wedge (((p \to q) \wedge (q \to p)) \to \mathrm{e}) \\ p \approx q & = & \mathrm{e} \wedge (p \equiv q). \end{array}$$

Reasoning semantically:

$$p \prec q = \begin{cases} \mathrm{e} & \text{if } p < q; \\ (p \wedge q) \wedge (\neg p \wedge \neg q) & \text{otherwise.} \end{cases}$$

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A $\prec$-*chain* over variables $V = \{p_1, \ldots, p_n\}$ is a formula

$$(\bot \bowtie_0 p_{\pi(1)}) \wedge (p_{\pi(1)} \bowtie_1 p_{\pi(2)}) \wedge \cdots \wedge (p_{\pi(n-1)} \bowtie_{n-1} p_{\pi(n)}) \wedge (p_{\pi(n)} \bowtie_n \top)$$

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A Normal Form

Lemma

Let $C(V)$ be the set of all $\prec$-chains over $V = \{p_1, \ldots, p_n\}$:

(i) $\vdash_{\text{IUML}} \bigvee_{\chi \in C(V)} \chi$;

(iii) *Each* $\varphi(p_1, \ldots, p_n)$ *is equivalent in* IUML *to some formula:*

$$\bigvee_{\chi \in C(V)} (\chi \cdot \psi_\chi)$$

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For any formulas φ, ψ, χ such that p does not occur in χ:

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where $\exists p.\varphi(p, \vec{q}) = \sup\{\varphi(p, \vec{q}) \mid p \in [0, 1]\}$.

We can then eliminate quantifiers, e.g., from

$$\begin{aligned} &\exists p.(((a \prec p) \wedge (p \prec b)) \cdot p) \\ &\exists p.(((a \prec p) \wedge (p \prec b)) \cdot \neg p) \\ &\exists p.((a \prec p) \wedge (p \prec b)). \end{aligned}$$

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Let $\varphi^* = \varphi \wedge \mathrm{e}$ and consider for $n \geq 3$:

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Then $\vdash_{\text{IUML}_{2n+1}} (\psi_n \equiv p_1^*) \to ((q^* \to p_1^*) \vee q^*)$ and an interpolant is equivalent to p_1^*, e, $\bot$, $(\neg p_1)^*$, or $(p_1 \to \bot)^*$, none of which works. □

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Deductive Interpolation

A UL-extension L has the *Deductive Interpolation Property* DIP if whenever

$$\varphi(\vec{p}, \vec{q}) \vdash_{\mathsf{L}} \psi(\vec{q}, \vec{r})$$

there exists a formula $\chi(\vec{q})$ such that

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In the presence of the *deduction theorem*: DIP $\Leftrightarrow$ CIP.

What happens for extensions of $\mathsf{RM}^{\perp,\mathrm{e}} = \mathsf{UL} + (\mathsf{INV}) + (\mathsf{ID})$?

Here we have a simple *local deduction theorem*:

$$\Gamma \cup \{\varphi\} \vdash_{\mathsf{L}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathsf{L}} (\varphi \wedge \mathrm{e}) \to \psi.$$

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Theorem

An extension of $\mathrm{RM}^{\perp,\mathrm{e}}$ *has the* DIP *iff it has the* CIP*.*

Proof.

If L is an extension of $\mathrm{RM}^{\perp,\mathrm{e}}$, then

$$\vdash_{\mathrm{L}} \varphi(\vec{p},\vec{q}) \to \psi(\vec{q},\vec{r}) \quad \Rightarrow \quad \varphi(\vec{p},\vec{q}) \vdash_{\mathrm{L}} \psi(\vec{q},\vec{r}) \text{ and } \neg\psi(\vec{q},\vec{r}) \vdash_{\mathrm{L}} \neg\varphi(\vec{p},\vec{q}).$$

Moreover, if L has the DIP, then for some $\chi_1(\vec{q}), \chi_2(\vec{q})$:

$$\varphi(\vec{p},\vec{q}) \vdash_{\mathrm{L}} \chi_1(\vec{q}) \vdash_{\mathrm{L}} \psi(\vec{q},\vec{r}) \quad \text{and} \quad \neg\psi(\vec{q},\vec{r}) \vdash_{\mathrm{L}} \chi_2(\vec{q}) \vdash_{\mathrm{L}} \neg\varphi(\vec{p},\vec{q})$$

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Deductive Interpolation

Theorem

An extension of $\mathrm{RM}^{\perp,\mathrm{e}}$ *has the* DIP *iff it has the* CIP*.*

Proof.

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Concluding Remarks

Recall the following

Conjecture: a UL-extension with the CIP must at least be *idempotent*.

We have made a little progress, but...

...can we can prove the conjecture even for MTL-extensions?

...does UML = UL + (ID) have the CIP?

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References

M. Baaz and H. Veith.
Interpolation in fuzzy logic.
Archive for Mathematical Logic, 38:461–489, 1999.

H. Kihara and H. Ono.
Interpolation properties, Beth definability properties and amalgamation properties for substructural logics.
Journal of Logic and Computation, to appear.

E. Marchioni and G. Metcalfe.
Interpolation properties for uninorm based logics.
Proceedings of ISMVL 2010, 2010.

F. Montagna.
Interpolation and Beth's property in propositional many-valued logics: a semantic investigation.
Annals of Pure and Applied Logic, 141(1-2):148-179, 2006.