First-order Substructural Logics: An Algebraic Approach

Petr Cintula[1] Carles Noguera[2]

[1]Institute of Computer Science, Czech Academy of Sciences
Prague, Czech Republic

[2]Artificial Intelligence Research Institute (IIIA - CSIC)
Bellaterra, Catalonia

Outline

1. Goals and basic definitions
2. Axiomatization and completeness theorem
3. Completeness w.r.t. finitely subdirectly irreducible models
4. Skolemization

Goals and design choices

GOAL: study a general theory of (almost) classical quantification over propositional substural logics

Design Choices:

- keep traditional notion of language and formula
- define semantics by generalizing existing cases
  - Rasiowa-Sikorski style Intuitionistic predicate logic
  - Rasiowa implicative predicate logics
  - Gödel-Dummett predicate logic
  - predicate fuzzy logics (Hájek, Esteva, Godo, . . .)
- interpret quantifiers as sups and infs w.r.t. some order which is intrinsic for the logic in question (for a fixed implication $\Rightarrow$).
- study a general notion of predicate logic over some propositional logic (as it is not unique even in our restricted setting, unlike in the classical logic!)

Syntax

- Predicate language $\mathcal{P} = \langle \mathbf{P},\ \mathbf{F},\ \mathbf{ar} \rangle$

  $\mathbb{PL}$ is the system of all predicate languages
- Quantifiers: $\forall$ and $\exists$
- $\mathcal{P}$-terms, $\langle \mathcal{L}, \mathcal{P} \rangle$-atomic formulae, $\langle \mathcal{L}, \mathcal{P} \rangle$–formulae
- free and bound occurrences of variables in formulae, sentences,
- substitutability of a term into a formula

Predicate logic

Definition (Predicate logic)

An $\mathcal{L}$-predicate logic is a system $\mathfrak{P} = \left(\vdash_{\mathfrak{P}}^{\mathcal{P}}\right)_{\mathcal{P}\in\mathbb{PL}}$, where $\vdash_{\mathfrak{P}}^{\mathcal{P}}$ is a consequence relation on the set of all $\langle\mathcal{L},\mathcal{P}\rangle$-formulae, satisfying the following condition:

- *for each $\mathcal{P}' \subseteq \mathcal{P}$: $\vdash_{\mathfrak{P}}^{\mathcal{P}'}$ is the $\mathcal{P}'$-fragment of $\vdash_{\mathfrak{P}}^{\mathcal{P}}$.*
- *?*

$\mathcal{P}_{\varphi}$ denotes the smallest language such that φ is a $\mathcal{P}$-formula.

Let $\mathcal{P} \supseteq \mathcal{P}_{T\cup\{\psi\}}$. Then $T \vdash_{\mathfrak{P}}^{\mathcal{P}_{T\cup\{\psi\}}} \psi$ iff $T \vdash_{\mathfrak{P}}^{\mathcal{P}} \psi$.

Definition (Constants Theorem)

$\mathfrak{P}$ enjoys the *Constants Theorem* if for each $\mathcal{P}$-theory $T \cup \{\varphi(x,\vec{z})\}$ and any constant $c \notin \mathcal{P}$ such that $T \vdash_{\mathfrak{P}} \varphi(c,\vec{z})$, we have $T \vdash_{\mathfrak{P}} \varphi(x,\vec{z})$.

Semantics—interpretation

Definition (Interpretation)

An $\langle \mathcal{L}, \mathcal{P} \rangle$-*interpretation* $\mathbf{I}$ is a triple $\langle \mathbf{A}, \mathbf{S}, \leq \rangle$, where

- $\mathbf{A}$ is an $\mathcal{L}$-matrix,
- $\leq$ is a partial order on A, (not necessarily complete!)
- $\mathbf{S}$ is a tuple $\langle S, (P_{\mathbf{I}})_{P \in \mathbf{P}}, (f_{\mathbf{I}})_{f \in \mathbf{F}} \rangle$, where
  - S is a non-empty domain,
  - $f_{\mathbf{I}}$ is a function $S^n \to S$ for each $f \in \mathbf{F}$,
  - $P_{\mathbf{I}}$ is a mapping $S^n \to A$ for each $P \in \mathbf{P}$.

Definition (Evaluation)

Let $\mathbf{I} = \langle \mathbf{A}, \mathbf{S}, \leq \rangle$ be an interpretation. An $\mathbf{I}$-*evaluation* v is a mapping from the set of object variables into S.

Semantics—truth definition

Definition (Truth definition)

Let $\mathbf{I} = \langle \mathbf{A}, \mathbf{S}, \leq \rangle$ be an interpretation and v an $\mathbf{I}$-evaluation. We define *values* of the terms and *truth values* of the formulae in $\mathbf{I}$ for an evaluation v as:

$$
\begin{aligned}
\|x\|^{\mathbf{I}}_{\mathrm{v}} &= \mathrm{v}(x)\,, \\
\|f(t_1,\dots,t_n)\|^{\mathbf{I}}_{\mathrm{v}} &= f_{\mathbf{I}}(\|t_1\|^{\mathbf{I}}_{\mathrm{v}},\ \dots,\ \|t_n\|^{\mathbf{I}}_{\mathrm{v}})\,, && \text{for } f \in \mathbf{F} \\
\|P(t_1,\dots,t_n)\|^{\mathbf{I}}_{\mathrm{v}} &= P_{\mathbf{I}}(\|t_1\|^{\mathbf{I}}_{\mathrm{v}},\ \dots,\ \|t_n\|^{\mathbf{I}}_{\mathrm{v}})\,, && \text{for } P \in \mathbf{P} \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\mathbf{I}}_{\mathrm{v}} &= c_{\mathbf{A}}(\|\varphi_1\|^{\mathbf{I}}_{\mathrm{v}},\ \dots,\ \|\varphi_n\|^{\mathbf{I}}_{\mathrm{v}})\,, && \text{for } c \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\mathbf{I}}_{\mathrm{v}} &= \inf_{\leq}\{\|\varphi\|^{\mathbf{I}}_{\mathrm{v}[x\to a]} \mid a \in S\}\,, \\
\|(\exists x)\varphi\|^{\mathbf{I}}_{\mathrm{v}} &= \sup_{\leq}\{\|\varphi\|^{\mathbf{I}}_{\mathrm{v}[x\to a]} \mid a \in S\}\,.
\end{aligned}
$$

If an infimum or a supremum does not exist, we consider its value undefined. We say that a $\mathbf{I}$ is *safe* iff $\|\varphi\|^{\mathbf{I}}_{\mathrm{v}}$ is defined for each formula φ and each $\mathbf{I}$-evaluation v.

Semantics—model

Definition (Model)

An $\langle \mathcal{L}, \mathcal{P} \rangle$-interpretation $\mathbf{I} = \langle \mathbf{A}, \mathbf{S}, \leq \rangle$ is an $\langle \mathrm{L}, \mathcal{P} \rangle$-*model* of Σ if

- $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$,
- $\leq \, = \, \leq_{\mathbf{A}}$,
- $\mathbf{I}$ *is safe and* $\mathbf{I} \models \Sigma$.

Since in the $\langle \mathrm{L}, \mathcal{P} \rangle$-model $\langle \mathbf{A}, \mathbf{S}, \leq \rangle$ the order is determined by L and $\mathbf{A}$, we identify it with the pair $\langle \mathbf{A}, \mathbf{S} \rangle$.

Semantics—consequence

Definition (Semantical consequence)

Let $\mathbb{I}$ be a class of L-models and $T \cup \{\varphi\}$ a theory. $T \models_{\mathbb{I}} \varphi$ if $\mathbf{I} \models \varphi$ for each $\mathcal{P}_{T\cup\{\varphi\}}$-model $\mathbf{I} \in \mathbb{I}$ of T.

Definition

- An $\langle \mathrm{L}, \mathcal{P}\rangle$-model $\mathbf{I} \in \mathbb{I}(\mathfrak{P})$ iff $\vdash^{\mathcal{P}}_{\mathfrak{P}} \subseteq\ \models_{\mathbf{I}}$.
- A class $\mathbb{I}$ is *logical* if $\mathbb{I}\forall = \left\{ \models^{\mathcal{P}}_{\mathbb{I}} \right\}_{\mathcal{P}\in\mathbb{PL}}$ is a predicate logic.
- $\mathbb{M}\forall$ is the class of all L-models $\langle \mathbf{A}, \mathbf{S}\rangle$ where $\mathbf{A} \in \mathbb{M} \subseteq \mathbf{MOD}^*(\mathrm{L})$.

Theorem

$\mathbb{M}\forall$ *is a logical class of models and* $\mathbb{M}\forall$ *enjoys the Constants Theorem.*

Outline

Axiomatization

Definition (Minimal predicate logic)

The *minimal $\mathcal{L}$-predicate logic over* L, denoted as $\mathrm{L}\forall^m$, is defined in each of its components $\vdash^{\mathcal{P}}_{\mathrm{L}\forall^m}$ as:

- (P) the rules resulting from the consecutions of L by the substitution of the propositional variables by $\langle\mathcal{L},\mathcal{P}\rangle$-formulae,
- $(\forall 0)$ $\varphi(x,\vec{z}) \vdash^{\mathcal{P}}_{\mathrm{L}\forall^m} \varphi(t,\vec{z})$, where t is substitutable for x in φ,
- $(\forall 1)$ $\vdash^{\mathcal{P}}_{\mathrm{L}\forall^m} (\forall x)\varphi(x,\vec{z}) \Rightarrow \varphi(t,\vec{z})$, where t is substitutable for x in φ,
- $(\exists 1)$ $\vdash^{\mathcal{P}}_{\mathrm{L}\forall^m} \varphi(t,\vec{z}) \Rightarrow (\exists x)\varphi(x,\vec{z})$, where t is substitutable for x in φ,
- $(\forall 2)$ $\chi \Rightarrow \varphi \vdash^{\mathcal{P}}_{\mathrm{L}\forall^m} \chi \Rightarrow (\forall x)\varphi$, where x is not free in χ,
- $(\exists 2)$ $\varphi \Rightarrow \chi \vdash^{\mathcal{P}}_{\mathrm{L}\forall^m} (\exists x)\varphi \Rightarrow \chi$, where x is not free in χ.

A predicate logic extending $\mathrm{L}\forall^m$ is called a *predicate logic over* L.

Properties of the minimal predicate logic

Theorem

Let $\Rightarrow$ *be an implication with a unit* $\mathbf{1}$. *Then:*

$$(\forall 0)' \quad \varphi \vdash_{\mathrm{L}\forall^m} (\forall x)\varphi.$$

Lemma

Any predicate logic over L *proves:*

$(T1)$	$\varphi \Rightarrow \psi \vdash (\forall x)\varphi \Rightarrow (\forall x)\psi$,	
$(T2)$	$\varphi \Rightarrow \psi \vdash (\exists x)\varphi \Rightarrow (\exists x)\psi$,	
$(T3)$	$\vdash \varphi \leftrightarrow (\forall x)\varphi$	*if* x *is not free in* φ,
$(T4)$	$\vdash (\exists x)\varphi \leftrightarrow \varphi$	*if* x *is not free in* φ,
$(T5)$	$\vdash (\forall x)\varphi(x, \vec{z}) \leftrightarrow (\forall x')\varphi(x', \vec{z})$	*if* x *and* x' *do not occur in* $\varphi(y, \vec{z})$,
$(T6)$	$\vdash (\exists x)\varphi(x, \vec{z}) \leftrightarrow (\exists x')\varphi(x', \vec{z})$	*if* x *and* x' *do not occur in* $\varphi(y, \vec{z})$,
$(T7)$	$\vdash (\forall x)(\forall y)\varphi \leftrightarrow (\forall y)(\forall x)\varphi$,	
$(T8)$	$\vdash (\exists x)(\exists y)\varphi \leftrightarrow (\exists y)(\exists x)\varphi$.	

Properties of the minimal predicate logic (cont.)

Corollary (Substitution rule)

Let φ, ψ, χ *be formulae,* φ *a subformula of* χ *and* $\mathfrak{P}$ *a predicate logic over* L*. Then*

$$\varphi \leftrightarrow \psi \vdash_{\mathfrak{P}} \chi \leftrightarrow \chi'$$

where χ' *is obtained from* χ *by replacing some occurrences of* φ *by* ψ*.*

Theorem (Constants Theorem)

$\mathrm{L}\forall^m$ *enjoys the Constants Theorem.*

Soundness

Theorem

$\mathbf{MOD}^*(\mathrm{L})\forall$ *is a predicate logic over* L.

Corollary

Let $\mathbb{I}$ *be a logical class of* L*-models. Then* $\mathbb{I}\forall$ *is a predicate logic over* L.

Corollary

Let $\mathbb{M} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Then* $\mathbb{M}\forall$ *is a predicate logic over* L.

Completeness

Theorem (Completeness for $\mathfrak{P}$)

Let $\mathfrak{P}$ be a predicate logic over L enjoying the Constants Theorem. Then $\mathfrak{P} = \mathbb{I}(\mathfrak{P})\forall$.

Corollary (Completeness for $\mathrm{L}\forall^m$)

The following are equivalent for any predicate language $\mathcal{P}$ and $\mathcal{P}$-formulae $T \cup \{\varphi\}$:

- $T \vdash^{\mathcal{P}}_{\mathrm{L}\forall^m} \varphi$.
- $\mathbf{I} \models \varphi$ *for each* $\langle \mathrm{L}, \mathcal{P}\rangle$*-model* $\mathbf{I}$ *of* T.

More syntax

Let us fix a predicate logic $\mathfrak{P}$.

Definition ($\forall$-Henkin)

A $\mathcal{P}$-theory T is *$\forall$-Henkin* in $\mathcal{P}$ if for each $\mathcal{P}$-formula ψ such that $T \nvdash_{\mathfrak{P}} \psi(x)$ there is an object constant c in $\mathcal{P}$ such that $T \nvdash_{\mathfrak{P}} \psi(c)$.

Lemma (1st Fundamental lemma)

If $\mathfrak{P}$ enjoys the Constants Theorem, then any $\mathcal{P}$-theory T such that $T \nvdash_{\mathfrak{P}} \varphi$ can be expanded into a $\forall$-Henkin $\mathcal{P}'$-theory $T' \supseteq T$ such that $T' \nvdash_{\mathfrak{P}} \varphi$.

Completeness—a hint of (the usual) proof

Let $\mathfrak{P}$ a predicate logic over L and T a $\mathcal{P}$-theory.

$$[\varphi]_T = \{\psi \mid \psi \text{ a } \mathcal{P}\text{-sentence and } T \vdash^{\mathcal{P}}_{\mathfrak{P}} \varphi \leftrightarrow \psi\}$$

Definition

The Lindenbaum matrix of T ($\mathbf{Lind}^{\mathfrak{P}}_T$) has the domain

$$L_T = \{[\varphi]_T \mid \varphi \text{ a } \mathcal{P}\text{-sentence}\},$$

operations

$$c_{\mathbf{Lind}^{\mathfrak{P}}_T}([\varphi_1]_T, \dots, [\varphi_n]_T) = [c(\varphi_1, \dots, \varphi_n)]_T$$

and the filter

$$F = \{[\varphi]_T \mid \varphi \text{ a } \mathcal{P}\text{-sentence and } T \vdash^{\mathcal{P}}_{\mathfrak{P}} \varphi\}.$$

1. $[\varphi]_T \leq_{\mathbf{Lind}^{\mathfrak{P}}_T} [\psi]_T$ iff $T \vdash^{\mathcal{P}}_{\mathfrak{P}} \varphi \Rightarrow \psi$
2. $\mathbf{Lind}^{\mathfrak{P}}_T \in \mathbf{MOD}^*(\mathrm{L})$.

Completeness—a hint of (the usual) proof

Let $\mathfrak{P}$ be a predicate logic over L and T a $\forall$-Henkin $\mathcal{P}$-theory:

Lemma

- $[(\forall x)\varphi(x)]_T = \inf_{\leq_{\mathbf{Lind}_T^{\mathfrak{P}}}} \{[\varphi(c)]_T \mid c \in \mathbf{C}_{\mathcal{P}}\}$.
- $[(\exists x)\varphi(x)]_T = \sup_{\leq_{\mathbf{Lind}_T^{\mathfrak{P}}}} \{[\varphi(c)]_T \mid c \in \mathbf{C}_{\mathcal{P}}\}$.

$\mathbf{C}_{\mathcal{P}}$ denotes the set of all closed $\mathcal{P}$-terms.

Definition (Canonical model)

The *canonical* $\langle \mathrm{L}, \mathcal{P}\rangle$*-model* of T over $\mathfrak{P}$, denoted by $\mathbf{CM}^{\mathfrak{P}}(T)$, is an interpretation $\langle \mathbf{Lind}_T^{\mathfrak{P}}, \mathbf{S}, \leq_{\mathbf{Lind}_T^{\mathfrak{P}}}\rangle$ s.t. the domain of $\mathbf{S}$ is $\mathbf{C}_{\mathcal{P}}$ and

- $f_{\mathbf{I}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ *for each* $f \in \mathcal{P}$,
- $P_{\mathbf{I}}(t_1, \ldots, t_n) = [P(t_1, \ldots, t_n)]_T$ *for each* $P \in \mathcal{P}$.

Completeness—a hint of (the usual) proof

Theorem

Let $\mathfrak{P}$ be a predicate logic over L and T a $\forall$-Henkin $\mathcal{P}$-theory. Then for each $\mathcal{P}$-formula φ and $\vec{c} \in \mathbf{C}_{\mathcal{P}}$ we have:

1. $\|\varphi(\vec{c})\|^{\mathbf{CM}^{\mathfrak{P}}(T)} = [\varphi(\vec{c})]_T$.
2. $\mathbf{CM}^{\mathfrak{P}}(T) \models \varphi[\vec{c}]$ *iff* $T \vdash_{\mathfrak{P}} \varphi(\vec{c})$.

Thus $\mathbf{CM}^{\mathfrak{P}}(T)$ is an $\langle \mathrm{L}, \mathcal{P} \rangle$-model of T.

Outline

Finitely subdirectly irreducible models

Recall: $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ is the set of all relatively finitely subdirectly irreducible reduced models of L and

$$\mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$$

Question: for which predicate logic $\mathfrak{P}$ over L holds:

$$\mathfrak{P} = \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}\forall \quad ?$$

It is non-trivial: consider Gödel-Dummett propositional logic G. Then

$$\mathrm{G}\forall^m \subsetneq \mathbf{MOD}^*(\mathrm{G})_{\mathrm{RFSI}}\forall = \mathbf{MOD}^\ell(\mathrm{G})\forall$$

Counterexample is e.g. the constant domains axioms:

$$(\forall x)(\varphi \vee \nu) \to (\forall x)\varphi \vee \nu$$

We would need (PCP)

Let us fix a p-disjunctional logic L with a p-disjunction ∇ and a predicate logic $\mathfrak{P}$ over L with presentations $(\mathcal{AS}_\mathcal{P})_{\mathcal{P}\in\mathbb{PL}}$.

$\mathfrak{P}$ has the *Proof by Cases Property* if:

$$\frac{T, \varphi \vdash_{\mathfrak{P}} \chi \qquad T, \psi \vdash_{\mathfrak{P}} \chi}{T, \varphi\nabla\psi \vdash_{\mathfrak{P}} \chi}$$

By $(\Gamma \rhd \varphi)^{\nabla}_{\mathcal{P}}$ we denote $\{\Gamma\nabla\chi \rhd \delta \mid \chi$ a $\mathcal{P}$-sentence and $\delta \in \varphi\nabla\chi\}$.

Theorem

$\mathfrak{P}$ enjoys (PCP) iff $R^{\nabla}_{\mathcal{P}} \subseteq \vdash_{\mathfrak{P}}$ for each $\mathcal{P} \in \mathbb{PL}$ and $R \in \mathcal{AS}_\mathcal{P}$.
The weakest predicate logic extending $\mathfrak{P}$ which enjoys (PCP) ($\mathfrak{P}^{\nabla}$) is axiomatized by $(\mathcal{AS}_\mathcal{P} \cup \{R^{\nabla}_{\mathcal{P}} \mid R \in \mathcal{AS}_\mathcal{P}\})_{\mathcal{P}\in\mathbb{PL}}$.

Notice: $(\varphi \vdash (\forall x)\varphi)^{\vee}$ is $\varphi \vee \psi \vdash (\forall x)\varphi \vee \psi$ and so $\mathrm{G}\forall^{m\vee} = \mathrm{G}\forall$.

Why do we need (PCP)?

Theorem

Let $\mathfrak{P}$ be a predicate logic enjoying (PCP) *and Constants Theorem. Let* T *be a* $\mathcal{P}$*-theory and* ψ *a formula such that* $T \nvdash_{\mathfrak{P}}^{\mathcal{P}} \psi$*. Then there is* $\mathcal{P}' \supseteq \mathcal{P}$ *and a* ∇*-prime* $\forall$*-Henkin* $\mathcal{P}'$*-theory* $T' \supseteq T$ *and* $T' \nvdash_{\mathfrak{P}}^{\mathcal{P}'} \psi$*.*

Corollary

Let $\mathfrak{P}$ be a predicate logic over L *with* (PCP) *and Constants Theorem such that* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}\forall \subseteq \mathbb{I}(\mathfrak{P})$*. Then* $\mathfrak{P} = \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}\forall$*.*

Corollary

Let L *be a p-disjunctional logic s.t.* $\mathrm{L}\forall^{m\nabla}$ *has Constants Theorem. Then*

$$\mathrm{L}\forall^{m\nabla} = \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}\forall.$$

Hint of proof

In Bool: if $T \nvdash \varphi$ iff $T, \neg\varphi \nvdash \bot$. Thus for any consistent extension $S \supseteq T, \neg\varphi$ also $S \nvdash \varphi$. (deduction theorem and double negation)

Motto: Life is hard in non-classical logics.

Let Ψ be a set of theories and T a theory. We write $T \not\Vdash \Psi$ whenever $T \nvdash S$ for each $S \in \Psi$.

A set of theories Ψ is *deductively directed* if for each $T, S \in \Psi$ there is $R \in \Psi$ s.t. $T \vdash_{\mathfrak{P}} R$ and $S \vdash_{\mathfrak{P}} R$.

Definition (Restricted Henkin theory)

Take $\mathcal{P} \subseteq \mathcal{P}'$. A $\mathcal{P}'$-theory T is: $\mathcal{P}$-$\forall$-*Henkin* in $\mathcal{P}'$ if for each $\mathcal{P}$-formula $\varphi(x)$ such that $T \nvdash_{\mathfrak{P}}^{\mathcal{P}'} \varphi(x)$ there is a constant $c \in \mathcal{P}'$ such that $T \nvdash_{\mathfrak{P}}^{\mathcal{P}'} \varphi(c)$.

Fundamental Lemma

Lemma (Fundamental Lemma)

Let $\mathfrak{P}$ be a finitary predicate logic enjoying Constants Theorem and (PCP), *$\mathcal{P}$ a predicate language, T a $\mathcal{P}$-theory, and Ψ a deductively directed set of closed $\mathcal{P}$-theories such that $T \nvdash_{\mathfrak{P}} \Psi$. Then:*

1. *there is a predicate language $\mathcal{P}' \supseteq \mathcal{P}$, a $\mathcal{P}'$-theory $T' \supseteq T$, and a deductively directed set of closed $\mathcal{P}'$-theories $\Psi' \supseteq \Psi$, such that*
   - $T' \nvdash_{\mathfrak{P}} \Psi'$,
   - *for each theory $S \supseteq T'$ in arbitrary language if $S \nvdash_{\mathfrak{P}} \Psi'$, then S is $\mathcal{P}$-$\forall$-Henkin.*
2. *there is a ∇-prime $\mathcal{P}$-theory $T' \supseteq T$ such that $T' \nvdash_{\mathfrak{P}} \Psi$.*

Outline

$\exists$-Henkin theories and preSkolem logics

Definition ($\exists$-Henkin theory)

Let $\mathcal{P} \subseteq \mathcal{P}'$ be predicate languages and $\mathfrak{P}$ a predicate logic. We say that a $\mathcal{P}'$-theory T is:

- *$\mathcal{P}$-$\exists$-Henkin* in $\mathcal{P}'$ if for each $\mathcal{P}$-formula $\varphi(x)$ s.t. $T \vdash_{\mathfrak{P}} (\exists x)\varphi(x)$ there is a constant $c \in \mathcal{P}'$ and $T \vdash_{\mathfrak{P}} \psi(c)$.
- *Henkin* in $\mathcal{P}'$ if it is $\mathcal{P}$-$\exists$-Henkin and $\forall$-Henkin in $\mathcal{P}'$.

Definition

Let $\mathfrak{P}$ be a predicate logic over L. We say that $\mathfrak{P}$ is preSkolem if $T \cup \{\varphi(c)\}$ is a conservative expansion of $T \cup \{(\exists x)\varphi(x)\}$ for each language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x)$ and any constant $c \notin \mathcal{P}$.

Predicate logics $\mathrm{MTL}\forall$ or $\mathrm{Int}\forall$ are preSkolem.

Fundamental Lemma for $\exists$-Henkin theories

Lemma (Fundamental Lemma)

Let $\mathfrak{P}$ a preSkolem predicate logic, T a $\mathcal{P}$-theory, and Ψ a deductively directed set of closed $\mathcal{P}$-theories such that $T \nVdash_{\mathfrak{P}}^{\mathcal{P}} \Psi$. Then there is $\mathcal{P}' \supseteq \mathcal{P}$ and a $\mathcal{P}'$-theory $T' \supseteq T$ such that

- $T' \nVdash_{\mathfrak{P}} \Psi$,
- *for each theory $S \supseteq T'$ in arbitrary language if $S \nVdash_{\mathfrak{P}} \Psi$, then S is $\mathcal{P}$-$\exists$-Henkin.*

Theorem

Let $\mathfrak{P}$ be a preSkolem predicate logic enjoying (PCP) and Constants Theorem, T a $\mathcal{P}$-theory, and Ψ a deductively directed set of closed $\mathcal{P}$-theories such that $T \nVdash_{\mathfrak{P}}^{\mathcal{P}} \Psi$. Then there is $\mathcal{P}' \supseteq \mathcal{P}$ and a ∇-prime Henkin $\mathcal{P}'$-theory $T' \supseteq T$ and $T' \nVdash_{\mathfrak{P}} \Psi$.

Skolemization

Theorem

Let $\mathfrak{P}$ be a predicate logic over L enjoying (PCP) and Constants Theorem. Then:

1. *$\mathfrak{P}$ is preSkolem.*
2. *For each $\mathcal{P}$-theory T, φ such that $T \nvdash_{\mathfrak{P}} \varphi$ there is $\mathcal{P}' \supseteq \mathcal{P}$ and a Henkin $\mathcal{P}'$-theory $T' \supseteq T$ and $T' \nvdash_{\mathfrak{P}} \varphi$.*
3. *$T \cup \{(\forall \vec{y})\varphi(f_\varphi(\vec{y}), \vec{y})\}$ is a conservative expansion of $T \cup \{(\forall \vec{y})(\exists x)\varphi(x, \vec{y})\}$ for each language $\mathcal{P}$, each $\mathcal{P}$-theory T, each $\mathcal{P}$-formula $\varphi(x, \vec{y})$ and any functional symbol $f_\varphi \notin \mathcal{P}$ of a proper arity.*

Thank you for your attention!