Regular completions of residuated lattices

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Algebraic semantics

We will discuss completions of algebras in relation to algebraic completeness of substructural predicate logics (QSLs).

In our algebraic semantics, quantifiers $\forall$ and $\exists$ will be interpreted by infinite meets and joins, resp., following Mostowski, Rasiowa and Sikorski etc.

A structure $\langle \mathbf{A}, V \rangle$ is an *algebraic frame* for QSLs when $\mathbf{A}$ is a complete residuated lattices and V a non-empty set, called the individual domain. In each valuation f, quantifiers are interpreted as:

- $f(\forall x \varphi(x)) = \bigwedge \{f(\varphi(w)) : w \in V\}$
- $f(\exists x \varphi(x)) = \bigvee \{f(\varphi(w)) : w \in V\}$

Here, algebras are complete when the join $\bigwedge X$ and the meet $\bigvee X$ exist always for each subset X.

There are obvious limitations of this algebraic semantics. For instance,

Proposition (HO 1973)

There exist uncountably many algebraically incomplete, superintuitionistic predicate logics.

Still, it will be interesting and worth while

- to study how much algebraic notions and methods will provide us understanding of logical properties, including those of *quantifiers*,
- to understand meanings of results on predicate logics in terms of algebra.

This study has originated from our joint work with Majid Alizadeh on Visser's basic algebras.

♣ M. Alizadeh, "Completions of Basic algebras", LNAI 5514, 2009.

Residuated lattices

For the simplicity's sake, we will consider commutative case. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \rightarrow, 1 \rangle$ is a *commutative residuated lattice* (CRL), iff it satisfies the following:

- $\langle A, \vee, \wedge \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a commutative monoid,
- for all $x, y, z \in A$, $x \cdot y \leq z$ iff $y \leq x \rightarrow z$.

We say that $(\cdot, \rightarrow)$ forms a *residuated pair*.

An $\mathbf{FL_e}$-*algebra* is a CRL with a fixed element 0. Using 0, we can define the *negation* $-$ by $-x = x \to 0$.

A *QSL* **L** is *complete* with respect to a class $\mathcal{K}$ of complete $\mathbf{FL_e}$-algebras, when for each formula φ, $\varphi \in \mathbf{L}$ iff

$\mathbf{A} \models f(\varphi) \geq 1$ for every $\mathbf{A} \in \mathcal{K}$ and every valuation f.

Completions

A *completion* of a given CRL ($\mathbf{FL_e}$-algebra) $\mathbf{A}$ is a pair $(\mathbf{C}, h)$ of a complete CRL ($\mathbf{FL_e}$-algebra) $\mathbf{C}$ and an embedding h from $\mathbf{A}$ to $\mathbf{C}$.

Note: Often we omit h and say simply that $\mathbf{C}$ is a completion of an algebra $\mathbf{A}$, whenever h is clear from the context.

Regular embeddings

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Notes:

1. Embeddings must be regular to show the algebraic completeness of predicate logics, since *interpretations of quantifiers* must be preserved.
2. Canonical extensions are never regular.

Nuclei

Let $\mathbf{P}$ be a (commutative) p.o. monoid. An operation E on $\wp(\mathbf{P})$ is a *nucleus* iff it is a closure operator such that $E(X) \cdot E(Y) \subseteq E(X \cdot Y)$. Here, $U \cdot V = \{u \cdot v : u \in U \text{ and } v \in V\}$.

Let $\wp(P)_E$ be the set of all *E-closed* subsets X of P, i.e. subsets X such that $E(X) = X$. Define $\mathbf{P}^E = \langle \wp(P)_E, \vee, \wedge, \circ, \Rightarrow \rangle$, where for all $X, Y \in \wp(P)$

- $X \vee_E Y = E(X \cup Y)$ and $X \wedge Y = X \cap Y$
- $X \circ_E Y = E(X \cdot Y)$
- $X \Rightarrow Y = \{z : z \cdot x \in Y \text{ for all } x \in X\}$.

Then, $\mathbf{P}^E$ forms a *complete* CRL.

The monoid operation with the partial order of a given CRL forms a p.o. monoid. Hereafter we will identify a CRL with its p.o.monoid-reduct.

MacNeille completions

For each subset X of a CRL $\mathbf{A}$, $U(X)$ ($L(X)$) denotes the set of all upper bounds (lower bounds, resp.) of X. Define $M(X) = L(U(X))$.

Then, M is a nucleus, and hence $\mathbf{A}^M$ is a complete CRL. Moreover, the mapping h defined by $h(a) = (a] = \{x : x \leq a\}$, called the *principal mapping*, is a regular (RL-) embedding from $\mathbf{A}$ to $\mathbf{A}^M$.

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The complete CRL $\mathbf{A}^M$ (with the mapping h) is called the *MacNeille completion* of a CRL $\mathbf{A}$.

Distributivity

There are many interesting substructural logics with the distributive law, like relevant logics, many-valued logics and fuzzy logics. On the other hand, the distributive law is not always preserved under MacNeille completions (cf. Funayama 1944).

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There are many interesting substructural logics with the distributive law, like relevant logics, many-valued logics and fuzzy logics. On the other hand, the distributive law is not always preserved under MacNeille completions (cf. Funayama 1944).

Then, how can we show the algebraic completeness of predicate logics with the distributive law?

Consider the case for intuitionistic predicate logic **QInt**. MacNeille completions of Heyting algebras are always Heyting algebras, and thus are distributive. Therefore, we get the algebraic completeness of **QInt** (cf. Rasiowa 1951).

But, why MacNeille completions can preserve always the distributivity in this case?

Because $(\wedge, \rightarrow)$ forms a residuated pair, i.e. $\rightarrow$ is a Heyting implication.

The law of residuation is preserved under MacNeille completions. In particular, in the present case, Heyting implication is preserverd.

Join infinite distributivity

For any algebra with a Heyting implication, a stronger form of the distributivity, i.e. the join infinite distributivity (JID) holds, where the (JID) (or $(\wedge, \bigvee)$-Dis) is:

$$\text{(JID)} : \; \bigvee_i a_i \wedge b = \bigvee_i (a_i \wedge b)$$

which means that if $\bigvee_i a_i$ exists then $\bigvee_i (a_i \wedge b)$ exists also and the equality holds.

Note that conversely when the (JID) holds in a *complete* CRL, a Heyting implication $\rightsquigarrow$ can be defined by

$$a \rightsquigarrow b = \bigvee\{c : a \wedge c \leq b\}.$$

This observation led us to show the algebraic completeness of the relevant predicate logic $\mathbf{QDFL}_e$, an extension of $\mathbf{QFL_e}$ with the distributive axiom and $(\wedge, \exists)$: $\exists x\varphi(x) \wedge \psi \rightarrow \exists x(\varphi(x) \wedge \psi)$.

♣ HO, "Algebraic semantics for predicate logics and their completeness", in: Logic at Work,1999.

Outline

1. Add a *Heyting implication* $\rightsquigarrow$ to the language of $\mathbf{QDFL}_e$, and its axioms which says that $\wedge$ and $\rightsquigarrow$ forms a residuated pair. Call this logic $(\mathbf{QDFL}_e)^{\rightsquigarrow}$.

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2. Let $\mathbf{A}^M$ be the MacNeille completion of Lindenbaum algebra $\mathbf{A}$ of $(\mathbf{QDFL}_e)^{\rightsquigarrow}$ and let $\mathbf{A}^*$ be its $\rightsquigarrow$-less reduct. Then $\mathbf{A}^*$ is a complete $\mathbf{FL_e}$-algebra with the (JID), and thus it validates all axioms of $(\mathbf{QDFL}_e)^{\rightsquigarrow}$.

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3. To complete our proof, it remains to show moreover that $(\mathbf{QDFL}_e)^{\rightsquigarrow}$ is a *conservative extension* of $\mathbf{QDFL}_e$. This is done in a proof-theoretic way, using cut elimination.

Consevativity of Heyting implication

For related topics,

♣ M. Theunissen and Y. Venema, "MacNeille completions of lattice expansions", Algebra Universalis, 2007.

♣ R. Goldblatt, "Conservativity of Heyting implication over relevant quantification", Review of Symbolic Logic 2, 2009.

Now, we will search for regular completions which preserve (infinite) distributivity. Actually, complete ideal completions work well, as they preserve the (JID).

Thus, we can show the algebraic completeness of QSLs with the distributive law and the axiom $(\wedge, \exists)$.

Ideal completions

A nonempty subset X of a CRL $\mathbf{A}$ is an *ideal* of $\mathbf{A}$ iff

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- If $a, b \in X$ then $a \vee b \in X$.

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For each non-empty subset Y of $\mathbf{A}$, let $J(Y)$ be the smallest ideal including Y. It is easy to see that J is a nucleus, and therefore $\mathbf{A}^J$ is a complete CRL. Moreover, the principal mapping h is an embedding from $\mathbf{A}$ to $\mathbf{A}^J$.

We call $\mathbf{A}^J$, the *ideal completion* of $\mathbf{A}$.

Join infinite distributivity

The mapping h in the ideal completion is not always regular though existing infinite meets are preserved. On the other hand, the distributivity is preserved under ideal completion. In fact, we have the following (see e.g. Birkhoff).

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Proposition

If the distributive law holds in a CRL then the join infinite distributivity (JID) holds in its ideal completion.

$$\text{(JID)}: \ \bigvee_i a_i \wedge b = \bigvee_i (a_i \wedge b)$$

Note

- Preservation under ideal completions has been discussed in G. Birkhoff 1948 for lattice identities, and also in A. García-Cerdaña 2008, and in R. Goldblatt 2009 in the setting of substructural propositional logics.

Complete ideal completions

A nonempty subset X of a CRL $\mathbf{A}$ is an *complete ideal* of $\mathbf{A}$ iff

- X is downward closed
- If $a_j \in X$ for each J and moreover $\bigvee_i a_i$ exists, then $\bigvee_i a_i \in X$

Complete ideal completions

A nonempty subset X of a CRL **A** is an *complete ideal* of **A** iff

- X is downward closed
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For each non-empty subset Y of **A**, let $K(Y)$ be the smallest ideal including Y. We can show that K is a nucleus, and therefore $\mathbf{A}^K$ is a complete CRL.

Moreover, the principal mapping h is a *regular* embedding from $\mathbf{A}$ to $\mathbf{A}^K$.

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♣ P. Crawley, "Regular embeddings which preserve lattice structure", Proc. Amer. Math. Soc. 13, 1962.

♣ W.H. Cornish, "Crawley's completion of a conditionally upper continuous lattice", Pacific Journal of Math., 51, 1974.

Explicit representation

For each non-empty subset Y of $\mathbf{A}$, let $K_0(Y) =$

$\{z : z \leq \bigvee_i y_i$ for existing $\bigvee_i y_i$ such that $y_i \in Y\downarrow$ for each $i\}$.

Explicit representation

For each non-empty subset Y of **A**, let $K_0(Y) =$

$$\{z : z \leq \bigvee_i y_i \text{ for existing } \bigvee_i y_i \text{ such that } y_i \in Y\downarrow \text{ for each } i\}.$$

Then,

- In general, $K_0(Y) \subseteq K(Y)$,
- the (JID) implies $K_0(Y) = K(Y)$.

Hence, an explicit representation of K is given by K_0, when the (JID) holds.

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Theorem

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Theorem

For any CRL $\mathbf{A}$*, the (JID) holds in the MacNeille completion* $\mathbf{A}^M$ $\Rightarrow$ *the (JID) holds in* $\mathbf{A}$.

MacNeille vs complete ideal

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It is easy to see that every normal ideal, i.e. every M-closed subset, is a complete ideal. The converse holds when the (JID) holds in the MacNeille completion $\mathbf{A}^M$ (cf. Cornish 1974 for lattices). In fact,

Theorem

For a CRL $\mathbf{A}$, the following are equivalent:

- *The (JID) holds in the MacNeille completion $\mathbf{A}^M$.*
- *The (JID) holds in $\mathbf{A}$, and every complete ideal of $\mathbf{A}$ is normal (i.e. $\mathbf{A}^M = \mathbf{A}^K$).*

As noted before, a Heyting implication can be preserved under MacNeille completions. Thus, the following well-known result is obtained as a corollary.

Corollary

If $\mathbf{A}$ *is a CRL with a Heyting implication, then* $\mathbf{A}^M = \mathbf{A}^K$.

Algebraic completeness

Algebraic completeness of predicate logics with the distributive law and the axiom $(\wedge, \exists)$ can be shown by using complete ideal completions. Note that

$(\wedge, \exists)$: $\exists x\varphi(x) \wedge \psi \rightarrow \exists x(\varphi(x) \wedge \psi)$ *expresses the (JID).*

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In fact, by using complete ideal completions, we have:

- algebraic completeness of **QIL** (see e.g. a proof in the book by Troelstra-van Dalen).

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$(\wedge, \exists)$: $\exists x \varphi(x) \wedge \psi \to \exists x(\varphi(x) \wedge \psi)$ *expresses the (JID).*

In fact, by using complete ideal completions, we have:

- algebraic completeness of **QIL** (see e.g. a proof in the book by Troelstra-van Dalen).
- a direct proof of algebraic completeness of relevant predicate logic $\mathbf{QDFL}_e$

Heyting implication

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By combining our previous results, we can give a sufficient condition for the conservativity of Heyting implication over a logic **L**. That is,

- the schema $(\wedge, \exists)$ holds in **L**,
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Relevant predicate logics **MQX** in Goldblatt (2009) are examples of these logics if we add moreover the axiom $(\wedge, \exists)$ to them.