State operators on bounded residuated ℓ-monoids

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AsubL4

Nomi, June 8-10, 2010

A state (= an analogue of a probabilistic measure) on an MV-algebra (Mundici 1995, Kopka and Chovanec 1994), a mapping into $[0,1] \subseteq \mathbb{R}$.
States on BL-algebras, pseudo BL-algebras and bounded $R\ell$-monoids (Riečan 2000, Georgescu 2004, Dvurečenskij and J. R. 2006).
Further, Kühr and Mundici 2007, Kroupa 2006, Panti 2008, Riečan and Mundici 2002).
States ≡ averaging process for formulas in many valued logics.

Flaminio and Montagna 2007, 2009 - another approach to states on MV-algebras: a unary operation, an internal state or a state operator.
State MV-algebras = MV-algebras + state operators.
Di Nola and Dvurečenskij 2009, state-morphism MV-algebras, stronger variations of state MV-algebras.
J. R. and Šalounová 2010, state and state-morphism GMV-algebras.
Ciungu, Dvurečenskij and Hyčko 2010, analogous operators on BL-algebras.

Here: General state and state-morphism bounded $R\ell$-monoids.

Definition

A *bounded $R\ell$-monoid* is an algebra $M = (M;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0, \rangle$ satisfying:

(i) $(M;\ \odot, 1)$ is a monoid (need not be commutative).

(ii) $(M;\ \vee, \wedge, 0, 1)$ is a bounded lattice.

(iii) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x, y \in M$.

(iv) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$.

$R\ell$-monoid = bounded $R\ell$-monoid.

$$x^- := x \to 0, \qquad x^\sim := x \rightsquigarrow 0.$$

An $R\ell$-monoid M is called

good if $x^{-\sim} = x^{\sim -}$

normal if $(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}$ and $(x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}$.

If M is good, then put $x \oplus y := (y^\sim \odot x^\sim)^-$ $(= (y^- \odot x^-)^\sim)$.

$R\ell$-monoids = bounded non-commutative residuated lattices satisfying the identities of divisibility ≡ bounded integral generalized BL-algebras ≡ FL_w-algebras satisfying the identities of divisibility ≡ bounded divisible pseudo-$BCK(pP)$ lattices

An $R\ell$-monoid M is

a. a pseudo-BL-algebra if and only if M satisfies the identities of pre-linearity
$(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;
b. equivalent to a GMV-algebra (pseudo-MV-algebra) if and only if M fulfils the identities
$x^{-\sim} = x = x^{\sim-}$;
c. a Heyting algebra if and only if the operations "$\odot$" and "$\wedge$" coincide on M.

Definition

A *state $R\ell$-monoid* is a pair (M, σ) such that M is a bounded $R\ell$-monoid and $\sigma : M \longrightarrow M$ is a mapping, called a *state operator*, on M such that

(1) $\sigma(0) = 0$;

(2) $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y)$, $\sigma(x \rightsquigarrow y) = \sigma(x) \rightsquigarrow \sigma(x \wedge y)$;

(3) $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \rightsquigarrow (x \odot y)) = \sigma(y \to (x \odot y)) \odot \sigma(y)$;

(4) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$;

(5) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$, $\sigma(\sigma(x) \rightsquigarrow \sigma(y)) = \sigma(x) \rightsquigarrow \sigma(y)$.

Denote $\mathrm{Ker}(\sigma) = \{x \in M : \sigma(x) = 1\}$, the *kernel* of σ.
A state operator σ on M is called *faithful* if $\mathrm{Ker}(\sigma) = \{1\}$.

$x, y \in M$ are called *orthogonal*, $x \perp y$, if $x^{-\sim} \leq y^{\sim}$ (iff $y^{\sim -} \leq x^{-}$).

Proposition

If (M, σ) is a state $R\ell$-monoid, then for any $x, y \in M$:

1. $\sigma(1) = 1$;
2. $\sigma(x^-) = \sigma(x)^-, \quad \sigma(x^\sim) = \sigma(x)^\sim$;
3. $x \leq y \implies \sigma(x) \leq \sigma(y)$;
4. $\sigma(x \odot y) \geq \sigma(x) \odot \sigma(y)$,
   $x \perp y \implies \sigma(x \odot y) = \sigma(x) \odot \sigma(y)$;
5. $\sigma(x \wedge y) = \sigma(x) \odot \sigma(x \rightsquigarrow y) = \sigma(y \to x) \odot \sigma(y)$;
6. $\sigma(x \to y) \leq \sigma(x) \to \sigma(y), \quad \sigma(x \rightsquigarrow y) \leq \sigma(x) \rightsquigarrow \sigma(y)$;
7. If σ is faithful, then $x < y$ implies $\sigma(x) < \sigma(y)$;
8. $\sigma(\sigma(x)) = \sigma(x)$;
9. $\sigma(M) = \{x \in M : \sigma(x) = x\}$.

Proposition

Let (M, σ) be a state $R\ell$-monoid. Then $\sigma(M)$ is a subalgebra of M.

M ... Rℓ-monoid, $\sigma : M \longrightarrow M$, $x, y \in M$:

(3') $\sigma(x \odot y) = \sigma(x) \odot \sigma(x^{\sim} \vee y) = \sigma(x \vee y^{-}) \odot \sigma(y)$.

$\sigma : M \longrightarrow M$ is called a *strong state operator* on M if σ satisfies conditions (1), (2), (3'), (4), (5), (6).

(M, σ) ... a *strong state $R\ell$-monoid*.

Proposition

Every strong state $R\ell$-monoid is a state $R\ell$-monoid.

Every state operator on a pseudo-MV-algebra is strong and every state operator on a linear $R\ell$-monoid is strong, but we do not know whether every state operator on a BL-algebra is strong. On the other hand, every state operator on a finite MV-algebra is strong and an endomorphism.

Question

Is every state operator on a finite $R\ell$-monoid strong and, moreover, an $R\ell$-endomorphism?

Proposition

Let σ be a strong state operator on an $R\ell$-monoid M. If, for all $x, y \in M$, $x^{\sim} \leq y$ or $y^{-} \leq x$, then $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$.

Proposition

Let σ be a strong state operator on a linearly ordered $R\ell$-monoid M. Then σ is an $R\ell$-endomorphism such that $\sigma^2 = \sigma$.

The result will be strenthened in one of the following theorems.

M=(M; $\odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1$) ... $R\ell$-monoid satisfying $x^{-\sim} = x^{\sim -} = x$, $x \oplus y := (x^{\sim} \odot y^{\sim})^{-}$. Then $(M; \oplus, ^{-}, ^{\sim}, 0, 1)$ is a GMV-algebra (and $x \rightarrow y := x^{-} \oplus y := (x \odot y^{\sim})^{-})$.

$A = (A; \oplus, ^{-}, ^{\sim}, 0, 1)$ ... a GMV-algebra. Define $x \odot y := (x^{-} \oplus y^{-})^{\sim}$, $x \vee y := x \oplus (y \odot x^{-})$, $x \wedge y := x \odot (y \oplus x^{\sim})$, $x \rightarrow y := x^{-} \oplus y := (x \odot y^{\sim})^{-}$, $x \rightsquigarrow y := y \oplus x^{\sim} = (y^{-} \odot x)^{\sim}$. Then $\overline{A} = (A; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded $R\ell$-monoid satisfying $x^{-\sim} = x^{\sim -} = x$.

Theorem

Let $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ be a good $R\ell$-monoid such that $(M; \oplus, ^{-}, ^{\sim}, 0, 1)$ is a GMV-algebra. Then a mapping $\sigma : M \longrightarrow M$ is a state operator on M if and only if σ is a strong state operator on M.

Definition

σ ... state operator on an $R\ell$-monoid M. Then σ is called a *weak state-morphism operator* on M if for every $x, y \in M$:

(7) $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$.

(M, σ) ... *weak state-morphism $R\ell$-monoid*.

Definition

An $R\ell$-endomorphism $\sigma : M \longrightarrow M$ is said to be a *state-morphism operator* if $\sigma^2 = \sigma$.

(M, σ) ... *state-morphism $R\ell$-monoid*.

Example

a) If M is any bounded $R\ell$-monoid, then the identity id_M is a state-morphism operator on M.

b) Let M be a bounded $R\ell$-monoid and $A = M \times M$. Then the mappings $\sigma_1 : A \longrightarrow A$, $\sigma_2 : A \longrightarrow A$ such that $\sigma_1(x_1, x_2) = (x_1, x_1)$, $\sigma_2(x_1, x_2) = (x_2, x_2)$ are state-morphism operators on the $R\ell$-monoid A.

c) Let M be the MV-algebra of all continuous and piecewise linear functions with real coefficients from $[0, 1]$ into itself. If we set $\sigma(f) = \int_{[0,1]} f(x)dx$, $f \in M$, then σ is a strong state operator on M that is not a weak state-morphism operator (Flaminio and Montagna, 2009).

Proposition

If σ is a state operator on an $R\ell$-monoid M preserving $\rightarrow$, or equivalently $\rightsquigarrow$, then σ is a weak state-morphism operator.

Proposition

(1) Every weak state-morphism operator σ on an $R\ell$-monoid M is a strong state operator. (The converse is not true.)
(2) If a state operator σ on an $R\ell$-monoid M preserves $\vee$, it preserves $\rightarrow$. If an $R\ell$-monoid is a pseudo-BL-algebra, then σ preserves $\rightarrow$ if and only if it preserves $\vee$.
(3) Every weak state-morphism operator σ on a pseudo-BL-algebra M that preserves $\rightarrow$ is a state-morphism operator.

Definition

Let $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ be an $R\ell$-monoid and $\emptyset \neq F \subseteq M$. Then F is called a *filter* of M if

(i) $x, y \in F \implies x \odot y \in F$;

(ii) $x \in F,\ y \in M,\ x \leq y \implies y \in F$.

A filter F is called *normal* if for each $x, y \in M$

(iii) $x \rightarrow y \in F \iff x \rightsquigarrow y \in F$.

normal filters of $M \longleftrightarrow$ congruences on M

Proposition

If σ is a state operator on an $R\ell$-monoid M, then $\mathrm{Ker}(\sigma)$ is a normal filter of M.

If (M, σ) is a state $R\ell$-monoid and F is a filter of M, then F is called a *σ-filter* of (M, σ) if $\sigma(x) \in F$ for every $x \in F$.

Definition

If (M, σ) is a state $R\ell$-monoid and θ is a congruence on M, then θ is called a *σ-congruence* on (M, σ) if $(x, y) \in \theta$ implies $(\sigma(x), \sigma(y)) \in \theta$.

Theorem

Let (M, σ) be a state $R\ell$-monoid. Then there is a one-to-one correspondence between normal σ-filters and σ-congruences of (M, σ) and the lattices of normal σ-filters and σ-congruences are isomorphic.

Theorem

Let F be a proper normal σ-filter of a state $R\ell$-monoid (M, σ). Then F is a maximal σ-filter of M if and only if for every $x \in M \setminus F$ there is $n \in \mathbb{N}$ such that $(\sigma(x)^n)^\sim \in F$, equivalently, for every $x \in M \setminus F$ there is $m \in \mathbb{N}$ such that $(\sigma(x)^m)^- \in F$.

Definition

Let M be an $R\ell$-monoid. Then a mapping $s: M \longrightarrow [0,1] \subseteq \mathbb{R}$ is called a *Bosbach state* on M if for any $x, y \in M$,

(B1) $s(x) + s(x \to y) = s(y) + s(y \to x)$;
(B2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$;
(B3) $s(0) = 0, \ s(1) = 1$.

Let M be a good $R\ell$-monoid. We define the partial binary operation "$+$" on M as follows:
If $x, y \in M$ then $x + y$ exists iff $x \perp y$ (i.e. $x^{-\sim} \le y^{\sim}$) and in this case $x + y := x \oplus y$ ($:= (y^{-} \odot x^{-})^{\sim} = (y^{\sim} \odot x^{\sim})^{-}$).

Definition

A mapping $s: M \longrightarrow [0,1] \subseteq \mathbb{R}$ is called a *Riečan state* on M if for any $x, y \in M$,

(R1) $s(1) = 1$;
(R2) $s(x + y) = s(x) + s(y)$ whenever $x + y$ is defined.

It was proved (Dvurečenskij and J. R., 2006) that for good $R\ell$-monoids Bosbach and Riečan states coincide. Hence we will call them simply *states*.

Denote by $\mathcal{S}(M)$ the set of all states on M. In contrast to commutative $R\ell$-monoids, $\mathcal{S}(M)$ can be empty (Dvurečenskij, 2001).
The state space $\mathcal{S}(M)$ is a convex set, i.e., if s_1, $s_2 \in \mathcal{S}(M)$ and $\lambda \in [0,1]$ then $s = \lambda s_1 + (1-\lambda)s_2 \in \mathcal{S}(M)$. A state s is called *extremal* if the equality $s = \lambda s_1 + (1-\lambda)s_2$, where s_1, $s_2 \in \mathcal{S}(M)$ and $\lambda \in (0,1)$, implies $s_1 = s_2$. A net $\{s_\alpha\}$ of states *converges weakly* to a state s if $s(x) = \lim_\alpha s_\alpha(x)$ for each $x \in M$. By the Krein-Mil'man theorem, every state on M is a weak limit of convex combinations of extremal states.

Theorem

If (M, σ) is a good state $R\ell$-monoid and s is a state on the $R\ell$-monoid $\sigma(M)$, then $s_\sigma : M \longrightarrow [0, 1] \subseteq \mathbb{R}$ such that $s_\sigma(x) := s(\sigma(x))$ for every $x \in M$, is a state on M.

Corollary

If M is a bounded $R\ell$-monoid and if there is at least one state operator σ on M such that $\mathcal{S}(\sigma(M)) \neq \emptyset$, then $\mathcal{S}(M) \neq \emptyset$.

Theorem

If (M, σ) is a good weak state-morphism $R\ell$-monoid and s is an extremal state on the $R\ell$-monoid $\sigma(M)$, then s_σ is an extremal state on M.

Theorem

Let (M, σ) be a good weak state-morphism $R\ell$-monoid. If F is a maximal σ-filter of (M, σ) which is normal, then $\sigma(F)$ is a normal and maximal filter of the $R\ell$-monoid $\sigma(M)$ and F is also a maximal filter of the $R\ell$-monoid M.

Definition

Let (M, σ) be a state $R\ell$-monoid and s be a state on M. Then s is called *σ-compatible* if $\sigma(x) = \sigma(y)$ implies $s(x) = s(y)$ for every $x, y \in M$.

Theorem

Let (M, σ) be a good state $R\ell$-monoid. Then there is a bijection between σ-compatible states on M and states on $\sigma(M)$.

If s is a σ-compatible state on M, put $\varphi(s)(\sigma(x)) := s(x)$.
If s is a state on $\sigma(M)$, put $\psi(s)(x) := s_\sigma(x) = s(\sigma(x))$.

Then φ and ψ are bijective mappings (between σ-compatible states on M and states on $\sigma(M)$) and $\psi = \varphi^{-1}$.

$x \oslash y := x \odot y^{\sim}, \quad x \varobslash y := y^{-} \odot x,$

Definition

Let $M = (M; \oplus, ^{-}, ^{\sim}, 0, 1)$ be a GMV-algebra and $\sigma : M \longrightarrow M$ be a mapping. Then σ is called a *state operator on M* if for any $x, y \in M$:

(1) $\sigma(1) = 1$;
(2) $\sigma(x^{-}) = \sigma(x)^{-}$, $\sigma(x^{\sim}) = \sigma(x)^{\sim}$;
(3) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \oslash (x \odot y)) = \sigma(x \varobslash (x \odot y)) \oplus \sigma(y)$;
(4) $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.

Proposition

Let $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ be a good $R\ell$-monoid such that $\overline{M} = (M; \oplus, ^{-}, ^{\sim}, 0, 1)$ is a GMV-algebra and $\sigma : M \longrightarrow M$ be a mapping. Then σ is a state operator on the $R\ell$-monoid M if and only if σ is a state operator on the GMV-algebra $\overline{M}$.

If M is an $R\ell$-monoid, set

$$MV(M) := \{x \in M : x^{-\sim} = x^{\sim -} = x\}.$$

If M is a good $R\ell$-monoid, then $MV(M) = (MV(M); \oplus, ^-, ^\sim, 0, 1)$, where $\oplus$, $^-$ and $^\sim$ are the restrictions of the corresponding operations from M, is a GMV-algebra. For the multiplication "$\odot_{MV}$" defined in $MV(M)$ by $x \odot_{MV} y := (y^- \oplus x^-)^\sim$ we have $x \odot_{MV} y = (x \odot y)^{-\sim}$.

Theorem

If σ is a state operator on a normal $R\ell$-monoid M, then its restriction onto $MV(M)$ is a state operator on the GMV-algebra $MV(M)$.

Theorem

Let M be a good normal $R\ell$-monoid. If τ is a state operator on the GMV-algebra $MV(M)$, then $\overline{\tau} : M \longrightarrow M$ such that $\overline{\tau}(x) := \tau(x^{-\sim})$ for any $x \in M$ is a state operator on the $R\ell$-monoid M.

Corollary

If M is a good pseudo-BL-algebra and τ is a state operator on the GMV-algebra $MV(M)$, then $\overline{\tau}$ introduced in the preceeding theorem is a state operator on M.

Every linearly ordered $R\ell$-monoid is always a pseudo-BL-algebra.

Definition

A *pseudo hoop* is an algebra $(M; \odot, \to, \rightsquigarrow, 1)$ of type $\langle 2, 2, 2, 0 \rangle$ such that, for all $x, y, z \in M$,

(i) $x \odot 1 = x = 1 \odot x$;

(ii) $x \to x = 1 = x \rightsquigarrow x$;

(iii) $(x \odot y) \to z = x \to (y \to z)$;

(iv) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;

(v) $(x \to y) \odot x = (y \to x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

Definition

A pseudo hoop M is said to be *basic* if, for all $x, y, z \in M$,

(B1) $(x \to y) \to z \leq ((y \to x) \to z) \to z$;

(B2) $(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$.

Any linearly ordered pseudo hoop and hence any representable pseudo hoop is basic.
Not every pseudo hoop is basic.

Definition

A pseudo hoop M is said to be *Wajsberg* if, for all $x, y \in M$,

(W1) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x$;

(W2) $(x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$.

Any pseudo Wajsberg hoop is a basic pseudo hoop. A pseudo hoop is said to be *bounded* if M admits a least element 0.

Examples

(1) Let $(G;\ \cdot, \vee, \wedge, ^{-1}, e)$ be an ℓ-group written multiplicatively. For the negative cone $G^- = \{g \in G :\ g \leq e\}$ we define $a \odot b = a \cdot b$, $a \to b = (b \cdot a^{-1}) \wedge e$, and $a \rightsquigarrow b = (a^{-1} \cdot b) \wedge e$. Then $(G^-; \odot, \to, \wedge, e)$ is a Wajsberg unbounded cancellative pseudo hoop.

(2) Let $(G; \cdot, \vee, \wedge, ^{-1}, e)$ be an ℓ-group written multiplicatively. Choose $u \geq e$ and we endow the interval $[u^{-1}, e]$ with $x \odot y = (x \cdot y) \vee u^{-1}$, $x \to y = (y \cdot x^{-1}) \wedge e$, $x \rightsquigarrow y = (x^{-1} \cdot y) \wedge e$, $x, y \in [u^{-1}, e]$. Then $([u^{-1}, e]; \odot, \to, \rightsquigarrow, u)$ is a bounded pseudo Wajsberg hoop with the least element u^{-1} which is term-equivalent with a pseudo-MV-algebra.

Every pseudo Wajsberg hoop is just one of the cases described in the preceeding example (Dvurečenskij, 2007).

Definition

Let $\{M_i : \ i \in I\}$ be a system of pseudo hoops with a linearly ordered index set $(I; \leq)$ such that $M_i \cap M_j = \{1\}$ for all $i \neq j$, $i, j \in I$. We set $M = \bigcup_{i \in I} M_i$ and on M we define the operation $\odot$, $\to$ and $\rightsquigarrow$ as follows

$$x \odot y = \begin{cases} x \odot_i y & \text{if } \ x, y \in M_i, \\ x & \text{if } \ x \in M_i \setminus \{1\}, \ y \in M_j, \ i < j, \\ y & \text{if } \ x \in M_i, \ y \in M_j \setminus \{1\}, \ i > j, \end{cases}$$

$$x \to y = \begin{cases} x \to_i y & \text{if } \ x, y \in M_i, \\ y & \text{if } \ x \in M_i, \ y \in M_j \setminus \{1\}, \ i > j, \\ 1 & \text{if } \ x \in M_i \setminus \{1\}, \ y \in M_j, \ i < j, \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} x \rightsquigarrow_i y & \text{if } \ x, y \in M_i, \\ y & \text{if } \ x \in M_i, \ y \in M_j \setminus \{1\}, \ i > j, \\ 1 & \text{if } \ x \in M_i \setminus \{1\}, \ y \in M_j, \ i < j. \end{cases}$$

Then M with 1, $\odot$, $\to$ and $\rightsquigarrow$, denotation $\bigoplus_i M_i$, is a pseudo hoop called the *ordinal sum* of $\{M_i : \ i \in I\}$.

If I has a minimum, 0, and M_0 is a bounded pseudo hoop, then $\bigoplus_i M_i$ is bounded. If all M_i's are linear so is $\bigoplus_i M_i$. If all M_i's are linear pseudo hoops and M_0 is bounded, then $\bigoplus_i M_i$ is a linear pseudo BL-algebra.

Theorem (Dvurečenskij, 2007)

Every linear pseudo-BL-algebra can be uniquely represented as the ordinal sum of a family of linear pseudo Wajsberg hoops whose first component is a linear bounded pseudo Wajsberg hoop.

Consequently, every linearly ordered pseudo-BL-algebra is the ordinal sum of a system whose each component is either the negative cone of a linear ℓ-group or an interval in a linear unital ℓ-group with strong unit.

Theorem

Every state operator σ on a linearly ordered pseudo-BL-algebra M preserves both $\odot$ and $\to$, and it is a state-morphism operator.

Question

Is every weak state-morphism operator on a bounded $R\ell$-monoid a state-morphism operator?

Connections among the kinds of state operators

(a) state operator

(b) strong state operator

(c) weak state-morphism operator

(d) state-morphism operator

(d) $\Longrightarrow$ (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a)

(b) $\not\Longrightarrow$ (c)

questions: (a) $\Longrightarrow$ (b) ? , (c) $\Longrightarrow$ (d) ?

For linearly ordered pseudo-BL-algebras: (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d)