State operators on bounded residuated $\ell$-monoids

Jiří Rachůnek

Palacký University in Olomouc

Czech Republic

AsubL4

Nomi, June 8-10, 2010


Here: General state and state-morphism bounded $R\ell$-monoids.
Definition

A \textit{bounded }R\ell\text{-monoid} is an algebra \(M = (M; \odot, \lor, \land, \rightarrow, \rightsquigarrow, 0, 1)\) of type \(\langle 2, 2, 2, 2, 2, 0, 0, \rangle\) satisfying:

(i) \((M; \odot, 1)\) is a monoid (need not be commutative).

(ii) \((M; \lor, \land, 0, 1)\) is a bounded lattice.

(iii) \(x \odot y \leq z\) iff \(x \leq y \rightarrow z\) iff \(y \leq x \rightsquigarrow z\) for any \(x, y \in M\).

(iv) \((x \rightarrow y) \odot x = x \land y = y \odot (y \rightsquigarrow x)\).

\(R\ell\)-monoid = bounded \(R\ell\)-monoid.

\[x^- := x \rightarrow 0, \quad x^\sim := x \rightsquigarrow 0.\]

An \(R\ell\)-monoid \(M\) is called \textit{good} if \(x^\sim^\sim = x^-^-\)
\textit{normal} if \((x \odot y)^\sim^\sim = x^-^- \odot y^\sim^\sim\) and \((x \odot y)^\sim^- = x^-^- \odot y^\sim^-\).

If \(M\) is good, then put \(x \oplus y := (y^\sim \odot x^\sim)^-\) (\(= (y^- \odot x^-)^\sim\)).
$R\ell$-monoids $\cong$ bounded non-commutative residuated lattices satisfying the identities of divisibility $\cong$ bounded integral generalized $BL$-algebras $\cong$ $FL_w$-algebras satisfying the identities of divisibility $\cong$ bounded divisible pseudo-$BCK(pP)$ lattices

An $R\ell$-monoid $M$ is

- a pseudo-$BL$-algebra if and only if $M$ satisfies the identities of pre-linearity
  \[(x \rightarrow y) \lor (y \rightarrow x) = 1 = (x \sim y) \lor (y \sim x);\]

- equivalent to a $GMV$-algebra (pseudo-$MV$-algebra) if and only if $M$ fulfils the identities
  \[x \sim \sim x = x = x \sim \sim;\]

- a Heyting algebra if and only if the operations "$\odot$" and "$\land$" coincide on $M$. 
A state $R\ell$-monoid is a pair $(M, \sigma)$ such that $M$ is a bounded $R\ell$-monoid and $\sigma : M \rightarrow M$ is a mapping, called a state operator, on $M$ such that

1. $\sigma(0) = 0$;
2. $\sigma(x \rightarrow y) = \sigma(x) \rightarrow \sigma(x \land y)$, $\sigma(x \leadsto y) = \sigma(x) \leadsto \sigma(x \land y)$;
3. $\sigma(x \circ y) = \sigma(x) \circ \sigma(x \leadsto (x \circ y)) = \sigma(y \rightarrow (x \circ y)) \circ \sigma(y)$;
4. $\sigma(\sigma(x) \circ \sigma(y)) = \sigma(x) \circ \sigma(y)$;
5. $\sigma(\sigma(x) \rightarrow \sigma(y)) = \sigma(x) \rightarrow \sigma(y)$, $\sigma(\sigma(x) \leadsto \sigma(y)) = \sigma(x) \leadsto \sigma(y)$.

Denote $\text{Ker}(\sigma) = \{x \in M : \sigma(x) = 1\}$, the kernel of $\sigma$. A state operator $\sigma$ on $M$ is called faithful if $\text{Ker}(\sigma) = \{1\}$.

$x, y \in M$ are called orthogonal, $x \perp y$, if $x \sim \leq y \sim$ (iff $y \sim \leq x \sim$).
Proposition

If \((M, \sigma)\) is a state \(R\ell\)-monoid, then for any \(x, y \in M\):

1. \(\sigma(1) = 1\);
2. \(\sigma(x^-) = \sigma(x)^-, \quad \sigma(x^\sim) = \sigma(x)^\sim\);
3. \(x \leq y \implies \sigma(x) \leq \sigma(y)\);
4. \(\sigma(x \odot y) \geq \sigma(x) \odot \sigma(y), \quad \sigma(x \perp y) \implies \sigma(x \odot y) = \sigma(x) \odot \sigma(y)\);
5. \(\sigma(x \land y) = \sigma(x) \odot \sigma(x \leadsto y) = \sigma(y \rightarrow x) \odot \sigma(y)\);
6. \(\sigma(x \rightarrow y) \leq \sigma(x) \rightarrow \sigma(y), \quad \sigma(x \leadsto y) \leq \sigma(x) \leadsto \sigma(y)\);
7. If \(\sigma\) is faithful, then \(x < y\) implies \(\sigma(x) < \sigma(y)\);
8. \(\sigma(\sigma(x)) = \sigma(x)\);
9. \(\sigma(M) = \{x \in M : \sigma(x) = x\}\).

Proposition

Let \((M, \sigma)\) be a state \(R\ell\)-monoid. Then \(\sigma(M)\) is a subalgebra of \(M\).
\(M\) ... \(R\ell\)-monoid, \(\sigma : M \rightarrow M, x, y \in M:\)

\((3')\) \(\sigma(x \odot y) = \sigma(x) \odot \sigma(x^\sim \lor y) = \sigma(x \lor y^-) \odot \sigma(y)\).

\(\sigma : M \rightarrow M\) is called a **strong state operator** on \(M\) if \(\sigma\) satisfies conditions (1), (2), (3'), (4), (5), (6).

\((M, \sigma)\) ... a **strong state \(R\ell\)-monoid**.

**Proposition**

Every strong state \(R\ell\)-monoid is a state \(R\ell\)-monoid.

Every state operator on a pseudo-\(MV\)-algebra is strong and every state operator on a linear \(R\ell\)-monoid is strong, but we do not know whether every state operator on a \(BL\)-algebra is strong. On the other hand, every state operator on a finite \(MV\)-algebra is strong and an endomorphism.

**Question**

Is every state operator on a finite \(R\ell\)-monoid strong and, moreover, an \(R\ell\)-endomorphism?
Proposition

Let $\sigma$ be a strong state operator on an $R\ell$-monoid $M$. If, for all $x, y \in M$, $x \sim \leq y$ or $y^- \leq x$, then $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$.

Proposition

Let $\sigma$ be a strong state operator on a linearly ordered $R\ell$-monoid $M$. Then $\sigma$ is an $R\ell$-endomorphism such that $\sigma^2 = \sigma$.

The result will be strengthened in one of the following theorems.
\( M = (M; \odot, \lor, \land, \rightarrow, \sim, 0, 1) \) is a \( R\ell \)-monoid satisfying \( x^{\sim \sim} = x^{\sim \sim} = x \), \( x \oplus y := (x^{\sim} \odot y^{\sim})^- \). Then \( (M; \oplus, -, \sim, 0, 1) \) is a GMV-algebra (and \( x \rightarrow y := x^- \ominus y := (x \odot y^-)^- \)).

A = (A; \oplus, -, \sim, 0, 1) ... a GMV-algebra. Define \( x \odot y := (x^- \oplus y^-)^-, x \lor y := x \oplus (y \odot x^-), x \land y := x \odot (y \oplus x^-), x \rightarrow y := x^- \ominus y := (x \odot y^-)^-, x \rightsquigarrow y := y \oplus x^- = (y^- \odot x)^- \). Then \( \overline{A} = (A; \odot, \lor, \land, \rightarrow, \rightsquigarrow, 0, 1) \) is a bounded \( R\ell \)-monoid satisfying \( x^{\sim \sim} = x^{\sim \sim} = x \).

**Theorem**

Let \( M = (M; \odot, \lor, \land, \rightarrow, \rightsquigarrow, 0, 1) \) be a good \( R\ell \)-monoid such that \( (M; \oplus, -, \sim, 0, 1) \) is a GMV-algebra. Then a mapping \( \sigma : M \rightarrow M \) is a state operator on \( M \) if and only if \( \sigma \) is a strong state operator on \( M \).
Definition

σ ... state operator on an $R\ell$-monoid $M$. Then $\sigma$ is called a **weak state-morphism operator** on $M$ if for every $x, y \in M$:

\[(7) \quad \sigma(x \circ y) = \sigma(x) \circ \sigma(y).\]

$(M, \sigma)$ ... weak state-morphism $R\ell$-monoid.

Definition

An $R\ell$-endomorphism $\sigma : M \rightarrow M$ is said to be a **state-morphism operator** if $\sigma^2 = \sigma$.

$(M, \sigma)$ ... state-morphism $R\ell$-monoid.
Example

a) If $M$ is any bounded $R\ell$-monoid, then the identity $id_M$ is a state-morphism operator on $M$.

b) Let $M$ be a bounded $R\ell$-monoid and $A = M \times M$. Then the mappings

$\sigma_1 : A \rightarrow A, \sigma_2 : A \rightarrow A$ such that

$\sigma_1(x_1, x_2) = (x_1, x_1), \sigma_2(x_1, x_2) = (x_2, x_2)$ are state-morphism operators on the $R\ell$-monoid $A$.

c) Let $M$ be the $MV$-algebra of all continuous and piecewise linear functions with real coefficients from $[0, 1]$ into itself. If we set $\sigma(f) = \int_{[0,1]} f(x)dx$, $f \in M$, then $\sigma$ is a strong state operator on $M$ that is not a weak state-morphism operator (Flaminio and Montagna, 2009).
**Proposition**

If $\sigma$ is a state operator on an $R\ell$-monoid $M$ preserving $\rightarrow$, or equivalently $\sim\rightarrow$, then $\sigma$ is a weak state-morphism operator.

**Proposition**

(1) Every weak state-morphism operator $\sigma$ on an $R\ell$-monoid $M$ is a strong state operator. (The converse is not true.)

(2) If a state operator $\sigma$ on an $R\ell$-monoid $M$ preserves $\lor$, it preserves $\rightarrow$. If an $R\ell$-monoid is a pseudo-$BL$-algebra, then $\sigma$ preserves $\rightarrow$ if and only if it preserves $\lor$.

(3) Every weak state-morphism operator $\sigma$ on a pseudo-$BL$-algebra $M$ that preserves $\rightarrow$ is a state-morphism operator.
Definition

Let $M = (M; \odot, \lor, \land, \to, \iff, 0, 1)$ be an $R\ell$-monoid and $\emptyset \neq F \subseteq M$. Then $F$ is called a filter of $M$ if

(i) $x, y \in F \implies x \odot y \in F$;
(ii) $x \in F$, $y \in M$, $x \leq y \implies y \in F$.

A filter $F$ is called normal if for each $x, y \in M$

(iii) $x \to y \in F \iff x \iff y \in F$.

normal filters of $M \iff$ congruences on $M$

Proposition

If $\sigma$ is a state operator on an $R\ell$-monoid $M$, then $\text{Ker}(\sigma)$ is a normal filter of $M$. 
If \((M, \sigma)\) is a state \(R\ell\)-monoid and \(F\) is a filter of \(M\), then \(F\) is called a \(\sigma\)-filter of \((M, \sigma)\) if \(\sigma(x) \in F\) for every \(x \in F\).

**Definition**

If \((M, \sigma)\) is a state \(R\ell\)-monoid and \(\theta\) is a congruence on \(M\), then \(\theta\) is called a \(\sigma\)-congruence on \((M, \sigma)\) if \((x, y) \in \theta\) implies \((\sigma(x), \sigma(y)) \in \theta\).

**Theorem**

Let \((M, \sigma)\) be a state \(R\ell\)-monoid. Then there is a one-to-one correspondence between normal \(\sigma\)-filters and \(\sigma\)-congruences of \((M, \sigma)\) and the lattices of normal \(\sigma\)-filters and \(\sigma\)-congruences are isomorphic.

**Theorem**

Let \(F\) be a proper normal \(\sigma\)-filter of a state \(R\ell\)-monoid \((M, \sigma)\). Then \(F\) is a maximal \(\sigma\)-filter of \(M\) if and only if for every \(x \in M \setminus F\) there is \(n \in \mathbb{N}\) such that \((\sigma(x)^n) \sim \in F\), equivalently, for every \(x \in M \setminus F\) there is \(m \in \mathbb{N}\) such that \((\sigma(x)^m)^- \in F\).
Let $M$ be an $R\ell$-monoid. Then a mapping $s : M \rightarrow [0, 1] \subseteq \mathbb{R}$ is called a **Bosbach state** on $M$ if for any $x, y \in M$,

(B1) $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$;
(B2) $s(x) + s(x \bowtie y) = s(y) + s(y \bowtie x)$;
(B3) $s(0) = 0$, $s(1) = 1$.

Let $M$ be a good $R\ell$-monoid. We define the partial binary operation "+" on $M$ as follows:

If $x, y \in M$ then $x + y$ exists iff $x \perp y$ (i.e. $x^{-\bowtie} \leq y^{-\bowtie}$) and in this case $x + y := x \oplus y$ (:= $(y^{-\bowtie} \circ x^{-\bowtie})^{-\bowtie} = (y^{-\bowtie} \circ x^{-\bowtie})^{-\bowtie}$).

**Definition**

A mapping $s : M \rightarrow [0, 1] \subseteq \mathbb{R}$ is called a **Riečan state** on $M$ if for any $x, y \in M$,

(R1) $s(1) = 1$;
(R2) $s(x + y) = s(x) + s(y)$ whenever $x + y$ is defined.
It was proved (Dvurečenskij and J. R., 2006) that for good $R\ell$-monoids Bosbach and Riečan states coincide. Hence we will call them simply states.

Denote by $S(M)$ the set of all states on $M$. In contrast to commutative $R\ell$-monoids, $S(M)$ can be empty (Dvurečenskij, 2001). The state space $S(M)$ is a convex set, i.e., if $s_1, s_2 \in S(M)$ and $\lambda \in [0, 1]$ then $s = \lambda s_1 + (1 - \lambda) s_2 \in S(M)$. A state $s$ is called extremal if the equality $s = \lambda s_1 + (1 - \lambda) s_2$, where $s_1, s_2 \in S(M)$ and $\lambda \in (0, 1)$, implies $s_1 = s_2$. A net $\{s_\alpha\}$ of states converges weakly to a state $s$ if $s(x) = \lim_\alpha s_\alpha(x)$ for each $x \in M$. By the Krein-Mil’man theorem, every state on $M$ is a weak limit of convex combinations of extremal states.
Theorem
If \((M, \sigma)\) is a good state \(R\ell\)-monoid and \(s\) is a state on the \(R\ell\)-monoid \(\sigma(M)\), then \(s_{\sigma} : M \rightarrow [0, 1] \subseteq \mathbb{R}\) such that \(s_{\sigma}(x) := s(\sigma(x))\) for every \(x \in M\), is a state on \(M\).

Corollary
If \(M\) is a bounded \(R\ell\)-monoid and if there is at least one state operator \(\sigma\) on \(M\) such that \(S(\sigma(M)) \neq \emptyset\), then \(S(M) \neq \emptyset\).

Theorem
If \((M, \sigma)\) is a good weak state-morphism \(R\ell\)-monoid and \(s\) is an extremal state on the \(R\ell\)-monoid \(\sigma(M)\), then \(s_{\sigma}\) is an extremal state on \(M\).

Theorem
Let \((M, \sigma)\) be a good weak state-morphism \(R\ell\)-monoid. If \(F\) is a maximal \(\sigma\)-filter of \((M, \sigma)\) which is normal, then \(\sigma(F)\) is a normal and maximal filter of the \(R\ell\)-monoid \(\sigma(M)\) and \(F\) is also a maximal filter of the \(R\ell\)-monoid \(M\).
Definition

Let \((M, \sigma)\) be a state \(R\ell\)-monoid and \(s\) be a state on \(M\). Then \(s\) is called \(\sigma\)-compatible if \(\sigma(x) = \sigma(y)\) implies \(s(x) = s(y)\) for every \(x, y \in M\).

Theorem

Let \((M, \sigma)\) be a good state \(R\ell\)-monoid. Then there is a bijection between \(\sigma\)-compatible states on \(M\) and states on \(\sigma(M)\).

If \(s\) is a \(\sigma\)-compatible state on \(M\), put \(\varphi(s)(\sigma(x)) := s(x)\).

If \(s\) is a state on \(\sigma(M)\), put \(\psi(s)(x) := s_\sigma(x) = s(\sigma(x))\).

Then \(\varphi\) and \(\psi\) are bijective mappings (between \(\sigma\)-compatible states on \(M\) and states on \(\sigma(M)\)) and \(\psi = \varphi^{-1}\).
\[ x \otimes y := x \odot y^\sim, \quad x \oslash y := y^\sim \odot x, \]

**Definition**

Let \( M = (M; \oplus, -, \sim, 0, 1) \) be a GMV-algebra and \( \sigma : M \to M \) be a mapping. Then \( \sigma \) is called a **state operator on** \( M \) if for any \( x, y \in M \):

1. \( \sigma(1) = 1 \);
2. \( \sigma(x^-) = \sigma(x)^-, \quad \sigma(x^\sim) = \sigma(x)^\sim \);
3. \( \sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \odot (x \odot y)) = \sigma(x \odot (x \odot y)) \oplus \sigma(y) \);
4. \( \sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y) \).

**Proposition**

Let \( M = (M; \odot, \lor, \land, \to, \twoheadrightarrow, 0, 1) \) be a good \( R\ell \)-monoid such that \( \overline{M} = (M; \oplus, -, \sim, 0, 1) \) is a GMV-algebra and \( \sigma : M \to M \) be a mapping. Then \( \sigma \) is a state operator on the \( R\ell \)-monoid \( M \) if and only if \( \sigma \) is a state operator on the GMV-algebra \( \overline{M} \).
If $M$ is an $R\ell$-monoid, set

$$MV(M) := \{x \in M : x^{\sim\sim} = x^{\sim\sim} = x\}.$$  

If $M$ is a good $R\ell$-monoid, then $MV(M) = (MV(M); \oplus, -, \sim, 0, 1)$, where $\oplus$, $-$ and $\sim$ are the restrictions of the corresponding operations from $M$, is a $GMV$-algebra. For the multiplication $\circ_{MV}$ defined in $MV(M)$ by $x \circ_{MV} y := (y^{-} \oplus x^{-})^{\sim}$ we have $x \circ_{MV} y = (x \circ y)^{\sim}$. 

**Theorem**

If $\sigma$ is a state operator on a normal $R\ell$-monoid $M$, then its restriction onto $MV(M)$ is a state operator on the $GMV$-algebra $MV(M)$. 


Theorem
Let $M$ be a good normal $R\ell$-monoid. If $\tau$ is a state operator on the $GMV$-algebra $MV(M)$, then $\overline{\tau} : M \rightarrow M$ such that $\overline{\tau}(x) := \tau(x^{\sim})$ for any $x \in M$ is a state operator on the $R\ell$-monoid $M$.

Corollary
If $M$ is a good pseudo-$BL$-algebra and $\tau$ is a state operator on the $GMV$-algebra $MV(M)$, then $\overline{\tau}$ introduced in the preceding theorem is a state operator on $M$. 
Every linearly ordered $R\ell$-monoid is always a pseudo-$BL$-algebra.

**Definition**

A *pseudo hoop* is an algebra $(M; \odot, \rightarrow, \leadsto, 1)$ of type $\langle 2, 2, 2, 0 \rangle$ such that, for all $x, y, z \in M$,

(i) $x \odot 1 = x = 1 \odot x$;
(ii) $x \rightarrow x = 1 = x \leadsto x$;
(iii) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
(iv) $(x \odot y) \leadsto z = y \leadsto (x \leadsto z)$;
(v) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \leadsto y) = y \odot (y \leadsto x)$.

**Definition**

A pseudo hoop $M$ is said to be *basic* if, for all $x, y, z \in M$,

(B1) $(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z$;
(B2) $(x \leadsto y) \leadsto z \leq ((y \leadsto x) \leadsto z) \leadsto z$. 
Any linearly ordered pseudo hoop and hence any representable pseudo hoop is basic. 
Not every pseudo hoop is basic.

**Definition**

A pseudo hoop $M$ is said to be *Wajsberg* if, for all $x, y \in M$,

(W1) $(x \rightarrow y) \bowtie y = (y \rightarrow x) \bowtie x$;
(W2) $(x \bowtie y) \rightarrow y = (y \bowtie x) \rightarrow x$.

Any pseudo Wajsberg hoop is a basic pseudo hoop. A pseudo hoop is said to be *bounded* if $M$ admits a least element 0.
Examples

(1) Let \((G; \cdot, \vee, \wedge, -1, e)\) be an \(\ell\)-group written multiplicatively. For the negative cone \(G^{-} = \{g \in G : g \leq e\}\) we define \(a \circ b = a \cdot b\), \(a \rightarrow b = (b \cdot a^{-1}) \wedge e\), and \(a \xrightarrow{\sim} b = (a^{-1} \cdot b) \wedge e\). Then \((G^{-}; \circ, \rightarrow, \wedge, e)\) is a Wajsberg unbounded cancellative pseudo hoop.

(2) Let \((G; \cdot, \vee, \wedge, -1, e)\) be an \(\ell\)-group written multiplicatively. Choose \(u \geq e\) and we endow the interval \([u^{-1}, e]\) with \(x \circ y = (x \cdot y) \vee u^{-1}\), \(x \rightarrow y = (y \cdot x^{-1}) \wedge e\), \(x \xrightarrow{\sim} y = (x^{-1} \cdot y) \wedge e\), \(x, y \in [u^{-1}, e]\). Then \(([u^{-1}, e]; \circ, \rightarrow, \xrightarrow{\sim}, u)\) is a bounded pseudo Wajsberg hoop with the least element \(u^{-1}\) which is term-equivalent with a pseudo-MV-algebra.

Every pseudo Wajsberg hoop is just one of the cases described in the preceding example (Dvurečenskij, 2007).
Definition

Let \( \{ M_i : i \in I \} \) be a system of pseudo hoops with a linearly ordered index set \((I; \leq)\) such that \( M_i \cap M_j = \{1\} \) for all \( i \neq j, i, j \in I \). We set \( M = \bigcup_{i \in I} M_i \) and on \( M \) we define the operation \( \circ, \to \) and \( \rightsquigarrow \) as follows

\[
x \circ y = \begin{cases} 
x \circ_i y & \text{if } x, y \in M_i, 
x & \text{if } x \in M_i \setminus \{1\}, y \in M_j, i < j, 
y & \text{if } x \in M_i, y \in M_j \setminus \{1\}, i > j,
\end{cases}
\]

\[
x \to y = \begin{cases} 
x \to_i y & \text{if } x, y \in M_i, 
y & \text{if } x \in M_i, y \in M_j \setminus \{1\}, i > j, 
1 & \text{if } x \in M_i \setminus \{1\}, y \in M_j, i < j,
\end{cases}
\]

\[
x \rightsquigarrow y = \begin{cases} 
x \rightsquigarrow_i y & \text{if } x, y \in M_i, 
y & \text{if } x \in M_i, y \in M_j \setminus \{1\}, i > j, 
1 & \text{if } x \in M_i \setminus \{1\}, y \in M_j, i < j.
\end{cases}
\]

Then \( M \) with \( 1, \circ, \to \) and \( \rightsquigarrow \), denotation \( \bigoplus_i M_i \), is a pseudo hoop called the \textit{ordinal sum} of \( \{ M_i : i \in I \} \).
If \( I \) has a minimum, 0, and \( M_0 \) is a bounded pseudo hoop, then \( \bigoplus_i M_i \) is bounded. If all \( M_i \)'s are linear so is \( \bigoplus_i M_i \). If all \( M_i \)'s are linear pseudo hoops and \( M_0 \) is bounded, then \( \bigoplus_i M_i \) is a linear pseudo BL-algebra.

**Theorem (Dvurečenskij, 2007)**

Every linear pseudo-BL-algebra can be uniquely represented as the ordinal sum of a family of linear pseudo Wajsberg hoops whose first component is a linear bounded pseudo Wajsberg hoop.

Consequently, every linearly ordered pseudo-BL-algebra is the ordinal sum of a system whose each component is either the negative cone of a linear \( \ell \)-group or an interval in a linear unital \( \ell \)-group with strong unit.
Theorem

Every state operator $\sigma$ on a linearly ordered pseudo-$BL$-algebra $M$ preserves both $\odot$ and $\rightarrow$, and it is a state-morphism operator.

Question

Is every weak state-morphism operator on a bounded $R\ell$-monoid a state-morphism operator?
Connections among the kinds of state operators

(a) state operator
(b) strong state operator
(c) weak state-morphism operator
(d) state-morphism operator

(d) \implies (c) \implies (b) \implies (a)
(b) \not\implies (c)

questions: (a) \implies (b) ?, (c) \implies (d) ?

For linearly ordered pseudo-BL-algebras: (a) \iff (b) \iff (c) \iff (d)