Weird Models of Linear Logic

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Linear logic as resource logic

- **Linear Logic without exponentials** is a substructural logic (no contraction and weakening) = Involutive commutative residuated lattices with bounds
- Conjunction and disjunction exist in two different flavors: multiplicative ($\otimes$ and 1, $\parr$ and $\perp$) and additive (& and $\top$, $\oplus$ and 0).
- Linear negation $(-)^{\perp}$ is involutive.
- In linear logic without exponentials a formula is used only one time, it cannot be neither duplicate neither created. In other words, a proof uses "linearly" its hypotheses.
- **Linear Logic with exponentials**: Exponential modalities ! and ? give a logical status to structural rules. Exponentials are responsible for the possibility of erasing and copying data, which is of course essential during computations

  Summing up, a formula F can be used linearly (only one time), while $!F$ non-linearly (an arbitrary number of times)

Lambda calculus as resource calculus

- Algebraic similarity type Σ:
  - Nullary operators: $a, b, c, \ldots$ (names = variables of λ-calculus)
  - Binary operator: $\bullet$ (application)
  - Unary operators: $\lambda a, \lambda b, \lambda c, \ldots$ (λ-abstractions)

- A λ-term is a ground Σ-term (no algebraic variable involved)

  $\lambda a(a \cdot b)$

  Notation: $\lambda a.ab$ for $\lambda a(a \cdot b)$

- Bound and free names: $\lambda a.ab$

- Substitution for names (with α-conversion)

$$(\lambda a.ab)[b := a] = \lambda c.ca$$

How to compute (informally) with λ-terms

- β-rule: $(\lambda a.a)b \to b$
- Looping terms need infinite resources:

$$(\lambda a.aa)(\lambda a.aa) \to (\lambda a.aa)(\lambda a.aa) \to \ldots$$

- β-rule in general:

$$(\lambda a.M)N \to M[a := N]$$

- Resource Lambda Calculus: A function uses an argument just one time:

$$(\lambda a.aa)[\lambda a.aa, \lambda a.aa] \to (\lambda a.(\lambda a.aa)a)[\lambda a.aa] + (\lambda a.a[\lambda a.aa])[\lambda a.aa] \to$$

$$(\lambda a.aa)[\lambda a.aa] + (\lambda a.aa)[\lambda a.aa] \to 0$$

- Differential Lambda Calculus is an extension of lambda calculus with a linear application. A function uses an argument either linearly (one time) or non-linearly (an arbitrary number of times), as in linear logic!

What is a model of linear logic? And of lambda calculus?

- Algebraic semantics:
  Linear logic: girals (Ursini)
  Linear logic without exponentials: arabesques (Ursini)
  λ-calculus: λ-abstraction algebras (Pigozzi-Salibra)
  Resource λ-calculus: resource λ-abstraction algebras (Carraro-Ehrhard-S.)

- Computer scientists like category theory. Why?

Linear Logic

- Interested in proofs, not in formulas
- Formulas are interpreted as objects of a Seely* category
- Proofs as morphisms: a proof π of the sequent $\phi_1, \dots, \phi_n \vdash \psi_1, \dots, \psi_k$ is interpreted as a morphism $|\pi|$ from the object $|\phi_1| \otimes \cdots \otimes |\phi_n|$ into the object $|\psi_1| \parr \cdots \parr |\psi_k|$.
- The structure of the Seely categories implies a completeness theorem:

  $|\pi| = |\sigma|$ in every Seely category iff $\pi = \sigma$ (modulo cut elimination).

What is a model of lambda calculus?

- Computer scientists prefer category theory. Why?

Lambda Calculus

  - Every λ-term is a "function", but it is also a possible argument of a function (untyped world). Roughly speaking, a model is a set X of element which are also functions:

$$X^X \lhd X$$

  - No model in the category of sets.
  - A model is a reflexive object X in a cartesian closed category: i.e., there is a retraction (Ap, λ) from X^X into X:

$$\lambda : X^X \to X \text{ and } Ap : X \to X^X \text{ such that } Ap \circ \lambda = id_{X^X}.$$

Every reflexive object determines a λ-abstraction algebra and vice versa.

The REL model of linear logic

- $S^\perp = S$
- $S \otimes T = S \times T$ (cartesian product is a tensor in REL) and $1 = \{*\}$
- and so $S \parr T = S \times T$ and $\perp = \{*\}$, because the involution $^\perp$ is the identity and $\parr$ is the de Morgan dual of $\otimes$: $S \parr T = (S^\perp \otimes T^\perp)^\perp$
- $S \,\&\, T = S + T$ (disjoint union is the product in REL) and $\top = \emptyset$
- and so $S \oplus T = S + T$ and $\top = \emptyset$, because the involution $^\perp$ is the identity and & is the de Morgan dual of $\oplus$: $S \,\&\, T = (S^\perp \oplus T^\perp)^\perp$
- $!S = \mathcal{M}_{fin}(S)$ (the finite multisets of elements of S)

The exponential is a *comonad*.

The Kleisli category of a comonad is cartesian closed

The exponential is a *comonad*:

- $! : \text{REL} \to \text{REL}$ is a functor
- The functor ! has a structure of comonad: dereliction $d : ! \Rightarrow id$ and digging $p : ! \Rightarrow !!$ are natural transformations satisfying.....

The Kleisli category $\text{REL}_!$ is cartesian closed

- $\text{Ob}(\text{REL}_!) = \text{Sets}$
- $\text{REL}_!(\text{S},\text{T}) = \text{REL}(!\text{S},\text{T})$
- $f :!S \to T$ and $g :!T \to U$ then $g \cdot f :!S \to U$:

$$g \cdot f =_{def} !S \to^{p_S} !!S \to^{!f} !T \to^{g} U$$

and

$$id_S :!S \to S =_{def} d_S :!S \to S$$

A model of λ-calculus in $\mathbf{REL}_!$

- Let D be any set such that

$$D = (!(D^{\mathbb{N}}))^{\perp}$$

- D is a model of lambda calculus in $\mathbf{REL}_!$:

$$\begin{array}{rcl} D \Rightarrow D & = & !D \multimap D \\ & = & (!D)^{\perp} \parr D \\ & = & (!D)^{\perp} \parr (!(D^{\mathbb{N}}))^{\perp} \\ & = & (!D \otimes !(D^{\mathbb{N}}))^{\perp} \\ & = & (!(D \,\&\, D^{\mathbb{N}}))^{\perp} \\ & = & (!(D^{\mathbb{N}}))^{\perp} \\ & = & D \end{array}$$

In this way we get a λ-calculus model in the Kleisli ccc of an arbitrary categorical model of linear logic such that the equation $X = (!(X^{\mathbb{N}}))^{\perp}$ has a solution.

The theory of the model D

- The interpretation of a closed λ-term

$$|M| : !\top \to D$$

in REL is a subset of D.

- In REL$_!$ $|M|$ is equal to the Taylor expansion of M into the resource lambda calculus. Define

  1. $\mathcal{T}(a) = \{a\}$;
  2. $\mathcal{T}(\lambda a.M) = \{\lambda a.t : t \in \mathcal{T}(M)\}$;
  3. $\mathcal{T}(MN) = \{t[s_1, \ldots, s_n] : t \in \mathcal{T}(M), s_1, \ldots, s_n \in \mathcal{T}(N)\}$

- $|M| = \bigcup_{t \in \mathcal{T}(M)} |t|$ (The semantical validation of the Taylor formula)

- Since *looping terms* ask for infinite resources and resource λ-terms provide only finite resources, then $|M| = \bigcup_{t \in \mathcal{T}(M)} |t|$ implies that $|M| = \emptyset$, for all looping terms (i.e., the equational theory is sensible).

- How to get non-sensible models of λ-calculus? In other words, how to get infinite resources in such a kind of model?

Multisets with infinite multiplicity

Multisets with infinite multiplicity!

- A multiset $m \in !S = \mathcal{M}_{fin}(S)$ is a finite function from S into $\mathbb{N}$. If we add to $\mathbb{N}$ an element ω such that

$$\omega + n = n + \omega = \omega; \quad 0 \cdot \omega = \omega \cdot 0 = 0; \quad n \cdot \omega = \omega \cdot n = \omega$$

  then $m(x) = \omega$ (for some $x \in S$) means that the resource x can be used an infinite number of times.

- We change semiring. We choose any semiring $\mathcal{R}$ such that

  $!_{\mathcal{R}} S = \mathcal{M}^{\mathcal{R}}_{fin}(S)$, where $m \in \mathcal{M}^{\mathcal{R}}_{fin}(S)$ is a finite function from S into $\mathcal{R}$.

  is a comonad.

- Is it possible to axiomatize these semirings? Answer: YES

Multiplicity semi-rings

- A semi-ring $\mathcal{R}$ is a *multiplicity semi-ring* if it is commutative, has a multiplicative unit and satisfies
  - $x + y = 0 \;\Rightarrow\; x = y = 0$ (we say that $\mathcal{R}$ is *positive*)
  - $x + y = 1 \;\Rightarrow\; x = 0$ or $y = 0$ (we say that $\mathcal{R}$ is *discrete*)
  - $x_1 + x_2 = y_1 + y_2 \;\Rightarrow\; \exists$ a matrix $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ such that

    $$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \qquad \begin{pmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

    (we say that M has the *additive splitting property*)
  - $px = y_1 + y_2 \Rightarrow \exists$ a vector $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and a matrix $\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ such that

    $$p = p_1 + p_2; \quad \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \quad \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix}$$

    (we say that M has the *multiplicative splitting property*).

The main theorem

Theorem 1 *If $\mathcal{R}$ is a multiplicity semi-ring, then the $\mathcal{R}$-valued multiset exponential $!_{\mathcal{R}}$ is a comonad in REL.*

New models of linear logic! But does it exists a multiplicity semi-ring? Yes, here three examples:

1. Natural numbers constitute a multiplicity semi-ring.

2. Completed natural numbers with an infinite element ω constitute a multiplicity semi-ring.

3. A more interesting example is determined by the ordinal numbers of type $n\omega^d$ $(n, d \in \mathbb{N})$, where $0\omega^d$ is identified with 0. Addition is defined as follows

$$n\omega^d + k\omega^e = \begin{cases} (n+k)\omega^d & \text{if } d = e \\ n\omega^d & \text{if } n \neq 0 \text{ and } e < d \\ k\omega^e & \text{if } k \neq 0 \text{ and } d < e \end{cases}$$

   and multiplication is defined by $n\omega^d k\omega^e = nk\omega^{d+e}$.

A non-sensible model of λ-calculus in $\mathrm{REL}_{!_{\mathcal{R}}}$

- Let $\mathcal{R}$ be a multiplicity semi-ring with an infinite element ω satisfying

$$\omega + 1 = \omega.$$

- $D = (!(D^{\mathbb{N}}))^{\perp}$ is a model of lambda calculus in $\mathrm{REL}_{!_{\mathcal{R}}}$. Moreover,

$$D = !_{\mathcal{R}} D \to D = (!_{\mathcal{R}} D)^{\perp} \parr D = !_{\mathcal{R}} D \parr D = \mathcal{M}^{\mathcal{R}}_{fin} D \times D$$

We can construct D in such a way that there is an element $x \in D$ such that

$$x = ([\omega x], x)$$

Let $M = (\lambda a.aa)(\lambda a.aa)$ be the famous looping term.

Theorem 2 *$x \in |M|$. Then the model D does not satisfy the Taylor formula:*

$$|M| \neq \bigcup_{t \in \mathcal{T}(M)} |t|$$

and the theory of D is not sensible.