Filter theory of bounded residuated lattice ordered monoids

Dana Šalounová

VŠB–Technical University Ostrava
Czech Republic

dana.salounova@vsb.cz

Algebra and Substructural Logics
Nomi, June 8–10 2010

Outline

- Background
- Filters and congruences
- Types of filters of non-commutative $R\ell$-monoids
- Examples

Bounded $R\ell$-monoids

Definition

A bounded $R\ell$-monoid is an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying:

- $(M; \odot, 1)$ is a monoid;
- $(M; \vee, \wedge, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$;
- $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$.

Bounded $R\ell$-monoids

Remarks

- $(M; \vee, \wedge)$ is distributive,
- $\odot$ distributes over the lattice operations,
- bounded $R\ell$-monoids form a variety of algebras,
- they are a particular case of bounded non-commutative residuated lattices (satisfying divisibility),
- they can be recognized as bounded integral generalized BL-algebras,
- they are FL_w-algebras satisfying divisibility.

Bounded $R\ell$-monoids

Additional unary operations:

$$x^- := x \to 0$$
$$x^\sim := x \rightsquigarrow 0$$

In what follows, an $R\ell$-monoid is a bounded $R\ell$-monoid.

If "$\odot$" is commutative then an $R\ell$-monoid is called commutative.
In such a case, " $\to$ " = " $\rightsquigarrow$ " and "$^-$" = "$^\sim$".

Bounded $R\ell$-monoids

Additional unary operations:

$$x^- := x \to 0$$
$$x^\sim := x \rightsquigarrow 0$$

In what follows, an $R\ell$-monoid is a bounded $R\ell$-monoid.

If "$\odot$" is commutative then an $R\ell$-monoid is called commutative.
In such a case, " $\to$ " = " $\rightsquigarrow$ " and "$^-$" = "$^\sim$".

Bounded $R\ell$-monoids

Additional unary operations:

$$x^- := x \to 0$$
$$x^\sim := x \rightsquigarrow 0$$

In what follows, an $R\ell$-monoid is a bounded $R\ell$-monoid.

If "$\odot$" is commutative then an $R\ell$-monoid is called commutative. In such a case, "$\to$" = "$\rightsquigarrow$" and "$^-$" = "$^\sim$".

Special cases of $R\ell$-monoids

Characterization of the algebras of propositional logics in the class of $R\ell$-monoids:

An $R\ell$-monoid M is

- a pseudo BL-algebra iff

$$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x);$$

algebras of Hájek's non-commutative basic fuzzy logic

- a GMV-algebra (pseudo MV-algebra) iff

$$x^{-\sim} = x = x^{\sim -};$$

algebras of non-commutative Łukasiewicz logic

- a Heyting algebra iff

$$x \odot x = x \quad ('' \odot '' = '' \wedge '').$$

algebras of intuitionistic logic

Special cases of $R\ell$-monoids

Characterization of the algebras of propositional logics in the class of $R\ell$-monoids:

An $R\ell$-monoid M is

- a pseudo BL-algebra iff

$$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x);$$

algebras of Hájek's non-commutative basic fuzzy logic

- a GMV-algebra (pseudo MV-algebra) iff

$$x^{-\sim} = x = x^{\sim -};$$

algebras of non-commutative Łukasiewicz logic

- a Heyting algebra iff

$$x \odot x = x \quad ('' \odot '' = '' \wedge '').$$

algebras of intuitionistic logic

Special cases of $R\ell$-monoids

Characterization of the algebras of propositional logics in the class of $R\ell$-monoids:

An $R\ell$-monoid M is

- a pseudo BL-algebra iff

  $$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x);$$

  algebras of Hájek's non-commutative basic fuzzy logic

- a GMV-algebra (pseudo MV-algebra) iff

  $$x^{-\sim} = x = x^{\sim -};$$

  algebras of non-commutative Łukasiewicz logic

- a Heyting algebra iff

  $$x \odot x = x \quad ('' \odot '' = '' \wedge '').$$

  algebras of intuitionistic logic

Special cases of $R\ell$-monoids

Characterization of the algebras of propositional logics in the class of $R\ell$-monoids:

An $R\ell$-monoid M is

- a pseudo BL-algebra iff

$$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x);$$

algebras of Hájek's non-commutative basic fuzzy logic

- a GMV-algebra (pseudo MV-algebra) iff

$$x^{-\sim} = x = x^{\sim -};$$

algebras of non-commutative Łukasiewicz logic

- a Heyting algebra iff

$$x \odot x = x \quad ('' \odot '' = '' \wedge '').$$

algebras of intuitionistic logic

Properties of bounded $R\ell$-monoids

1. $x \leq y \iff x \to y = 1 \iff x \rightsquigarrow y = 1.$
2. $x \leq y \implies z \to x \leq z \to y, \;\; z \rightsquigarrow x \leq z \rightsquigarrow y.$
3. $x \leq y \implies y \to z \leq x \to z, \;\; y \rightsquigarrow z \leq x \rightsquigarrow z.$
4. $x \to x = 1 = x \rightsquigarrow x, \quad 1 \to x = x = 1 \rightsquigarrow x,$
   $x \to 1 = 1 = x \rightsquigarrow 1.$
5. $(x \to y) \odot x \leq x \leq y \to (x \odot y),$
   $x \odot (x \rightsquigarrow y) \leq y \leq x \rightsquigarrow (x \odot y).$
6. $1^{-\sim} = 1 = 1^{\sim -}, \quad 0^{-\sim} = 0 = 0^{\sim -}.$
7. $x \leq x^{-\sim}, \quad x \leq x^{\sim -}.$
8. $x^{-\sim -} = x^{-}, \quad x^{\sim - \sim} = x^{\sim}.$
9. $x^{-} \odot x = 0 = x \odot x^{\sim}.$
10. $x \leq y \implies y^{-} \leq x^{-}, \quad y^{\sim} \leq x^{\sim}.$

Properties of bounded $R\ell$-monoids

12. $x \leq (x \to y) \rightsquigarrow y, \quad x \leq (x \rightsquigarrow y) \to y.$

13. $y \leq x \to y, \quad y \leq x \rightsquigarrow y.$

14. $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z).$

15. $x \to (y \to z) = (x \odot y) \to z,$
$x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z.$

16. $x \to y \leq (y \to z) \rightsquigarrow (x \to z),$
$x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z).$

17. $y \to z \leq (x \to y) \to (x \to z),$
$y \rightsquigarrow z \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z).$

18. $x \to y \leq y^- \rightsquigarrow x^-, \quad x \rightsquigarrow y \leq y^\sim \to x^\sim.$

19. $x \to (y \wedge z) = (x \to y) \wedge (x \to z),$
$x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge x \rightsquigarrow z).$

Filters

A non-empty subset F of an $R\ell$-monoid M is called a *filter* of M if

(F1) $x, y \in F$ imply $x \odot y \in F$;

(F2) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

References

Filters

A non-empty subset F of an $R\ell$-monoid M is called a *filter* of M if

(F1) $x, y \in F$ imply $x \odot y \in F$;

(F2) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

▸ References

Denote by $\mathcal{F}(M)$ the complete lattice of filters of M.

Infima in $\mathcal{F}(M)$ coincide with intersections.

$\mathcal{F}(M)$ is a complete Heyting algebra, hence

$$G \cap \bigvee_{i \in I} F_i = \bigvee_{i \in I} (G \cap F_i), \text{ for any } G,\ F_i \in \mathcal{F}(M),\ i \in I.$$

Filters and congruences

A non-empty subset F of an $R\ell$-monoid M is called a *filter* of M if

(F1) $x, y \in F$ imply $x \odot y \in F$;

(F2) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

▸ References

A filter F is called *normal* if for each $x, y \in M$

(F3) $x \to y \in F \iff x \rightsquigarrow y \in F$.

normal filters of M $\longleftrightarrow$ kernels of congruences on M;

If F is a normal filter of M, then F is the kernel of the unique congruence $\Theta(F)$ such that

$$\begin{aligned}\langle x, y\rangle \in \Theta(F) &\iff (x \to y) \wedge (y \to x) \in F \\ &\iff (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \in F.\end{aligned}$$

Filters and congruences

A non-empty subset F of an $R\ell$-monoid M is called a *filter* of M if

(F1) $x, y \in F$ imply $x \odot y \in F$;

(F2) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

References

A filter F is called *normal* if for each $x, y \in M$

(F3) $x \to y \in F \iff x \rightsquigarrow y \in F$.

normal filters of M $\longleftrightarrow$ kernels of congruences on M;

If F is a normal filter of M, then F is the kernel of the unique congruence $\Theta(F)$ such that

$$\begin{aligned} \langle x, y \rangle \in \Theta(F) &\iff (x \to y) \wedge (y \to x) \in F \\ &\iff (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \in F. \end{aligned}$$

Filters and deductive systems

Dvurečenskij A., Rachůnek J. (2006)
Probabilistic averaging in bounded $R\ell$-monoids. Semigroup Forum 72.

$H \subseteq M$ is a filter of M iff

H is a deductive system of M

Filters and deductive systems

A subset D of an $R\ell$-monoid M is called a *deductive system* of M if

(d1) $1 \in D$;

(d2) $x \in D,\ x \to y \in D$ imply $y \in D$.

Dvurečenskij A., Rachůnek J. (2006)
Probabilistic averaging in bounded $R\ell$-monoids. Semigroup Forum 72.

$H \subseteq M$ is a filter of M iff

H is a deductive system of M iff

H satisfies (1) and

(d2') $x \in D,\ x \rightsquigarrow y \in D$ imply $y \in D$.

Filters and deductive systems

A subset D of an $R\ell$-monoid M is called a *deductive system* of M if

(d1) $1 \in D$;

(d2) $x \in D,\ x \to y \in D$ imply $y \in D$.

Dvurečenskij A., Rachůnek J. (2006)
Probabilistic averaging in bounded $R\ell$-monoids. Semigroup Forum 72.

$H \subseteq M$ is a filter of M iff

H is a deductive system of M iff

H satisfies (1) and

(d2') $x \in D,\ x \rightsquigarrow y \in D$ imply $y \in D$.

Implicative filters

A subset F of an $R\ell$-monoid M is called an *implicative filter* of M if

(1) $1 \in F$;

(2) $x \to (y \to z) \in F,\ x \rightsquigarrow y \in F$ imply $x \to z \in F$,
$x \rightsquigarrow (y \rightsquigarrow z) \in F,\ x \to y \in F$ imply $x \rightsquigarrow z \in F$.

Proposition

Every implicative filter of an $R\ell$-monoid M is a filter of M.

Implicative filters

A subset F of an $R\ell$-monoid M is called an *implicative filter* of M if

(1) $1 \in F$;

(2) $x \to (y \to z) \in F,\ x \rightsquigarrow y \in F$ imply $x \to z \in F$,
$x \rightsquigarrow (y \rightsquigarrow z) \in F,\ x \to y \in F$ imply $x \rightsquigarrow z \in F$.

Proposition

Every implicative filter of an $R\ell$-monoid M is a filter of M.

Implicative filters

For a filter F of an $R\ell$-monoid M and $a \in M$:

$$M_a := \{x \in M : \ a \to x \in F\}.$$

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then F is an implicative filter of M if and only if M_a is a filter of M for every $a \in M$.

Implicative filters

For a filter F of an $R\ell$-monoid M and $a \in M$:

$$M_a := \{x \in M : \; a \to x \in F\}.$$

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then F is an implicative filter of M if and only if M_a is a filter of M for every $a \in M$.

Implicative filters

Proposition

If F is an implicative filter of an $R\ell$-monoid M, then $x \to (x \odot x) \in F$ and $x \rightsquigarrow (x \odot x) \in F$ for any $x \in M$.

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then F is implicative in M if and only if $x \to (x \odot x) \in F$, for any $x \in M$.

Theorem

If F_1 and F_2 are normal filters of an $R\ell$-monoid M, $F_1 \subseteq F_2$ and F_1 is an implicative filter of M, then F_2 is also an implicative filter of M.

Implicative filters

Proposition

If F is an implicative filter of an $R\ell$-monoid M, then $x \to (x \odot x) \in F$ and $x \rightsquigarrow (x \odot x) \in F$ for any $x \in M$.

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then F is implicative in M if and only if $x \to (x \odot x) \in F$, for any $x \in M$.

Theorem

If F_1 and F_2 are normal filters of an $R\ell$-monoid M, $F_1 \subseteq F_2$ and F_1 is an implicative filter of M, then F_2 is also an implicative filter of M.

Implicative filters – theorems

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then F is implicative in M if and only if $x \to (x \odot x) \in F$, for any $x \in M$.

1.

If F is a normal filter of an $R\ell$-monoid M, then F is an implicative filter of M if and only if the quotient $R\ell$-monoid M/F is a Heyting algebra.

2.

If M is an $R\ell$-monoid, then the following conditions are equivalent.

1. *M is a Heyting algebra.*
2. *Every normal filter of M is implicative.*
3. *$\{1\}$ is an implicative filter of M.*

Implicative filters – theorems

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then F is implicative in M if and only if $x \to (x \odot x) \in F$, for any $x \in M$.

1.

If F is a normal filter of an $R\ell$-monoid M, then F is an implicative filter of M if and only if the quotient $R\ell$-monoid M/F is a Heyting algebra.

2.

If M is an $R\ell$-monoid, then the following conditions are equivalent.

1. *M is a Heyting algebra.*
2. *Every normal filter of M is implicative.*
3. *$\{1\}$ is an implicative filter of M.*

Implicative filters – propositions

1.

If F is an implicative filter of an $R\ell$-monoid M, then

(b) $\forall x, y \in M$;

$$y \to (y \to x) \in F \Longrightarrow y \to x \in F,$$
$$y \rightsquigarrow (y \rightsquigarrow x) \in F \Longrightarrow y \rightsquigarrow x \in F.$$

2.

If F is a normal filter of an $R\ell$-monoid M satisfying (b), then

(c) $\forall x, y, z \in M$;

$$z \to (y \to x) \in F \Longrightarrow (z \rightsquigarrow y) \to (z \rightsquigarrow x) \in F,$$
$$z \rightsquigarrow (y \rightsquigarrow x) \in F \Longrightarrow (z \to y) \rightsquigarrow (z \to x) \in F.$$

Implicative filters – propositions

3.

If a filter F of an $R\ell$-monoid M satisfies (c), then

(d) $\forall x, y, z \in M;$

$$z \to (y \to (y \to x)) \in F,\ z \in F \Longrightarrow y \rightsquigarrow x \in F,$$

$$z \rightsquigarrow (y \rightsquigarrow (y \rightsquigarrow x)) \in F,\ z \in F \Longrightarrow y \to x \in F.$$

4.

If a normal filter F of an $R\ell$-monoid M satisfies (d), then F is implicative.

Implicative filters

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then the following conditions are equivalent.

1. F *is an implicative filter of* M.
2. $y \to (y \to x) \in F \implies y \to x \in F$,
   $y \rightsquigarrow (y \rightsquigarrow x) \in F \implies y \rightsquigarrow x \in F$.
3. $z \to (y \to x) \in F \implies (z \rightsquigarrow y) \to (z \rightsquigarrow x) \in F$,
   $z \rightsquigarrow (y \rightsquigarrow x) \in F \implies (z \to y) \rightsquigarrow (z \to x) \in F$.
4. $z \to (y \to (y \to x)) \in F,\ z \in F \implies y \rightsquigarrow x \in F$,
   $z \rightsquigarrow (y \rightsquigarrow (y \rightsquigarrow x)) \in F,\ z \in F \implies y \to x \in F$.

Boolean filters

A filter F of an $R\ell$-monoid M is called *Boolean* if for any $x \in M$,

(3) $\quad x \vee x^{-}$ and $x \vee x^{\sim} \in F$.

Theorem

If F_1 and F_2 are filters of an $R\ell$-monoid M, $F_1 \subseteq F_2$ and F_1 is a Boolean filter of M, then F_2 is also a Boolean filter of M.

Theorem

If M is an $R\ell$-monoid, then the following conditions are equivalent.

1. *M is a Boolean algebra.*
2. *Every filter of M is Boolean.*
3. *$\{1\}$ is a Boolean filter of M.*

Boolean filters

A filter F of an $R\ell$-monoid M is called *Boolean* if for any $x \in M$,

(3) $\quad x \vee x^{-}$ and $x \vee x^{\sim} \in F$.

Theorem

If F_1 and F_2 are filters of an $R\ell$-monoid M, $F_1 \subseteq F_2$ and F_1 is a Boolean filter of M, then F_2 is also a Boolean filter of M.

Theorem

If M is an $R\ell$-monoid, then the following conditions are equivalent.

1. *M is a Boolean algebra.*
2. *Every filter of M is Boolean.*
3. *$\{1\}$ is a Boolean filter of M.*

Boolean filters

Lemma

If M is an $R\ell$-monoid and $x \in M$ is such that

$$x \vee x^{-} = x \vee x^{\sim} = 1,$$

then $$x \wedge x^{-} = x \wedge x^{\sim} = 0.$$

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then the following conditions are equivalent.

1. *F is a Boolean filter of M.*
2. *M/F is a Boolean algebra.*

Theorem

Every normal Boolean filter of an $R\ell$-monoid M is implicative in M.

Boolean filters

Lemma

If M is an $R\ell$-monoid and $x \in M$ is such that

$$x \vee x^{-} = x \vee x^{\sim} = 1,$$

then $$x \wedge x^{-} = x \wedge x^{\sim} = 0.$$

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then the following conditions are equivalent.

1. *F is a Boolean filter of M.*
2. *M/F is a Boolean algebra.*

Theorem

Every normal Boolean filter of an $R\ell$-monoid M is implicative in M.

Boolean filters

Lemma

If M is an $R\ell$-monoid and $x \in M$ is such that

$$x \vee x^{-} = x \vee x^{\sim} = 1,$$

then $$x \wedge x^{-} = x \wedge x^{\sim} = 0.$$

Theorem

Let F be a normal filter of an $R\ell$-monoid M. Then the following conditions are equivalent.

1. *F is a Boolean filter of M.*
2. *M/F is a Boolean algebra.*

Theorem

Every normal Boolean filter of an $R\ell$-monoid M is implicative in M.

Boolean filters – theorems

1.

Let F be a filter of an $R\ell$-monoid M. Then the following conditions are equivalent.

1. F is a *Boolean filter* of M.
2. $x \to (y^- \rightsquigarrow y) \in F \implies x \to y \in F$,
   $x \rightsquigarrow (y^\sim \to y) \in F \implies x \rightsquigarrow y \in F$.
3. $(x \to y) \rightsquigarrow x \in F \implies x \in F$,
   $(x \rightsquigarrow y) \to x \in F \implies x \in F$.

2.

A filter F of an $R\ell$-monoid M is Boolean if and only

(4) $\forall x \in M;\ (x^- \rightsquigarrow x) \to x \in F$ and $(x^\sim \to x) \rightsquigarrow x \in F$.

Boolean filters – theorems

1.

Let F be a filter of an $R\ell$-monoid M. Then the following conditions are equivalent.

1. F is a *Boolean filter* of M.
2. $x \to (y^- \rightsquigarrow y) \in F \implies x \to y \in F$,
   $x \rightsquigarrow (y^\sim \to y) \in F \implies x \rightsquigarrow y \in F$.
3. $(x \to y) \rightsquigarrow x \in F \implies x \in F$,
   $(x \rightsquigarrow y) \to x \in F \implies x \in F$.

2.

A filter F of an $R\ell$-monoid M is Boolean if and only

(4) $\forall x \in M$; $(x^- \rightsquigarrow x) \to x \in F$ and $(x^\sim \to x) \rightsquigarrow x \in F$.

Positive implicative filters

A subset F of an $R\ell$-monoid M is called an *positive implicative filter* of M if

(1) $1 \in F$;

(5) $x \to ((y \to z) \rightsquigarrow y) \in F,\ x \in F$ imply $y \in F$,
$x \rightsquigarrow ((y \rightsquigarrow z) \to y \in F,\ x \in F$ imply $y \in F$.

A positive implicative filter of M is a filter of M.

Theorem

A filter F of an $R\ell$-monoid M is positive implicative if and only if F is a Boolean filter of M.

Positive implicative filters

A subset F of an $R\ell$-monoid M is called an *positive implicative filter* of M if

(1) $1 \in F$;

(5) $x \to ((y \to z) \rightsquigarrow y) \in F,\ x \in F$ imply $y \in F$,
$x \rightsquigarrow ((y \rightsquigarrow z) \to y \in F,\ x \in F$ imply $y \in F$.

A positive implicative filter of M is a filter of M.

Theorem

A filter F of an $R\ell$-monoid M is positive implicative if and only if F is a Boolean filter of M.

Positive implicative filters

A subset F of an $R\ell$-monoid M is called an *positive implicative filter* of M if

(1) $1 \in F$;

(5) $x \to ((y \to z) \rightsquigarrow y) \in F,\ x \in F$ imply $y \in F$,
$x \rightsquigarrow ((y \rightsquigarrow z) \to y \in F,\ x \in F$ imply $y \in F$.

A positive implicative filter of M is a filter of M.

Theorem

A filter F of an $R\ell$-monoid M is positive implicative if and only if F is a Boolean filter of M.

Corollary

Every normal positive implicative filter of an $R\ell$-monoid M is implicative in M.

Positive implicative filters

A subset F of an $R\ell$-monoid M is called an *positive implicative filter* of M if

(1) $1 \in F$;

(5) $x \to ((y \to z) \rightsquigarrow y) \in F,\ x \in F$ imply $y \in F$,
$x \rightsquigarrow ((y \rightsquigarrow z) \to y \in F,\ x \in F$ imply $y \in F$.

A positive implicative filter of M is a filter of M.

Theorem

A filter F of an $R\ell$-monoid M is positive implicative if and only if F is a Boolean filter of M.

Corollary

Every normal positive implicative filter of an $R\ell$-monoid M is implicative in M.

Maximal and prime filters

Let M be an $R\ell$-monoid and F be a filter of M. Then F is called

- *maximal* if F is a dual atom in the lattice $\mathcal{F}(M)$;
- *prime* if $F \neq M$ and $x \vee y \in F$ implies $x \in F$ or $y \in F$ for any $x, y \in M$.

Dvurečenskij A., Rachůnek J. (2006)
Probabilistic averaging in bounded $R\ell$-monoids. Semigroup Forum 72.

Maximal and prime filters

Let M be an $R\ell$-monoid and F be a filter of M. Then F is called

- *maximal* if F is a dual atom in the lattice $\mathcal{F}(M)$;
- *prime* if $F \neq M$ and $x \vee y \in F$ implies $x \in F$ or $y \in F$ for any $x, y \in M$.

Dvurečenskij A., Rachůnek J. (2006)
Probabilistic averaging in bounded $R\ell$-monoids. Semigroup Forum 72.

Remark

- Every maximal filter is prime.
- If F is a maximal and normal filter of M, then M/F is a linearly ordered MV-algebra.

Maximal and prime filters

Theorem

If F is a normal filter of an $R\ell$-monoid M, then the following conditions are equivalent.

1. F *is a maximal and Boolean filter of* M
2. F *is a maximal and implicative filter of* M.
3. $\forall x, y \in M \setminus F;\ x \to y \in F$ *and* $y \to x \in F$.
4. M/F *is a two-element Boolean algebra.*
5. F *is a prime and Boolean filter of* M.
6. F *is a proper filter of* M *such that* $x \in F$ *or* $x^- \in F$, *and* $x \in F$ *or* $x^\sim \in F$, *for any* $x \in M$.

Fantastic and GMV-filters

Let M be an $R\ell$-monoid and F be a filter of M. Then F is called

- *fantastic* if

$$\begin{aligned} (6) \qquad & y \to x \in F \implies ((x \to y) \rightsquigarrow y) \to x \in F, \\ & y \rightsquigarrow x \in F \implies ((x \rightsquigarrow y) \to y) \rightsquigarrow x \in F; \end{aligned}$$

Fantastic and GMV-filters

Let M be an $R\ell$-monoid and F be a filter of M. Then F is called

- *fantastic* if

$$(6) \qquad \begin{aligned} y \to x \in F &\implies ((x \to y) \rightsquigarrow y) \to x \in F, \\ y \rightsquigarrow x \in F &\implies ((x \rightsquigarrow y) \to y) \rightsquigarrow x \in F; \end{aligned}$$

Haveshki, M., Saeid, A.B., Eslami, E. (2006)
Some types of filters in BL-algebras. Soft Computing 10.

Fantastic and GMV-filters

Let M be an $R\ell$-monoid and F be a filter of M. Then F is called

- *fantastic* if

$$y \to x \in F \implies ((x \to y) \rightsquigarrow y) \to x \in F, \qquad (6)$$
$$y \rightsquigarrow x \in F \implies ((x \rightsquigarrow y) \to y) \rightsquigarrow x \in F;$$

- a *GMV-filter* if for any $x \in M$,

$$x^{-\sim} \to x \in F, \;\; x^{\sim -} \to x \in F. \qquad (7)$$

Proposition

If F is a fantastic filter of an $R\ell$-monoid M, then F is a GMV-filter of M.

Fantastic and GMV-filters

Let M be an $R\ell$-monoid and F be a filter of M. Then F is called

- *fantastic* if

$$(6) \qquad \begin{aligned} y \to x \in F &\implies ((x \to y) \rightsquigarrow y) \to x \in F, \\ y \rightsquigarrow x \in F &\implies ((x \rightsquigarrow y) \to y) \rightsquigarrow x \in F; \end{aligned}$$

- a *GMV-filter* if for any $x \in M$,

$$(7) \qquad x^{-\sim} \to x \in F, \; x^{\sim -} \to x \in F.$$

Proposition

If F is a fantastic filter of an $R\ell$-monoid M, then F is a GMV-filter of M.

Filters of good $R\ell$-monoids

An $R\ell$-monoid M is called *good* if it satisfies the identity

$$x^{-\sim} = x^{\sim -}.$$

Theorem

If F is a filter of a good $R\ell$-monoid M, then the following conditions are equivalent.

1. *F is a fantastic filter of M.*
2. *F is a GMV-filter of M.*
3. $x \to u \in F,\ y \to u \in F \implies ((x \to y) \rightsquigarrow y) \to u \in F,$
   $x \rightsquigarrow u \in F,\ y \rightsquigarrow u \in F \implies ((x \rightsquigarrow y) \to y) \rightsquigarrow u \in F.$
4. $z \to (y \rightsquigarrow x) \in F,\ z \in F \implies ((x \rightsquigarrow y) \to y) \rightsquigarrow x \in F,$
   $z \rightsquigarrow (y \to x) \in F,\ z \in F \implies ((x \to y) \rightsquigarrow y) \to x \in F.$

Boolean, GMV- and fantastic filters

Theorem

Every Boolean filter of an $R\ell$-monoid M (a good $R\ell$-monoid, respectively) is a GMV-filter (a fantastic filter, respectively) of M.

Theorem

Let M be an $R\ell$-monoid (a good $R\ell$-monoid, respectively). Then the following conditions are equivalent.

1. *M is a GMV-algebra.*
2. *Every filter of M is a GMV-filter (a fantastic filter, respectively) of M.*
3. *$\{1\}$ is a GMV-filter (a fantastic filter, respectively) of M.*

Boolean, GMV- and fantastic filters

Theorem

A normal filter F of an $R\ell$-monoid M (of a good $R\ell$-monoid M, respectively) is a GMV-filter (a fantastic filter, respectively) of M if and only if M/F is a GMV-algebra.

Theorem

If a maximal filter F of an $R\ell$-monoid M (of a good $R\ell$-monoid M, respectively) is normal, then F is a GMV-filter (a fantastic filter, respectively) and M/F is linearly ordered.

Examples

Example 1

Let $M_1 = \{0, a, b, 1\}$, the operations "$\odot$" and "$\rightarrow$" be defined by the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

- M_1 is a BL-algebra;
- $F_1 = \{b, 1\}$ is a filter of M_1;

Examples

Example 1

Let $M_1 = \{0, a, b, 1\}$, the operations "$\odot$" and "$\to$" be defined by the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

- M_1 is a BL-algebra;
- $F_1 = \{b, 1\}$ is a filter of M_1;
- $a \to (a \odot a) = a \notin F_1$, hence F_1 is not implicative.

Examples

Example 1

Let $M_1 = \{0, a, b, 1\}$, the operations "$\odot$" and "$\rightarrow$" be defined by the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

- M_1 is a BL-algebra;
- $F_1 = \{b, 1\}$ is a filter of M_1;
- $a \rightarrow (a \odot a) = a \notin F_1$, hence F_1 is not implicative.

Examples

Example 2

Consider the real interval $[0,1]$ and define

$$x \odot y = \min\{x, y\} = x \wedge y \ \text{ and } \ x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y. \end{cases}$$

- $M_2 = [0,1]$ is a BL-algebra;
- $F_2 = [\frac{1}{2}, 1]$ is a filter of M_2;

Examples

Example 2

Consider the real interval $[0,1]$ and define

$$x \odot y = \min\{x, y\} = x \wedge y \text{ and } x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y. \end{cases}$$

- $M_2 = [0,1]$ is a BL-algebra;
- $F_2 = [\frac{1}{2}, 1]$ is a filter of M_2;
- $x \to (x \odot x) = x \to (x \wedge x) = 1$ for any $x \in M_2$, it is implicative;

Examples

Example 2

Consider the real interval $[0,1]$ and define

$$x \odot y = \min\{x, y\} = x \wedge y \text{ and } x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y. \end{cases}$$

- $M_2 = [0,1]$ is a BL-algebra;
- $F_2 = [\frac{1}{2}, 1]$ is a filter of M_2;
- $x \to (x \odot x) = x \to (x \wedge x) = 1$ for any $x \in M_2$, it is implicative;
- if $0 < x < \frac{1}{2}$, then $x \vee x^- = x \notin F_2$,

Examples

Example 2

Consider the real interval $[0, 1]$ and define

$$x \odot y = \min\{x, y\} = x \wedge y \quad \text{and} \quad x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y. \end{cases}$$

- $M_2 = [0, 1]$ is a BL-algebra;
- $F_2 = [\frac{1}{2}, 1]$ is a filter of M_2;
- $x \to (x \odot x) = x \to (x \wedge x) = 1$ for any $x \in M_2$, it is implicative;
- if $0 < x < \frac{1}{2}$, then $x \vee x^{-} = x \notin F_2$, therefore F_2 is not Boolean, consequently it is not positive implicative.

Examples

Example 2

Consider the real interval $[0,1]$ and define

$$x \odot y = \min\{x,y\} = x \wedge y \text{ and } x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y. \end{cases}$$

- $M_2 = [0,1]$ is a BL-algebra;
- $F_2 = [\frac{1}{2}, 1]$ is a filter of M_2;
- $x \rightarrow (x \odot x) = x \rightarrow (x \wedge x) = 1$ for any $x \in M_2$, it is implicative;
- if $0 < x < \frac{1}{2}$, then $x \vee x^- = x \notin F_2$, therefore F_2 is not Boolean, consequently it is not positive implicative.

Examples

Example 3

Let $M_3 = \{0, a, b, 1\}$, the operations "$\odot$" and "$\rightarrow$" be defined by the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

- M_3 is a BL-algebra;
- $F_3 = \{1\}$ is a fantastic filter of M_3;

Examples

Example 3

Let $M_3 = \{0, a, b, 1\}$, the operations "$\odot$" and "$\to$" be defined by the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

- M_3 is a BL-algebra;
- $F_3 = \{1\}$ is a fantastic filter of M_3;
- e.g. $b \vee b^- = b \notin F_3$, hence F_3 is not positive implicative.

Examples

Example 3

Let $M_3 = \{0, a, b, 1\}$, the operations "$\odot$" and "$\rightarrow$" be defined by the following tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

- M_3 is a BL-algebra;
- $F_3 = \{1\}$ is a fantastic filter of M_3;
- e.g. $b \vee b^- = b \notin F_3$, hence F_3 is not positive implicative.

Examples

Remark

The operations "$\oplus$" and "$\odot$" in GMV-algebras are mutually dual and the dual algebra to any GMV-algebra is a GMV-algebra as well. The notions as "filter", "normal filter", etc., will be replaced by the dual notions "ideal", "normal ideal", etc.

Example 4

Let G be the multiplicative group of all matrices of the form

$$X = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \text{ where } x, y \in \mathbb{R},\ x > 0.$$

Denote $X = (x, y)$, then $X^{-1} = \left(\frac{1}{x}, -\frac{y}{x}\right)$, $E = (1, 0)$.

$G^+ = \{(x, y) :\ x > 1\} \cup \{(x, y) :\ x = 1,\ y \geq 0\}$.

Then G with G^+ is a linearly ordered group,

$U = (2, 0)$ is a strong unit of G.

Examples

Remark

The operations "$\oplus$" and "$\odot$" in GMV-algebras are mutually dual and the dual algebra to any GMV-algebra is a GMV-algebra as well. The notions as "filter", "normal filter", etc., will be replaced by the dual notions "ideal", "normal ideal", etc.

Example 4

Let G be the multiplicative group of all matrices of the form

$$X = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \text{ where } x, y \in \mathbb{R},\ x > 0.$$

Denote $X = (x, y)$, then $X^{-1} = \left(\frac{1}{x}, -\frac{y}{x}\right)$, $E = (1, 0)$.

$G^{+} = \{(x, y) : x > 1\} \cup \{(x, y) : x = 1, y \geq 0\}$.

Then G with G^{+} is a linearly ordered group,

$U = (2, 0)$ is a strong unit of G.

Examples

Remark

The operations "$\oplus$" and "$\odot$" in GMV-algebras are mutually dual and the dual algebra to any GMV-algebra is a GMV-algebra as well. The notions as "filter", "normal filter", etc., will be replaced by the dual notions "ideal", "normal ideal", etc.

Example 4

Let G be the multiplicative group of all matrices of the form

$$X = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \text{ where } x, y \in \mathbb{R},\ x > 0.$$

Denote $X = (x, y)$, then $X^{-1} = \left(\frac{1}{x}, -\frac{y}{x}\right)$, $E = (1, 0)$.

$G^+ = \{(x, y) :\ x > 1\} \cup \{(x, y) :\ x = 1,\ y \geq 0\}$.

Then G with G^+ is a linearly ordered group,

$U = (2, 0)$ is a strong unit of G.

Examples

$A = \Gamma(G, U)$ is a linearly ordered GMV-algebra.

$A = A_1 \cup A_2 \cup A_3$, where
$A_1 = \{(1, y) : \; y \geq 0\}, \; A_2 = \{(x, y) : \; 1 < x < 2\},$
$A_3 = \{(2, y) : \; y \leq 0\}.$
A_1 is a maximal and normal ideal of A.

Denote $M = A_1 \cup A_3$, M is a non-commutative subalgebra of A, the operations can be described as follows:

Let $(x, y), \; (v, w) \in M$. Then

$$(x, y) \oplus (v, w) = \begin{cases} (1, y + u), & \text{if } x = v = 1, \\ (2, xw + y), & \text{if } xv = 1 \text{ and } xw + y \leq 0, \\ (2, 0), & \text{otherwise}, \end{cases}$$

$$(x, y)^- = \left(\frac{2}{x}, -\frac{2y}{x}\right), \quad (x, y)^\sim = \left(\frac{2}{x}, -\frac{y}{x}\right).$$

Examples

$A = \Gamma(G, U)$ is a linearly ordered GMV-algebra.

$A = A_1 \cup A_2 \cup A_3$, where
$A_1 = \{(1, y) : \ y \geq 0\}$, $A_2 = \{(x, y) : \ 1 < x < 2\}$,
$A_3 = \{(2, y) : \ y \leq 0\}$.
A_1 is a maximal and normal ideal of A.

Denote $M = A_1 \cup A_3$, M is a non-commutative subalgebra of A, the operations can be described as follows:

Let (x, y), $(v, w) \in M$. Then

$$(x, y) \oplus (v, w) = \begin{cases} (1, y + u), & \text{if } x = v = 1, \\ (2, xw + y), & \text{if } xv = 1 \text{ and } xw + y \leq 0, \\ (2, 0), & \text{otherwise,} \end{cases}$$

$$(x, y)^- = \left(\frac{2}{x}, -\frac{2y}{x}\right), \ (x, y)^\sim = \left(\frac{2}{x}, -\frac{y}{x}\right).$$

Examples

A_1 is a normal ideal also in the GMV-algebra M.

The ideal A_1 is Boolean:

- If $(1, y) \in A_1$, then (since A_1 is an ideal of M) $(1, y) \wedge (1, y)^{-}, \ (1, y) \wedge (1, y)^{\sim} \in A_1$.
- Let $(2, y) \in A_3$, i.e. $y \leq 0$. Then $(2, y)^{-} = (1, -y)$, since $-y \geq 0$, we have $(2, y)^{-} \in A_1$.

  Similarly, $(2, y)^{\sim} = \left(1, -\frac{y}{2}\right), \ -\frac{y}{2} \geq 0$, hence $(2, y)^{\sim} \in A_1$.

That means A_1 is a normal Boolean ideal of M and, moreover, since A_1 is a maximal ideal of M, we get that the quotient algebra M/A_1 is a two-element Boolean algebra.

Examples

A_1 is a normal ideal also in the GMV-algebra M.

The ideal A_1 is Boolean:

- If $(1, y) \in A_1$, then (since A_1 is an ideal of M) $(1, y) \wedge (1, y)^-, \ (1, y) \wedge (1, y)^\sim \in A_1$.
- Let $(2, y) \in A_3$, i.e. $y \leq 0$. Then $(2, y)^- = (1, -y)$, since $-y \geq 0$, we have $(2, y)^- \in A_1$.

  Similarly, $(2, y)^\sim = \left(1, -\frac{y}{2}\right), \ -\frac{y}{2} \geq 0$, hence $(2, y)^\sim \in A_1$.

That means A_1 is a normal Boolean ideal of M and, moreover, since A_1 is a maximal ideal of M, we get that the quotient algebra M/A_1 is a two-element Boolean algebra.

Examples

A_1 is a normal ideal also in the GMV-algebra M.

The ideal A_1 is Boolean:

- If $(1, y) \in A_1$, then (since A_1 is an ideal of M) $(1, y) \wedge (1, y)^-,\ (1, y) \wedge (1, y)^\sim \in A_1$.
- Let $(2, y) \in A_3$, i.e. $y \le 0$. Then $(2, y)^- = (1, -y)$, since $-y \ge 0$, we have $(2, y)^- \in A_1$.

  Similarly, $(2, y)^\sim = \left(1, -\frac{y}{2}\right),\ -\frac{y}{2} \ge 0$, hence $(2, y)^\sim \in A_1$.

That means A_1 is a normal Boolean ideal of M and, moreover, since A_1 is a maximal ideal of M, we get that the quotient algebra M/A_1 is a two-element Boolean algebra.

Examples

A_1 is a normal ideal also in the GMV-algebra M.

The ideal A_1 is Boolean:

- If $(1, y) \in A_1$, then (since A_1 is an ideal of M) $(1, y) \wedge (1, y)^-, \ (1, y) \wedge (1, y)^\sim \in A_1$.
- Let $(2, y) \in A_3$, i.e. $y \leq 0$. Then $(2, y)^- = (1, -y)$, since $-y \geq 0$, we have $(2, y)^- \in A_1$.

  Similarly, $(2, y)^\sim = \left(1, -\dfrac{y}{2}\right), \ -\dfrac{y}{2} \geq 0$, hence $(2, y)^\sim \in A_1$.

That means A_1 is a normal Boolean ideal of M and, moreover, since A_1 is a maximal ideal of M, we get that the quotient algebra M/A_1 is a two-element Boolean algebra.

Examples

A_1 is a normal ideal also in the GMV-algebra M.

The ideal A_1 is Boolean:

- If $(1, y) \in A_1$, then (since A_1 is an ideal of M) $(1, y) \wedge (1, y)^-, \ (1, y) \wedge (1, y)^\sim \in A_1$.
- Let $(2, y) \in A_3$, i.e. $y \leq 0$. Then $(2, y)^- = (1, -y)$, since $-y \geq 0$, we have $(2, y)^- \in A_1$.

  Similarly, $(2, y)^\sim = \left(1, -\dfrac{y}{2}\right), \ -\dfrac{y}{2} \geq 0$, hence $(2, y)^\sim \in A_1$.

That means A_1 is a normal Boolean ideal of M and, moreover, since A_1 is a maximal ideal of M, we get that the quotient algebra M/A_1 is a two-element Boolean algebra.

References

Turunen E. (2001)
Boolean deductive systems of BL-algebras. Arch. Math. Logic 40.

Haveshki M., Saeid A.B., Eslami E. (2006)
Some types of filters in BL-algebras. Soft. Comput. 10.

Kondo M., Dudek W.A. (2007)
Filter theory of BL-algebras. Soft. Comput.

Zhang, X.H., Li, W.H. (2006)
On pseudo-BL algebras and BCC-algebras. Soft Comput. 10.

Rachůnek J., Šalounová D. (2005)
Boolean deductive systems of bounded commutative residuated ℓ-monoids. Contrib. Gen. Algebra. 16.

filter