An algebraic proof of the γ-admissibility of relevant modal logics

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Relevant modal logics

- Relevant modal logic is modal logic based on relevant logic.
  - Relevant logics are regarded as substructural logics without weakening. In (propositional) relevant logics, if A→B is a theorem, then A and B contain common propositional variables.
  - Modal logics are formalized by a few kinds of necessity-like and possibility-like operators.
  - Recently, many studies on modal logics based on non-classical logics have appeared.

Ackermann's rule γ

$$\frac{\sim A \vee B \quad A}{B}$$

Note: In relevant logic,
$A \rightarrow B$
is ***not*** equivalent to
$\sim A \vee B$

- The γ-admissibility is one of the important problems in the area of relevant logics.
- The rule γ is admissible for relevant logics **E** and **R** (Meyer and Dunn, 1969), but is not for **LR** (Meyer et al, 1984).

Methods of proving γ-admissibility

- Algebraic method (Meyer and Dunn 1969)
- Normal model based on Routley-Meyer semantics (Routley and Meyer 1972, Routley and Meyer 1973, …)
- Metavaluations (Meyer 1976, Mares and Meyer 1992, Mares 1993, …)

Themes of this talk

- When Meyer and Dunn’s paper was published (1969), the algebraic technique on relevant logics had not developed enough.
  - They did not use constants and used matrices.
  - Residuated structures in relevant logics were not found at that time. (They were found in 1970s.)
- We discuss an application of Meyer and Dunn’s technique to modern algebraic models for (wider class of) relevant modal logics.

Language

a. Propositional variables
b. Logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\circ$, $\sim$
c. Modal operators: $\Box$, $\Diamond$
   (We don't adopt the usual definition $\Diamond A = \sim\Box\sim A$.)
d. Constant: **t**.

- Prop and Wff denote the set of propositional variables and formulas, respectively.

Relevant modal logic $\mathbf{G.N}_{\Box\Diamond}$ (1)

- Axioms

(A1) $A \to A$

(A2) $A \wedge B \to A$

(A3) $A \wedge B \to B$

(A4) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

(A5) $A \to A \vee B$

(A6) $B \to A \vee B$

(A7) $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

(A8) $A \wedge (B \vee C) \to (A \wedge B) \vee C$

(A9) $\sim\sim A \to A$

(A10) $A \vee \sim A$

(A11) $\mathbf{t}$

(A12) $\Box A \wedge \Box B \to \Box(A \wedge B)$

(A13) $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$

(A14) $\Box \mathbf{t}$

Relevant modal logic $\mathbf{G.N}_{\Box\Diamond}$ (2)

- Rules of inference

$$\text{(R1)}\ \frac{A\to B \qquad A}{B} \qquad \text{(R2)}\ \frac{A \qquad B}{A\wedge B} \qquad \text{(R3)}\ \frac{A\to B \qquad C\to D}{(B\to C)\to(A\to D)}$$

$$\text{(R4)}\ \frac{A\to(B\to C)}{A\circ B\to C} \qquad \text{(R5)}\ \frac{A\circ B\to C}{A\to(B\to C)} \qquad \text{(R6)}\ \frac{A\to\sim B}{B\to\sim A}$$

$$\text{(R7)}\ \frac{A}{\mathbf{t}\to A} \qquad \text{(R8)}\ \frac{A\to B}{\Box A\to\Box B} \qquad \text{(R9)}\ \frac{A\to B}{\Diamond A\to\Diamond B} \qquad \text{(R10)}\ \frac{A}{\Diamond A}$$

- Note that the rule of Necessitation is derivable in $\mathbf{G.N}_{\Box\Diamond}$ since $\Box\mathbf{t}$ is an axiom of $\mathbf{G.N}_{\Box\Diamond}$. On the other hand, $\Box(A\to B)\to(\Box A\to\Box B)$ is not a theorem of $\mathbf{G.N}_{\Box\Diamond}$.

$\mathbf{G.N}_{\Box\Diamond}$-algebra

- A structure $\mathbf{M} = \langle M, \cap, \cup, \rightarrow, \cdot . -, \Box, \Diamond, 1 \rangle$ is called an **$\mathbf{G.N}_{\Box\Diamond}$-algebra** if it satisfies the following postulates, where $\leq$ denotes the lattice order:

(a1) $\langle M, \cap, \cup \rangle$ is a distributive lattice.

(a2) $x \cdot (y \cup z) = (x \cdot y) \cup (x \cdot z)$

(a3) $(x \cup y) \cdot z = (x \cdot z) \cup (y \cdot z)$

(a4) $x \cdot y \leq z$ iff $x \leq y \rightarrow z$

(a5) $x \cup y = -(-x \cap -y)$

(a6) $1 \leq x \cup -x$

(a7) $1 \cdot x = x$

(a8) $\Box(x \cap y) = \Box x \cap \Box y$

(a9) $\Diamond(x \cup y) = \Diamond x \cup \Diamond y$

(a10) $1 \leq x \;\Rightarrow\; 1 \leq \Diamond x$

(a11) $1 \leq \Box 1$

Valuation and interpretation

- For any $\mathbf{G.N}_{\Box\Diamond}$-algebra $\mathbf{M} = \langle M, \cap, \cup, \rightarrow, \cdot, -, \Box, \Diamond, 1 \rangle$, a mapping v : Prop→M is called a valuation on $\mathfrak{M}$.
- Given a valuation v on $\mathbf{M}$, a mapping I : Wff→M, called the interpretation associated with v, is defined as follows:

(i) For p∈Prop, $I(p) = v(p)$

(ii) $I(A \wedge B) = I(A) \cap I(B)$

(iii) $I(A \vee B) = I(A) \cup I(B)$

(iv) $I(A \rightarrow B) = I(A) \rightarrow I(B)$

(v) $I(A \circ B) = I(A) \cdot I(B)$

(vi) $I({\sim}A) = -I(A)$

(vii) $I(\Box A) = \Box I(A)$

(viii) $I(\Diamond A) = \Diamond I(A)$

(ix) $I(\mathbf{t}) = 1$.

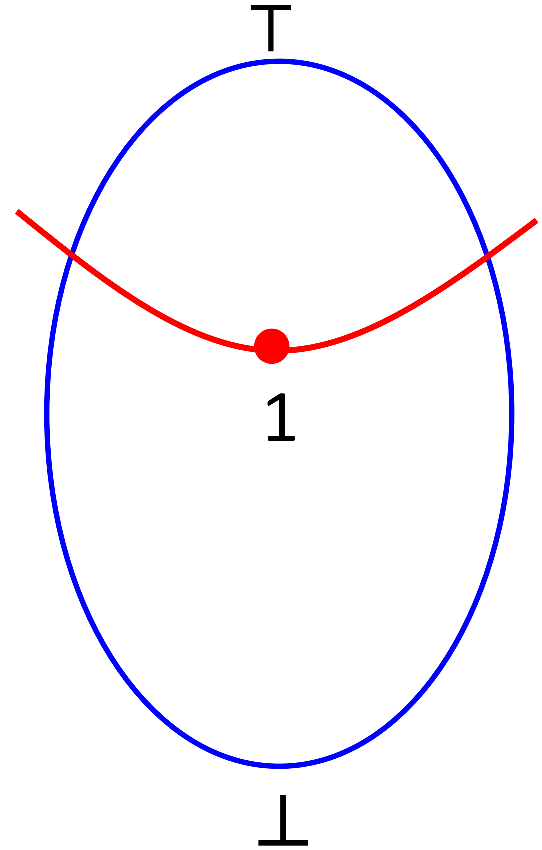
Validity

- Let $\mathbf{M}$ be an **G.N**$_{\Box\Diamond}$-algebra, *v* be a valuation on $\mathbf{M}$, and *I* be the interpretation associated with *v*. Then we say

(a) A is valid in *v* iff $1 \leq I(A)$.

(b) A is $\mathbf{M}$-valid iff

A is valid in any *v*.



Filters and prime filters

Let $\mathbf{M}$ = <M,∩,∪,→,·,−,□,◇,1> be an $\mathbf{G.N}_{\Box\Diamond}$-algebra.

- A non-empty subset H of M is called a filter if it satisfies the following. For any $x,y \in M$:
  - (1) $x \in H \ \&\ x \leq y \ \Rightarrow \ y \in H$
  - (2) $x \in H \ \&\ y \in H \ \Rightarrow \ x \cap y \in H$.
- A filter H of $\mathbf{M}$ is called prime if it satisfies the following. For any $x,y \in M$:

$$x \cup y \in H \ \Rightarrow \ x \in H \ \textit{or} \ y \in H.$$

Prime $\mathbf{G.N}_{\Box\Diamond}$-algebra and completeness

Let $\mathbf{M}$ = <M,∩,∪,→,·,−,□,◇,1> be an $\mathbf{G.N}_{\Box\Diamond}$-algebra.

- The filter { x∈M | 1 ≤ x } is called the P-filter.
- $\mathbf{M}$ is prime if its P-filter is prime.

<u>Theorem 1</u> The following are equivalent.

(i) A is a theorem of $\mathbf{G.N}_{\Box\Diamond}$.

(ii) A is $\mathbf{M}$-valid for all $\mathbf{G.N}_{\Box\Diamond}$-algebras $\mathbf{M}$.

(iii) A is $\mathbf{M}$-valid for all prime $\mathbf{G.N}_{\Box\Diamond}$-algebras $\mathbf{M}$.

Prime algebras are insufficient to prove the γ-admissibility

- Theorem 1 is insufficient for the proof of the γ-admissibility because the following situation might occur:

  Both A and $\sim A \vee B$ are $\mathbf{M}$-valid for all prime $\mathbf{G.N}_{\Box\Diamond}$-algebras $\mathbf{M}$, but there is an interpretation *I* for a prime $\mathbf{G.N}_{\Box\Diamond}$-algebra $\mathbf{M}$ on which B is not $\mathbf{M}$-valid.

- In that case both $1 \leq I(A)$ and $1 \leq -I(A)$ hold.

Avoid this situation

Normal algebra

Let $\mathbf{M}$ = <M,∩,∪,→,·,−,□,◇,1> be an $\mathbf{G.N}_{\Box\Diamond}$-algebra.

- $\mathbf{M}$ is prime iff its P-filter { x∈M | 1 ≤ x } is prime.
- A filter H of $\mathbf{M}$ is called consistent if −1∉H.
- $\mathbf{M}$ is consistent iff its P-filter is consistent.
- $\mathbf{M}$ is normal iff $\mathbf{M}$ is both prime and consistent.

Note: In a normal $\mathbf{G.N}_{\Box\Diamond}$-algebra $\mathbf{M}$ = <M,∩,∪,→,·,−,□,◇,1>, 1 ≤ −x iff 1 ≰ x for every x∈M.

From the logical view, ∼A is a theorem iff A is not a theorem.

Partitioning elements of prime **G.N**$_{\Box\Diamond}$-algebra

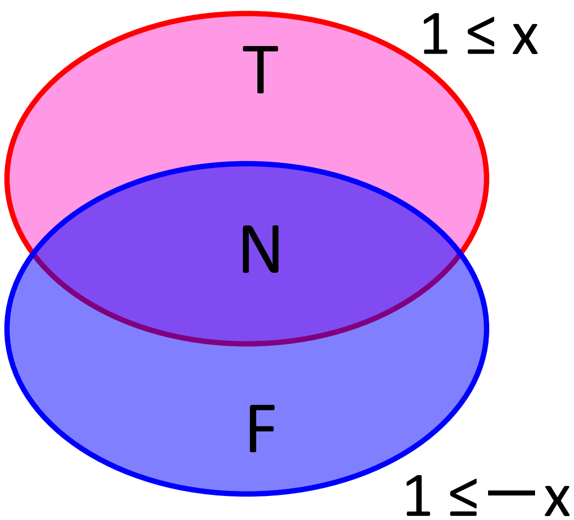
Let $\mathbf{M}$ = <M,∩,∪,→,·,−,□,◇,1> be a prime **G.N**$_{\Box\Diamond}$-algebra.

- Since $\mathbf{M}$ is prime and satisfies $1 \leq x \cup -x$, there is no $x \in M$ satisfying both $1 \not\leq x$ and $1 \not\leq -x$.
- All elements of M partition into the following:

$$T = \{ x \in M \mid 1 \leq x \ \&\ 1 \not\leq -x \}$$

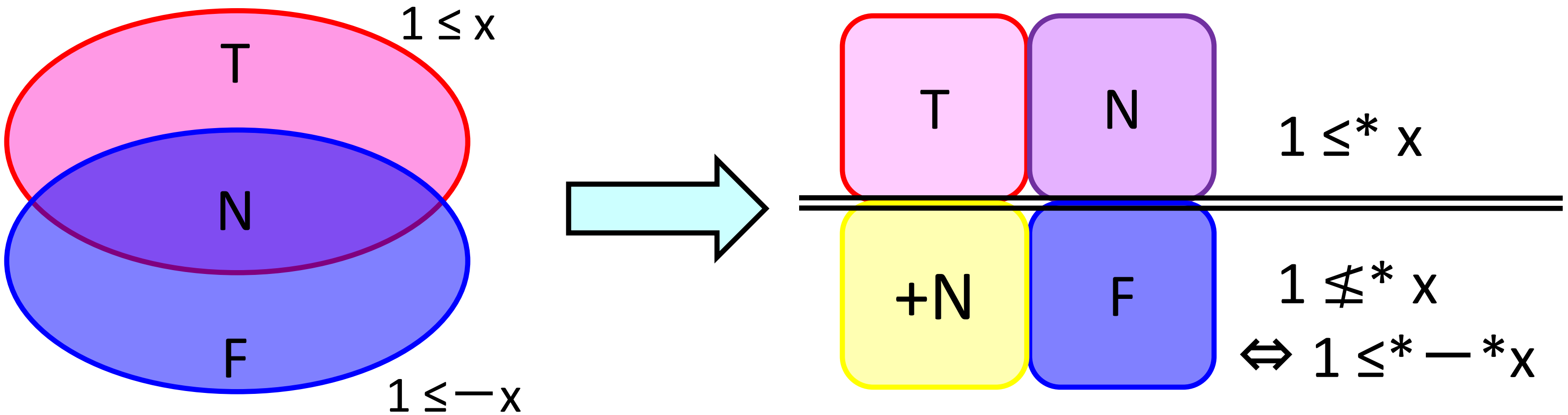
$$N = \{ x \in M \mid 1 \leq x \ \&\ 1 \leq -x \}$$

$$F = \{ x \in M \mid 1 \not\leq x \ \&\ 1 \leq -x \}$$



Normalization of prime **G.N**$_{\Box\Diamond}$-algebra

- Let +N = { +x | x∈N } and M* = M∪+N, where M∩+N = ∅.



- An onto function *h* : M*→M is defined as follows:

 (i) for x∈M, *h*(x) = x

(ii) for x∈+N, *h*(x) = y, where x = +y

Normalization of prime **G.N**$_{\Box\Diamond}$-algebra and four-valued truth tables (1)

- Operations $\cap^*, \cup^*, \rightarrow^*, \cdot^*, -^*, \Box^*, \Diamond^*$ on M* are defined as follows. For $x, y \in M^*$:

(i) $x \cap^* y = \begin{cases} +(h(x) \cap h(y)), & \text{if } (x \in +N \text{ or } y \in +N),\ x \notin F,\ y \notin F \\ h(x) \cap h(y), & \text{otherwise} \end{cases}$

(ii) $x \cup^* y = \begin{cases} +(h(x) \cup h(y)), & \text{if } (x \in +N \text{ or } y \in +N),\ x \notin T \cup N,\ y \notin T \cup N \\ h(x) \cup h(y), & \text{otherwise} \end{cases}$

∩*	T	N	+N	F
T	T	N	+N	F
N	N	N	+N	F
+N	+N	+N	+N	F
F	F	F	F	F

∪*	T	N	+N	F
T	T	T	T	T
N	T	N	N	N
+N	T	N	+N	+N
F	T	N	+N	F

Normalization of prime **G.N**$_{\Box\Diamond}$-algebra and four-valued truth tables (2)

(iii) $x \to^* y = \begin{cases} +(h(x) \to h(y)), & \text{if } x \in N,\ y \in +N,\ h(x) \to h(y) \in N \\ h(x) \to h(y), & \text{otherwise} \end{cases}$

(iv) $x \cdot^* y = \begin{cases} +(h(x) \cdot h(y)), & \text{if } (x \in +N \cup F \text{ or } y \in +N \cup F),\ h(x) \cdot h(y) \in N \\ h(x) \cdot h(y), & \text{otherwise} \end{cases}$

→*	T	N	+N	F
T	M	F	F	F
N	M	M	M*—N	F
+N	M	M	M	F
F	M	M	M	M

･*	T	N	+N	F
T	T	T∪N	T∪+N	M*—N
N	T	T∪N	T∪+N	M*—N
+N	T	T∪+N	T∪+N	M*—N
F	M*—N	[illegible]	M*—N	M*—N

If x∈N and y∈T, then x･*y∈T
(If 1 ≤* x, 1 ≤*—*x, 1 ≤* y and 1 ≰*—*y, then 1 ≤* x･*y and 1 ≰*—*(x･*y))

Normalization of prime $\mathbf{G.N}_{\Box\Diamond}$-algebra and four-valued truth tables (3)

(v) $-^*x = \begin{cases} +(-h(x)), \text{ if } x \in N \\ -h(x), \text{ otherwise} \end{cases}$

(vi) $\Box^*x = \begin{cases} +(\Box h(x)), \text{ if } x \in +N \\ \Box h(x), \text{ otherwise} \end{cases}$

(vii) $\Diamond^*x = \begin{cases} +(\Diamond h(x)), \text{ if } x \in +N \cup F,\ \Diamond h(x) \in N \\ \Diamond h(x), \text{ otherwise} \end{cases}$

	$-^*$	$\Box^*$	$\Diamond^*$
T	F	T∪N	T∪N
N	+N	T∪N	T∪N
+N	N	T∪+N	T∪+N
F	T	M	M*−N

<u>Proposition 2</u> Define a binary relation $\leq^*$ on M* by $x \leq^* y \Leftrightarrow x \cap^* y = x$. Then $1 \leq^* x$ iff $x \in T \cup N$.

Key Lemmas (1)

<u>Lemma 3</u> $\mathbf{M}^*$ = <M*,∩*,∪*,→*,·*,−*,□*,◇*,1> is a normal $\mathbf{G.N}_{\Box\Diamond}$-algebra.

Proof. We omit a proof that $\mathbf{M}^*$ is a $\mathbf{G.N}_{\Box\Diamond}$-algebra.

Let H =

Proposit

Note that without (a10) [1 ≤ x ⇒ 1 ≤ ◇x], a proof for checking (a9) [◇(x∪y) = ◇x∪◇y] does not work well.

To show that $\mathbf{M}^*$ is prime, suppose x∪*y∈H = T∪N. Then x∈T∪N = H *or* y∈T∪N = H.

- $\mathbf{M}^*$ is prime iff its P-filter H is prime.
  (i.e., x∪y∈H ⇒ x∈H *or* y∈H.)
- $\mathbf{M}^*$ is consistent iff its P-filter H is consistent.
  (i.e., −1∉H.)
- $\mathbf{M}^*$ is normal iff $\mathbf{M}^*$ is both prime and consistent.

∪*	T	N	+N	F
T	T	T	T	T
N	T	N	N	N
+N	T	N	+N	+N
F	T	N	+N	F

Key Lemmas (1)

Lemma 3 $\mathbf{M}^*$ = <M*,∩*,∪*,→*,·*,−*,□*,◇*> is a normal $\mathbf{G.N}_{\square\Diamond}$-algebra.

	−*
T	F
N	+N
+N	N
F	T

Proof. We omit a proof that $\mathbf{M}^*$ is a $\mathbf{G.N}_{\square\Diamond}$-algebra.

Let H = { x∈M* | 1 ≤* x } be the P-filter of $\mathbf{M}^*$. By Proposition 2, H = T∪N.

To show that $\mathbf{M}^*$ is prime, suppose x∪*y∈H = T∪N. Then x∈T∪N = H *or* y∈T∪N = H.

To show that $\mathbf{M}^*$ is consistent, assume −*1∈H = T∪N. Then 1∈+N∪F. This is a contradiction.

■

- $\mathbf{M}^*$ is prime iff its P-filter H is prime. (i.e., x∪y∈H ⇒ x∈H *or* y∈H.)
- $\mathbf{M}^*$ is consistent iff its P-filter H is consistent. (i.e., −1∉H.)
- $\mathbf{M}^*$ is normal iff $\mathbf{M}^*$ is both prime and consistent.

M∩+N = ∅
F = { x∈M | 1 ≰ x & 1 ≤ −x }

Key Lemmas (2)

Lemma 4 If A is not a theorem of $\mathbf{G.N}_{\Box\Diamond}$, then there exists a normal $\mathbf{G.N}_{\Box\Diamond}$-algebra $\mathbf{M}^*$ = <M*,∩*,∪*,→*, ·*,−*,□*, ◇*,1> such that A is not $\mathbf{M}^*$-valid.

Proof. By Theorem 1, there exists a prime $\mathbf{G.N}_{\Box\Diamond}$-algebra $\mathbf{M}$ and the interpretation *I* associated with a valuation *v* on $\mathbf{M}$ such that $1 \not\leq I(A)$.

Constructing $\mathbf{M}^*$ from $\mathbf{M}$ as discussed before, $\mathbf{M}^*$ is a normal $\mathbf{G.N}_{\Box\Diamond}$-algebra by Lemma 3.

Now a valuation v^* on $\mathbf{M}^*$ is defined by $v^*(p) = v(p)$ for every $p \in$ Prop. For the interpretation I^* associated with v^*, we have $I^*(B) = I(B)$ for every $B \in$ Wff. Then $1 \not\leq^* I^*(A)$. ■

Main Result

<u>Theorem 5</u> The following are equivalent.

(i) A is a theorem of $\mathbf{G.N}_{\Box\Diamond}$.

(ii) A is $\mathbf{M}$-valid for all normal $\mathbf{G.N}_{\Box\Diamond}$-algebras $\mathbf{M}$.

<u>Theorem 6</u> The rule γ is admissible in $\mathbf{G.N}_{\Box\Diamond}$.

Proof. Suppose that both A and $\sim A \vee B$ are theorems of $\mathbf{G.N}_{\Box\Diamond}$ but B is not a theorem of $\mathbf{G.N}_{\Box\Diamond}$.

By Theorem 5, there exists a normal $\mathbf{G.N}_{\Box\Diamond}$-algebra $\mathbf{M}$ and the interpretation I associated with a valuation on $\mathbf{M}$ such that $1 \not\leq I(B)$. Then $1 \leq I(A)$ and $1 \leq I(\sim A \vee B)$. Since $\mathbf{M}$ is normal, $1 \leq -I(A)$ or $1 \leq I(B)$. This is a contradiction. ■

Conclusion

- Meyer and Dunn’s algebraic method for proving the γ-admissibility has been revived in modern algebraic models of relevant logics and extended to relevant modal logics.

Concluding remarks (1)

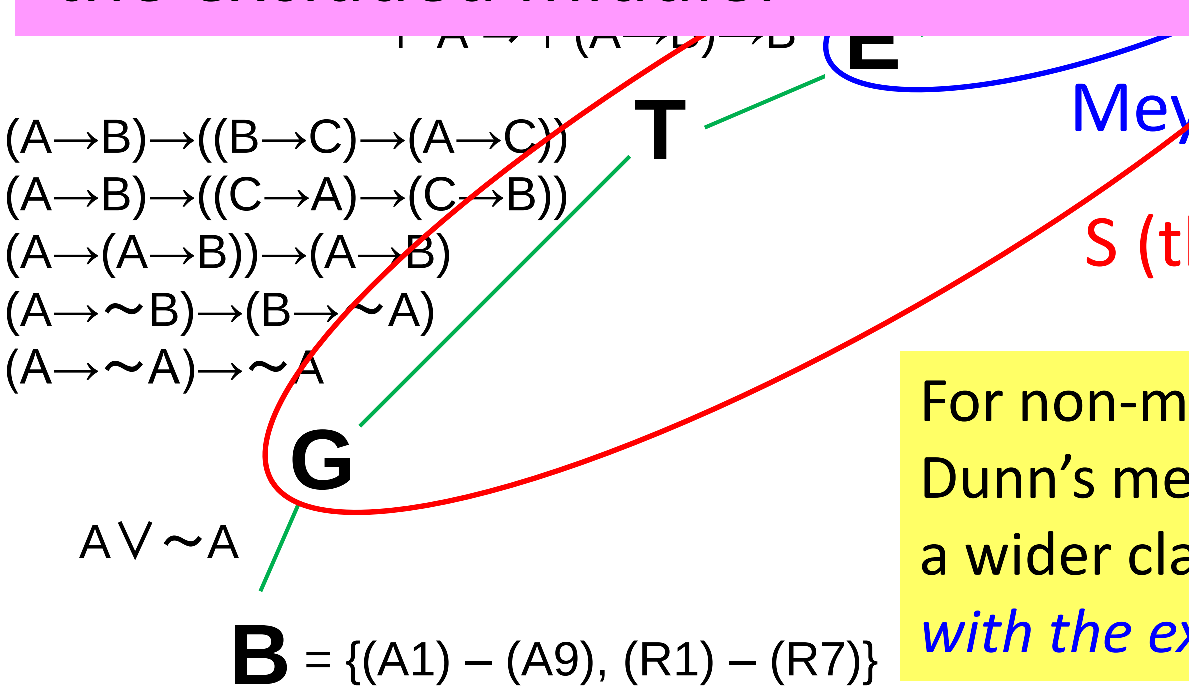
- This argument can be applied for logics obtained from $\mathbf{G.N}_{\Box\Diamond}$ by adding any set of the following axioms, for example:

$A \wedge (A \to B) \to B$	$\Diamond \Box A \to A$
$(A \to B) \to ((B \to C) \to (A \to C))$	$A \to \Box \Diamond A$
$(A \to B) \to ((C \to A) \to (C \to B))$	$\Box A \to \Box \Box A$
$A \to ((A \to B) \to B)$	$\Diamond \Diamond A \to \Diamond A$

Further Research : Consider a proof of the γ-admissibility of $\mathbf{G.N}_{\Box\Diamond}$ with Sahlqvist axioms by using our algebraic method.

Concluding remarks (2)

Further Research : Consider other algebraic proofs which can be applied to relevant logics without the excluded middle.



For non-modal part, Meyer and Dunn's method can be applied to a wider class of relevant logics *with the excluded middle*.

Concluding remarks (3)

- Let $\mathbf{G.C}_{\Box\Diamond}$ be {(A1) – (A13), (R1) – (R9)}.

Further Research : Consider a proof of the γ-admissibility of $\mathbf{G.C}_{\Box\Diamond}$ (or show γ fails in $\mathbf{G.C}_{\Box\Diamond}$)

- Algebraic method (S : this talk):

$$\mathbf{G.C}_{\Box\Diamond} + \{\Box\mathbf{t}, \frac{A}{\Diamond A}\} \quad (= \mathbf{G.N}_{\Box\Diamond})$$

- Method of normal models / Method using metavaluations (S : submitted):

$$\mathbf{G.C}_{\Box\Diamond} + \{\frac{C \vee A}{C \vee \sim(A \to \sim A)}, \frac{C \vee A}{C \vee \sim\Box\sim A}, \frac{C \vee A}{C \vee \Diamond A}\}$$