Logical connections in the many-sorted setting

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Plan of the talk

1. Classical modal logic (syntax, semantics, Jónsson-Tarski, Goldblatt-Thomason theorems) can be recovered from a logical conection.
2. A general nature of a logical connection.
3. Results: syntax, semantics, Jónsson-Tarski and Goldblatt-Thomason theorems.
4. More examples: what we can choose for propositional part of a logic.

(Bonsangue & Kurz, 2004)

Classical modal logic can be retrieved from the data

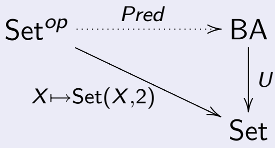
1. models will have sets of states = choice of the category Set of sets and mappings
2. Kripke frames = coalgebras for $P : \mathrm{Set} \to \mathrm{Set}$, one such is $c : X \to P(X)$
3. choice of the propositional part of the logic = the variety BA of Boolean algebras
4. an adjunction

$$\mathrm{Set}^{op} \underset{Pred}{\overset{Stone}{\rightleftarrows}} \mathrm{BA} \quad (\bot) \qquad \frac{X \to Stone(A)}{A \to Pred(X)}$$

of a special kind: it is given by an object $\Omega = 2$ living in both Set and BA (schizophrenic object) = logical connection

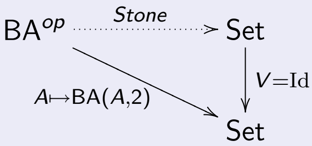
The adjunction *Stone* $\dashv$ *Pred* in more detail

The assignment $X \mapsto \text{Set}(X, 2)$ has a canonical lift to Boolean algebras



Intution: *Pred* gives truth distributions.

The assignment $A \mapsto \text{BA}(A, 2)$ has a canonical lift to sets



Intuition: *Stone* gives theories.

The syntax of the modal part

Given $P : \mathsf{Set} \to \mathsf{Set}$, define $L : \mathsf{BA} \to \mathsf{BA}$ as the composite[a]

$$\mathsf{BA} \xrightarrow{Stone} \mathsf{Set}^{op} \xrightarrow{P^{op}} \mathsf{Set}^{op} \xrightarrow{Pred} \mathsf{BA}$$

Algebras for L = modal algebras (= BAOs), one such is $\alpha : LA \to A$.

α computes flat terms:

$$(\heartsuit, a_0, \ldots, a_{n-1}) \in LA \mapsto \heartsuit(a_0, \ldots, a_{n-1}) \in A$$

[a] A tiny technicality here, if we want a finitary language.

Modalities

We obtain all finitary modalities, one such is a map

$$\heartsuit : P(2^n) \longrightarrow 2$$

i.e., it is a "truth table"

$0, \ldots, n-1$	$\heartsuit$
r	v

"rows" $r =$ subsets of 2^n, "values" $v =$ elements of $\Omega = 2$

Thus: the modal language of predicate liftings (Kupke, Kurz, Schröder, ...).

Modal algebras = BAOs = algebras for L

Denote their category by BA^L. There is a chain of adjunctions

$$\begin{array}{ccc}
\mathsf{BA}^L & F^L F(At) & \text{(BAO of modal formulas)} \\
F^L \uparrow \dashv \downarrow U^L & \uparrow & \\
\mathsf{BA} & F(At) & \text{(Boolean algebra of prop. formulas)} \\
F \uparrow \dashv \downarrow U & \uparrow & \\
\mathsf{Set} & At & \text{(set of atomic propositions)}
\end{array}$$

that is finitary and monadic = BAOs form a variety.

Semantic of the logic

There is, canonically, a commutative square

$$\begin{array}{ccc} (\mathrm{Set}_P)^{op} & \xrightarrow{\;Pred^\sharp\;} & \mathrm{BA}^L \\ {\scriptstyle (V_P)^{op}}\downarrow & & \downarrow{\scriptstyle U^L} \\ \mathrm{Set}^{op} & \xrightarrow[\;Pred\;]{} & \mathrm{BA} \end{array}$$

allowing for

$$\|-\|_c^{val} : F^L F(At) \to Pred^\sharp(X, c), \quad \varphi \mapsto \{x \in X \mid x \Vdash_c^{val} \varphi\}$$

from a given valuation

$$val : At \to UU^L Pred^\sharp(X, c), \quad p \mapsto \{x \in X \mid p \text{ holds in } x\}$$

for each coalgebra $c : X \to PX$.

Semantic map = homomorphism

By construction:

$$\|-\|_c^{val} : F^L F(At) \to Pred^\sharp(X, c)$$

is a homomorphism of modal algebras, i.e.,

$$\|\heartsuit.(\varphi_0, \dots, \varphi_{n-1})\|_c^{val} = [\![\heartsuit]\!]_{Pred^\sharp(X,c)}(\|\varphi_0\|_c^{val}, \dots, \|\varphi_{n-1}\|_c^{val})$$

holds.

Unravelling the definition of $\Vdash_c$ in $c : X \to P(X)$

To see whether

$$x \Vdash_c \heartsuit.(\varphi_0, \ldots, \varphi_{n-1})$$

holds:

1. In each $x \in X$ determine, whether $x \Vdash_c \varphi_j$, $j \in n$.
   This gives a map $X \to 2^n$, hence a map $P(X) \to P(2^n)$.
2. Compute the composite

$$X \xrightarrow{c} P(X) \longrightarrow P(2^n) \xrightarrow{\heartsuit} 2$$

   and evaluate at x.

Hence: the classical Kripke semantics.

Further results on classical modal logic

1. Jónsson-Tarski completeness theorem: the unit η of $Stone \dashv Pred$ is a (regular) mono and it lifts to BA^L (Kurz & Rosický, 2006)

$$\begin{array}{ccc} (\mathsf{Set}_P)^{op} & \underset{Pred^\sharp}{\overset{Stone^\sharp}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathsf{BA}^L \\ {\scriptstyle (V_P)^{op}}\downarrow & & \downarrow{\scriptstyle U^L} \\ \mathsf{Set}^{op} & \underset{Pred}{\overset{Stone}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathsf{BA} \end{array}$$

$$\eta^\sharp_A : (A, a) \to Pred^\sharp Stone^\sharp(A, a)$$

$$\eta_A : A \to Pred\,Stone(A)$$

Further results on classical modal logic

2. Goldblatt-Thomason theorem on modal definability: the counit $\varepsilon^\sharp$ of $Stone^\sharp \dashv Pred^\sharp$ provides us with the notion of ultrafilter extension

$$\varepsilon^\sharp_{(X,c)} : Stone^\sharp Pred^\sharp(X,c) \longrightarrow (X,c) \quad \text{in } (\mathsf{Set}_P)^{op}$$

or, by passing from $(\mathsf{Set}_P)^{op}$ to Set_P,

$$(\varepsilon^\sharp)^{op}_{(X,c)} : (X,c) \longrightarrow (Stone^\sharp)^{op}(Pred^\sharp)^{op}(X,c) \quad \text{in } \mathsf{Set}_P$$

(Kurz & Rosický, 2007)

What has been studied instead of $P : \mathsf{Set} \to \mathsf{Set}$

1. Kripke-polynomial functors on Set (Kupke, Kurz, Moss, Pattinson, Schröder, Venema, ...):

$$T ::= \mathrm{Id} \mid \mathrm{const}_X \mid T \times T \mid T + T \mid PT$$

   essentially: modal logics of automata.

2. measure-polynomial functors (Goldblatt, Moss, Viglizzo, ...):

$$T ::= \mathrm{Id} \mid \mathrm{const}_X \mid T \times T \mid T + T \mid \Delta T$$

   Hence: modal logic on the category Meas of measurable spaces.

The main idea

Replace

$$\begin{array}{ccc} \mathsf{Set} & & \mathsf{BA} \\ \Big\downarrow {\scriptstyle V=\mathrm{Id}} & & {\scriptstyle F}\Big\uparrow \dashv \Big\downarrow{\scriptstyle U} \\ \mathsf{Set} & \Omega = 2 \in \mathsf{Set} & \mathsf{Set} \end{array}$$

by a general situation

$$\begin{array}{ccc} \mathsf{Spa} & & \mathsf{Alg} \\ \Big\downarrow {\scriptstyle V} & & {\scriptstyle F}\Big\uparrow \dashv \Big\downarrow{\scriptstyle U} \\ [\mathscr{S}, \mathsf{Set}] & \Omega : \mathscr{S} \times \mathscr{A} \to \mathsf{Set} & [\mathscr{A}, \mathsf{Set}] \end{array}$$

with Ω schizophrenic, i.e., in both Spa and Alg.

Examples of Alg = "propositional" part of the logic

1. Any finitary variety: distributive lattices, Heyting algebras, residuated lattices.
2. $\mathscr{A}$=finite ordinals and all maps, H-algebras for

$$HX = X \times X + X + \delta X, \quad (\delta X)(n) = X(n+1)$$

   H-algebra = "1st order formulas"
3. $\mathscr{A}$=ingredients of a Kripke-polynomial $T : \mathsf{Set} \to \mathsf{Set}$. $\mathsf{Alg} = [\mathscr{A}, \mathsf{BA}]$. This gives many-sorted modal logic of Rössiger for description of automata.

Examples of Alg, cont.

4. H-monoids for a strong $H : [\mathscr{A}, \mathsf{Set}] \to [\mathscr{A}, \mathsf{Set}]$, where $\mathscr{A}$ is promonoidal.
   Example: $\mathscr{A}$=finite ordinals and all maps, syntax of λ-calculus with α-conversion (Fiore, Plotkin & Turi, 1999), $HX = X \times X + \delta X$.
   H-monoid = "presheaf of λ-terms".
   More generally: a syntax given by variable binding (Tanaka & Power, 2000). Examples include: linear λ-calculus, logic of bunched implications, . . .

What schizophrenic modules are

$\Omega : \mathscr{S} \times \mathscr{A} \to \mathsf{Set}$ such that

1. Ω "lives" in both Spa and Alg:

$$
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\;\Omega_{\mathsf{Spa}}\;} & \mathsf{Spa} \\
 & \searrow^{a \mapsto \Omega(-,a)} & \downarrow V \\
 & & [\mathscr{S}, \mathsf{Set}]
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{S} & \xrightarrow{\;\Omega_{\mathsf{Alg}}\;} & \mathsf{Alg} \\
 & \searrow^{s \mapsto \Omega(s,-)} & \downarrow U \\
 & & [\mathscr{A}, \mathsf{Set}]
\end{array}
$$

2. Roughly: for every a and s, there are canonical Spa- and Alg-structures on

$$\{A, \Omega(s,-)\} \quad \{X, \Omega(-,a)\}$$

for every "algebra" A and "space" X.

Results (Kurz & JV, 2010)

1. Schizophrenic modules Ω give rise to logical connections

$$\mathit{Stone} \dashv \mathit{Pred} : \mathsf{Spa}^{op} \to \mathsf{Alg}$$

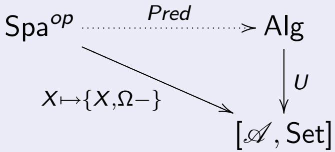
   One can even work enriched, i.e., with $\mathscr{V}$-categories (replace Set by a symmetric monoidal closed $\mathscr{V}$).
   Technique:[a] initial lifts along $U : \mathsf{Alg} \to [\mathscr{A}, \mathsf{Set}]$ and $V : \mathsf{Spa} \to [\mathscr{S}, \mathsf{Set}]$.

2. Given $T : \mathsf{Spa} \to \mathsf{Spa}$, one can define $L : \mathsf{Alg} \to \mathsf{Alg}$, giving rise to
   1. Modal algebras Alg^L forming a variety over $[\mathscr{A}, \mathsf{Set}]$, if Alg was a variety over $[\mathscr{A}, \mathsf{Set}]$. (Variety = the adjunction $F \dashv U$ is finitary and monadic.)
   2. Syntax and semantics can be defined. The rôle of Ω = "external" truth values.

[a] Generalizing Porst & Tholen, 1991

Intuition behind *Pred* and *Stone*

There is a canonical lift



Here, elements of $\{X, \Omega -\} : \mathscr{A} \to \mathrm{Set}$ are "truth-distributions" on $X : \mathscr{S} \to \mathrm{Set}$, one such is

$$a \mapsto \text{a natural transf. from } X \text{ to } \Omega(-, a), \text{ naturally in } a$$

Similarly for *Stone*, elements of *Stone* are "ultrafilters" (= "theories").

Examples of logical connections

1. Ω=any distributive/residuated lattice, it gives rise to

$$Stone \dashv Pred : \mathrm{Set}^{op} \to \mathrm{DL}, \quad Stone \dashv Pred : \mathrm{Set}^{op} \to \mathrm{RL}$$

2. Spa=measurable spaces, Alg=Boolean algebras, $\Omega = 2$.
The resulting logic is the logic of Harsanyi spaces (Moss & Viglizzo, 2004).
Modality = a measurable map

$$\heartsuit : T(2^n) \to 2$$

i.e., a measurable subset of $T(2^n)$.

More examples of logical connections, enriched

4. $\mathscr{V}$=posets, Ω=two-element chain, it gives rise to

$$Stone \dashv Pred : \mathsf{Pos}^{op} \to \mathsf{DL}$$

   The resulting logic is expressive (Kapulkin, Kurz & JV, 2010) for "good" $T : \mathsf{Pos} \to \mathsf{Pos}$.

5. $\mathscr{V}$=2, $\mathscr{V}$-categories are preorders. Any $\Omega : \mathscr{S} \otimes \mathscr{A} \to \mathscr{V}$ is a (monotone) binary relation and it is schizophrenic.
   Logical connection = Galois connection between upper sets of $\mathscr{S}$ and $\mathscr{A}$.

6. Another facet of the previous:
   $\mathscr{V}$=Abelian groups, $\mathscr{A}$, $\mathscr{S}$ rings with unit, Ω= any $\mathscr{A}$-$\mathscr{S}$-bimodule. The resulting logical connection is the one known from module theory.

General form of modalities

1. Arities are finitely presentable objects of $[\mathscr{A}, \mathrm{Set}]$.
2. For an f.p. $n : \mathscr{A} \longrightarrow \mathrm{Set}$, an n-ary modality is

$$\heartsuit : a \mapsto \heartsuit(a) : T\{n, \Omega\} \to \Omega(-, a), \text{ functorial in } a$$

   and every $\heartsuit(a)$ is natural in s, since Spa "lives over" $[\mathscr{S}, \mathrm{Set}]$.
3. n-ary $\heartsuit$ can be applied only to an n-ad in A, i.e., to

$$n \longrightarrow UA$$

   allowing for partially defined modalities.

More results (Nentvich, Petrişan & JV, 2010)

Given $T : \mathsf{Spa} \to \mathsf{Spa}$, the existence of

$$\begin{array}{ccc}
(\mathsf{Spa}_T)^{op} & \underset{Pred^\sharp}{\overset{Stone^\sharp}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathsf{Alg}^L \\
\Big\downarrow (V_P)^{op} & & \Big\downarrow U^L \\
\mathsf{Spa}^{op} & \underset{Pred}{\overset{Stone}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathsf{Alg}
\end{array}$$

allows to formulate and prove

1. Jónsson-Tarski theorem
2. Goldblatt-Thomason theorem

Some side conditions on $Stone \dashv Pred$ are needed, however.

References

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