Algebra and Substructural Logics – take 4
Schedule and Abstracts

Tuesday 8 June 2010	
09:20 - 09:30	Opening
09:30 - 10:10	Daniele Mundici *Deductive interpolation in Łukasiewicz logic and amalgamation* of MV-algebras
10:10 - 10:50	George Metcalfe *Craig Interpolation for Semilinear Varieties*
10:50 - 11:20	Break
11:20 - 12:00	Leonardo Cabrer, José Gil-Férez *Leibniz Interpolation Properties*
12:00 - 12:40	小野寛晰 (Hiroakira Ono) *Regular completions of residuated lattices*
12:40 - 14:10	Lunch break
14:10 - 14:50	林哲 (Zhe Lin) *Finite Embeddability Property of S4 modal residuated groupoids*
14:50 - 15:30	関隆宏 (Takahiro Seki) *An Algebraic Proof of the γ-admissibility of Relevant Modal Logics*
15:30 - 16:10	William Young, 小野寛晰 (Hiroakira Ono) *Modal substructural logics*
16:10 - 16:40	Break
16:40 - 17:20	Alberto Carraro *Resource combinatory algebras*
17:20 - 18:00	Sándor Jenei, 小野寛晰 (Hiroakira Ono) *On involutive FL_e algebras*
18:00 - 19:30	Reception

Wednesday 9 June 2010	
09:20 - 10:00	José Gil-Férez
	Modules over quantaloids and the Isomorphism Problem
10:00 - 10:40	Antonio Ledda
	A duality for quasi ordered structures
10:40 - 11:10	Break
11:10 - 11:50	Dana Šalounová
	Filter theory of bounded residuated lattice ordered monoids
11:50 - 12:30	Jiří Velebil
	Logical connections in the many-sorted setting
12:30 - 14:30	Lunch break
14:30 - 15:10	James Raftery
	Finiteness properties for idempotent residuated structures
15:10 - 15:50	Jiří Rachůnek
	State operators on bounded residuated l-monoids
15:50 - 16:20	Break
16:20 - 17:00	Tomasz Kowalski, Francesco Paoli
	Quasi-subtractive varieties
17:00 - 17:40	Jan Kühr
	Pre-ideals of basic algebras
19:30 - 21:30	Conference Dinner

Thursday 10 June 2010	
09:20 - 10:00	Félix Bou *Characterizing the generators of McNaughton functions evaluated* in a lattice filter
10:00 - 10:40	Simone Bova *Lewis Dichotomies in Many-Valued Logics*
10:40 - 11:10	Break
11:10 - 11:50	Matthias Baaz, Norbert Preining *Quantified Propositional Logics and Quantifier Elimination*
11:50 - 12:30	Christian Fermüller *A Generalization of Giles's Game*
12:30 - 14:30	Lunch break
14:30 - 15:10	Petr Cintula, Carles Noguera *Lattice disjunction is not a disjunction (in many substructural logics)*
15:10 - 15:50	Carles Noguera, Petr Cintula *First-Order Substructural Logics: An Algebraic Approach*
15:50 - 16:20	Break
16:20 - 17:00	照井一成 (Kazushige Terui) *The birth of linear logic*
17:00 - 17:40	Antonino Salibra *Weird models of linear logic*
17:40 - 17:50	Closing

Matthiaz Baaz

Quantified Propositional Logics and Quantifier Elimination

(joint work with Norbert Preining and Viennese colleagues)

In this lecture we discuss quantified propositional logics with linearly ordered truth values and conditions for the elimination of propositional quantifiers. Propositional quantifiers are defined in a natural way by suprema and infima. We concentrate on Gödel logics on closed subsets of [0,1] containing 0,1 and on Łukasiewicz logic.

For the quantified propositional Gödel logic on $[0, 1]$ we demonstrate eliminability of quantifiers and exhibit a finite axiomatization based on an axiomatic variant of the Takeuti-Titani rule. For the quantified propositional Gödel logics $\mathbf{G}_\downarrow$ (on $\{1/n \mid n \in \mathbb{N} \setminus \{0\}\} \cup \{0\}$) and $\mathbf{G}_\uparrow$ (on $\{1 - 1/n \mid n \in \mathbb{N} \setminus \{0\}\} \cup \{1\}$) we calculate the minimal propositional extension where elimination of quantifiers is possible and provide a finite axiomatization. As a corollary we prove, that Garrowup is the intersection of quantified propositional finitely valued Gödel logics.

For quantified propositional Łukasiewicz logic we calculate the minimal propositional extension admitting quantifier elimination. As a corollary we obtain, that quantified propositional Łukasiewicz logic in the original language is decidable, but not finitely axiomatizable.

Félix Bou

Characterizing the generators of McNaughton functions evaluated in a lattice filter

In a recent paper the author has axiomatized, using some axioms Ax and rules R, the set T_r of formulas that in the infinite-valued Łukasiewicz algebra always take a value above some fixed real number r. The purpose of this talk is to present a characterization of all formulas in one variable which also axiomatize (using the same rules R) this set T_r. An immediate application of this characterization is that while both formulas $p \vee \neg p$ and $((p \odot p) \oplus p) \vee ((\neg p \odot \neg p) \oplus \neg p)$ belong to the set $T_{0.5}$, only the first one axiomatizes the set $T_{0.5}$.

Simone Bova

Lewis Dichotomies in Many-Valued Logics

A classical result by Harry Lewis in 1979 shows that the computational complexity of the satisfiability problem of Boolean logic dichotomizes, depending on the Boolean operations available to formulate the instance: intractable (NP-complete) if negation of implication is definable, and tractable (in P) otherwise. Exploiting Post lattice and clone theory, we investigate Lewis dichotomies for a number of many-valued generalizations of Boolean propositional logic, namely Kleene, DeMorgan, and Gödel logics. We obtain a complete dichotomy classification for Kleene and Gödel logics. We give a lower bound for DeMorgan logic, and raise the dichotomy problem for finite-valued Łukasiewicz logic.

Leonardo Manuel Cabrer
Leibniz Interpolation Properties
(joint work José Gil-Férez)

In the framework of Abstract Algebraic Logic (see [2] for a general survey) a number of different interpolation properties have been considered: Maehara Interpolation Property, Robinson Interpolation Property, and Deductive Interpolation Property among others (see [1] as well as the references therein).

In [3], Kihara and Ono introduce some new notions of interpolation for substructural logics. For instance, one of the interpolation properties that they study is the following: for any arbitrary pair of formulas φ, ψ, if $\vdash (\varphi\backslash\psi)\wedge(\psi\backslash\varphi)$, then there exists a formula δ with variables $\mathrm{var}(\delta) \subseteq \mathrm{var}(\varphi)\cap\mathrm{var}(\psi)$ such that

$$\vdash (\varphi\backslash\delta) \wedge (\delta\backslash\varphi) \text{ and } \vdash (\delta\backslash\psi) \wedge (\psi\backslash\delta).$$

In the same paper they present algebraic characterizations for all these notions and investigate their relation with the usual interpolation properties. The authors also remark: "only a few properties specific to FL-algebras or residuated lattices are used in our discussion," and they propose the study of these interpolation properties in a more general setting. The objective of the present paper is to give an answer to that remark by exhibiting a general framework in which these interpolation properties can be expressed and characterized by means of algebraic tools.

First, we note that $\nabla(x, y) = \{(x\backslash y) \wedge (y\backslash x)\}$ is a set of equivalence formulas (in the sense of Abstract Algebraic Logic) for the subestructural logics, that is, it defines the Leibniz congruences of their models. In particular, for every theory Γ and every pair of formulas φ, ψ,

$$\Gamma \vdash \nabla(\varphi, \psi) \quad \Leftrightarrow \quad (\varphi, \psi) \in \mathbf{\Omega}\Gamma.$$

This motivates the choice of the equivalential logics, i.e., logics having a set of equivalence formulas, as our general setting.

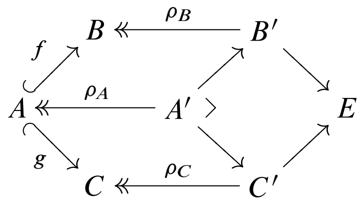
We introduce a family of what we call 'Leibniz' interpolation properties, in which we are mainly interpolating pairs of elements in the Leibniz congruence of a particular free algebra. Applied to the case of substructural logics, some of them coincide with those introduced in [3].

The first notion of interpolation that we introduce is the *Leibniz Interpolation Property* (LIP). A sentential logic $\mathcal{S}$ has the LIP if for every pair of formulas φ, ψ, if $(\varphi, \psi) \in \mathbf{\Omega}_{\mathcal{S}}$, then there exists a formula δ with variables $\mathrm{var}(\delta) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ such that $(\varphi, \delta), (\delta, \psi) \in \mathbf{\Omega}_{\mathcal{S}}$. Here, $\mathbf{\Omega}_{\mathcal{S}}$ stands for the Leibniz congruence of the set of theorems of $\mathcal{S}$.

By adding new conditions we can strengthen this property in several different ways. For instance, we say that a sentential logic $\mathcal{S}$ has the *Leibniz Interpolation Property for Theories* (LIPT) if for every set of formulas Γ and every pair of formulas φ, ψ, if $(\varphi, \psi) \in \mathbf{\Omega}\overline{\Gamma}$, then there exists a formula δ with variables $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma, \varphi) \cap \mathrm{var}(\Gamma, \psi)$ such that $(\varphi, \delta), (\delta, \psi) \in \mathbf{\Omega}\overline{\Gamma}$, where $\overline{\Gamma} = \mathrm{Cn}_{\mathcal{S}}(\Gamma)$ is the theory of $\mathcal{S}$ generated by Γ.

In the context of equivalential logics, we find characterizations for these interpolations properties in terms of categorical properties of their classes of reduced models, reduced algebras, or both. We say that a concrete category $\mathbf{C}$ has *covering pullbacks for (pairs of) monos* if whenever $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$

is a pair of monos in **C**, there exists a pullback and onto morphisms ρ_A, ρ_B, ρ_C rendering commutative the following diagram:



(1)

We obtaing the following result:

Theorem 1. *For an equivalential logic $\mathcal{S}$, the following statements are equivalent:*

(i) *$\mathcal{S}$ has the LIP,*

(ii) *the category* $\mathbf{Mod}^*\mathcal{S}$ *of its reduced models has covering pullbacks for monos,*

(iii) *the category* $\mathbf{Alg}^*\mathcal{S}$ *of its reduced algebras has covering pullbacks for monos.*

For the others interpolations properties we find similar characterizacions, by strengthening the conditions in Diagram (1). For instance, the LIPT for an equivalential logic $\mathcal{S}$ is equivalent to the existence in $\mathbf{Mod}^*\mathcal{S}$ of covering pullbacks for monos of the form of (1) where, moreover, ρ_A is an isomorphism.

Three of the notions that we introduce and deserve a special attention are the *Relative Leibniz*, *Robinson-Leibniz*, and *Maehara-Leibniz Interpolation Properties*. They are the natural generalizations of three of the interpolation properties considered in [3] for substructural logics. We investigate the relation of these properties with those studied by Czelakowsky and Pigozzi in [1] and obtain the following result for the equivalential logics satisfying the G-rule, that is, the regularly algebraizable logics.

Theorem 2. *For every regularly equivalential logic $\mathcal{S}$, we have the following implications:*

(i) *The Relative Interpolation Property implies Interpolation Property.*

(ii) *The Robinson-Leibniz Interpolation Property implies the Robinson Interpolation Property.*

(iii) *The Maehara-Leibniz Interpolation Property implies the Maehara Inerpolation Property.*

References

[1] J. Czelakowski, D. Pigozzi. Amalgamation and interpolation in abstract algebraic logic. In *Models, Algebras, and Proofs*, X. Caicedo and C. H. Montenegro (eds.) Vol. 203 of Lecture Notes in Pure and Applied Mathematics, 187-265 (1998).

[2] J. M. Font, R. Jansana, and D. Pigozzi. A survey of abstract algebraic logic. *Studia Logica*, 74(1-2):13–97, 2003. Abstract algebraic logic, Part II (Barcelona, 1997).

[3] H. Kihara, H. Ono, Interpolation Properties, Beth Definability Properties and Amalgamation Properties for Substructural Logics *Journal of Logic and Computation* DOI 10.1093/logcom/exn084.

Petr Cintula

Lattice disjunction is not a disjunction (in many substructural logics)

(joint work with Carles Noguera)

Disjunction is a crucial connective in both classical and intuitionistic logics. One of its most natural properties is the *proof by cases property* (PCP for short):

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}$$

In substructural logics (algebraizable finitary expansions of the implicational fragment of the logic of non-associative residuated lattices) the situation is quite different. We start by showing that their usual lattice connective $\vee$ is not actually a good disjunction (because it does not satisfy PCP) in neither FL nor FL_e nor FL_w (see [8] for definitions of these logics), while it works in all extensions of FL_{ew}. This, of course, entails several relevant questions:

1. Can we characterize those substructural logics where $\vee$ satisfies PCP?
2. Can we, for a given substructural logic, axiomatize its weakest extension where $\vee$ satisfies PCP?
3. Is it possible to find a *different* connective (say ∇) which would satisfy PCP?
4. Why do we care? Is disjunction (and PCP for that matter) really that important?

The goal of this talk is to answer these questions. Let us now show some hints to answer them (for full details see [3],[4]). We will see that the third question is formulated "too optimistically": it will not be enough to consider a different connective, but we will be forced to work with a *generalized* connective, similarly to the generalized equivalences well-known from abstract algebraic logic. However our useful notation will hide the generalized nature of our new disjunctions; indeed, given a parameterized set of formulae $\nabla(p, q, \overrightarrow{r})$ we define:

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}^{\leq \omega}\}.$$

We say that (parameterized) set of formulae ∇ is a *(p-)disjunction* in a substructural logic L, whenever it satisfies (a form of) PCP:

$$\Gamma, \varphi \vdash_{\mathrm{L}} \chi \text{ and } \Gamma, \psi \vdash_{\mathrm{L}} \chi \text{ if and only if, } \Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi.$$

This level of generality is actually needed as shown by the following examples:

- The connective $\varphi \nabla \psi = (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$ is a disjunction in FL_e.
- The set $\varphi \nabla \psi = \{(p \to q) \to q, (q \to p) \to p\}$ is a disjunction in the purely implicational fragment of the Gödel-Dummett logic (see [6]) and no single formula can define a disjunction in this logic.

- The infinite set with parameters

$$\varphi \nabla \psi = \{\gamma_1(\varphi \wedge \overline{1}) \vee \gamma_2(\psi \wedge \overline{1}) \mid \text{where } \gamma_1, \gamma_2 \text{ are iterated conjugates}\}$$

  is a p-disjunction in FL and no finite set of formulae can define a disjunction in this logic.

Now we can start answering the questions:

1. We can show that ∇ is a disjunction in a given logic iff it satisfies certain simple conditions and for each rule $\varphi_1, \ldots, \varphi_n \vdash \psi$ of some of its axiomatic systems the logic proves $\varphi_1 \nabla \chi, \ldots, \varphi_n \nabla \chi \vdash \psi \nabla \chi$. This, in particular, entails that the property of being a disjunction is preserved in axiomatic extensions of a given logic. We can also provide other characterizations (e.g. based on a generalized notion of prime filter).
2. All we have to do is to add the necessary rules required in point **1.** In particular the rule $\varphi_1 \vee \chi, \varphi_2 \vee \chi \vdash (\varphi_1 \vee \varphi_2) \vee \chi$ axiomatizes the weakest logic extending FL_e, where $\vee$ is disjunction.

3. It is a known result [5] that in each filter-distributive substructural logic some p-disjunction has to exist. Answer **1.** tells us how to recognize one. Other methods to construct a (p-)disjunction use the description of filters generated by a set or the (parameterized) local deduction theorem.

4. Why are disjunctions and PCP so important? Well, in logics with a (p-)disjunction ∇ we can use it to:

- axiomatize any of its extensions given by a universal class of its algebras, where ∇ is still a p-disjunction.
- axiomatize any of its extensions given by a positive-universal class of its algebras (a class given by positive universal sentences, i.e., disjuncions of identities), extending a known result by Galatos [7].
- axiomatize the extension given by its *linearly ordered algebras* (i.e. the minimum *semilinear*, or *fuzzy*, logic above it).
- characterize which of its extensions are complete w.r.t. *densely-linearly ordered algebras*, using the so-called *density rule* [1].
- axiomatize the intersections of its axiomatic extensions (or equivalently, the joints of (relative) subvarieties of the corresponding algebras).
- given a class $\mathbb{K}$ of its algebras, we can characterize strong completeness and finite strong completeness of our logic w.r.t. $\mathbb{K}$ in terms of embedding properties, generalizing results in [2].
- construct its first-order variant which remains complete w.r.t. finitely relatively-subdirectly irreducible algebras (i.e., the class of its linearly ordered algebras in the case of semilinear logics).

Furthermore we can solve several open problems (e.g. necessity of using $\triangle$ in axiomatizing $\mathrm{MTL}_{\sim}$) and give alternative proof of theorems (e.g. showing that the logic PŁ′ is complete w.r.t. chains) from the fuzzy logic literature.

References

[1] A. Ciabattoni, G. Metcalfe. Density elimination. *Theoretical Computer Science* 403(2-3):328–346, 2008.

[2] P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna, C. Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies. *Annals of Pure and Applied Logic*, 160(1):53–81, 2009.

[3] P. Cintula, C. Noguera. Implicational (semilinear) logics I: a new hierarchy. *Archive for Mathematical Logic*, 49:417–446, 2010.

[4] P. Cintula, C. Noguera. Implicational (semilinear) logics II: disjunction and completeness properties. In preparation.

[5] J. Czelakowski *Protoalgebraic logics*, vol. 10 of Trends in Logic, Kluwer, Dordrecht, 2001.

[6] M. Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 27:97–106, 1959.

[7] N. Galatos. Equational Bases for Joins of Residuated-lattice Varieties. *Studia Logica* 76(2):227–240, 2004.

[8] N. Galatos, P. Jipsen, T. Kowalski, H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

Alberto Carraro

Resource combinatory algebras

We initiate a purely algebraic study of Ehrhard and Regnier's resource lambda-calculus, by introducing three equational classes of algebras: resource combinatory algebras, resource lambda-algebras and resource lambda-abstraction algebras. We establish the relations between them, laying down foundations for a model theory of resource lambda-calculus. We also show that the ideal completion of a resource combinatory (resp. lambda-, lambda-abstraction) algebra induces a 'classical' combinatory (resp. lambda-, lambda-abstraction) algebra, and that any model of the classical lambda-calculus raising from a resource lambda-algebra determines a lambda-theory which equates all terms having the same Bohm tree

Christian Fermüller

A Generalization of Giles's Game

Already in the 1970s Robin Giles presented a combination of a dialogue game and a particular betting scheme that characterizes Łukasiewicz logic. Variants of this type of game semantics for other important fuzzy logics, including Gödel and Product logic have been described in the literature. However these versions of the game suffer from major drawbacks. In particular the one-one relation between risk values and truth values of Giles's game is lost. We will discuss a way to generalize Giles's game in a more conservative, yet at the same time also more open and systematic manner. Quite general conditions on the evaluation of final states and on the form of dialogues rules are stated. These conditions turn out to be sufficient to guarantee that optimal strategies for the proponent of a formula correspond to a truth functional evaluation in a many valued logic over some subset of the reals as truth values. We will also discuss to which extend our conditions are necessary to characterize a many valued logic.

José Gil-Férez

Modules over quantaloids and the Isomorphism Problem

(joint work with Nikolaos Galatos)

The Isomorphism Problem in the context of Abstract Algebraic Logic and of π-institutions is that of determining when the notions of syntactic and semantic equivalence among logics coincide. We present a solution for it in the general setting of categories of modules over quantaloids, which are the multi-signature version of the modules over residuated lattices, or quantales.

Sándor Jenei

On involutive FL_e algebras

(joint work with Hiroakira Ono)

$\mathcal{U} = \langle X, \circledast, \leq, e, f \rangle$ is called an *involutive FL_e-algebra* if $\mathcal{C} = \langle X, \leq \rangle$ is a poset, $\circledast$ is a uninorm over $\mathcal{C}$ with neutral element e, for every $x \in X$, $x \rightarrow_{\circledast} f = \max\{z \in X \mid x \circledast z \leq f\}$ exists, and for every $x \in X$, we have $(x \rightarrow_{\circledast} f) \rightarrow_{\circledast} f = x$.

We will give an overview on the structural description of involutive FL_e-algebras with respect to their underlying t-norm and t-conorm operations. A new construction, called twin-rotation will be introduced. Particular attention will be devoted to the finite case and to the complete, densely-ordered chain case.

Keywords: FL_e, uninorms, t-norms, t-conorms, co-residuation, twin-rotation.

Jan Kühr

Pre-ideals of basic algebras

A *basic algebra* [3] is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ that satisfies the identities

$$x \oplus 0 = x,$$
$$\neg\neg x = x,$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$$
$$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0.$$

These algebras have been introduced in [3] as a counterpart of bounded lattices with sectional antitone involutions and can be regarded as a common generalization of MV-algebras and orthomodular lattices, also including lattice effect algebras. More precisely, for every basic algebra $(A, \oplus, \neg, 0)$, the underlying order defined by $x \leq y$ iff $\neg x \oplus y = \neg 0$ makes A into a bounded lattice where, for each $a \in A$, the map $x \mapsto \neg x \oplus a$ is an antitone involution on the principal filter $[a)$. On the other hand, if we are given a bounded lattice with sectional antitone involutions $x \mapsto x^a$ $(a \in A)$, then the rule $x \oplus y := (x^0 \vee y)^y$ and $\neg x := x^0$ defines a basic algebra. Concerning the interconnection between basic algebras, MV-algebras and orthomodular lattices, MV-algebras are precisely the associative basic algebras (commutativity is then derivable from the other axioms), and orthomodular lattices may be characterized as basic algebras satisfying the identity $x \oplus (x \wedge y) = x$.

We should emphasize that our basic algebras are not much connected with Hájek's basic logic and BL-algebras; in fact, the intersection with BL-algebras are MV-algebras. We used the name 'basic algebra' just to express the fact that these algebras capture some basic common features of all the structures considered in [3].

In the present paper we study what we call *pre-ideals*, i.e., non-empty subsets that are closed under $\oplus$ and downwards closed. Since the variety of basic algebras is ideal determined, the term 'ideal' is reserved for the congruence kernels. It is not hard to show that the ideal lattice forms a complete sublattice of the lattice of pre-ideals.

We restrict ourselves to basic algebras satisfying the identity

$$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z) \tag{1}$$

or, more generally, the identity

$$\neg x \oplus (x \wedge y) = \neg x \oplus y. \tag{2}$$

Some results are also proved for commutative basic algebras. These algebras are much closer to MV-algebras; for instance, the induced lattice is distributive, and we prove that finite basic algebras satisfying (2) are automatically MV-algebras.

The first important observation is that basic algebras satisfying (2) have the following *Riesz decomposition property*: If $a \leq \tau(b_1, \ldots, b_n)$, where τ is an additive term, then there exist $c_i \leq b_i$ such that $a = \tau(c_1, \ldots, c_n)$.

Further, for every $a \in A$ and integer $n \geq 0$ we define $n \otimes a$ inductively: $0 \otimes a := 0$ and $n \otimes a := a \oplus ((n-1) \otimes a)$ for $n \geq 1$.

If A is finite and satisfies (2), then for every atom $a \in A$, the set $N(a) = \{n \otimes a \mid n \geq 0\}$ is a finite chain $0 < a < \ldots < \hat{a}$. The Riesz decomposition property entails $N(a) = [0, \hat{a}]$. Since intervals in basic algebras are basic algebras, it follows that $N(a)$ is a finite MV-chain.

Theorem 1. *Let A be a finite basic algebra satisfying (2). Then A is isomorphic to the direct product $\prod_{a \in M} N(a)$ where M denotes the set of atoms of A. Consequently, A is an MV-algebra.*

Given a basic algebra $(A, \oplus, \neg, 0)$ satisfying (2) and its pre-ideal I, we can define an equivalence relation θ_I on A as follows: $(x, y) \in \theta_I$ iff $x = a_1 \oplus (\ldots \oplus (a_m \oplus y') \ldots)$ and $y = b_1 \oplus (\ldots \oplus (b_n \oplus x') \ldots)$ for some $a_i, b_j \in I$ and $x', y' \in A$ with $x' \leq x$ and $y' \leq y$. Then θ_I is compatible with the meet-operation; the underlying order of A/θ_I is given by: $[x]_{\theta_I} \leq [y]_{\theta_I}$ iff $x = a_1 \oplus (\ldots \oplus (a_m \oplus y') \ldots)$ for some $a_i \in I$ and $y' \in A$ with $y' \leq y$. Moreover, if I is an ideal, i.e., $I = [0]_\phi$ for some congruence ϕ, then $\theta_I = \phi$.

Next, we focus on basic algebras satisfying (1). It is easy to show that the pre-ideal generated by $\emptyset \neq X \subseteq A$ consists of those $a \in A$ which are less than or equal to some finite sum of elements of X. The pre-ideal lattice is an algebraic distributive lattice, and we can characterize its meet-prime elements that we call *prime pre-ideals* of A.

Theorem 2. *Let $(A, \oplus, \neg, 0)$ be a basic algebra that satisfies (1). Then for every pre-ideal P, the following are equivalent:*

(i) P is prime;

(ii) for all $x, y \in A$, if $x \wedge y \in P$, then $x \in P$ or $y \in P$;

(iii) for all $x, y \in A$, if $x \wedge y = 0$, then $x \in P$ or $y \in P$;

(iv) for all $x, y \in A$, $\neg(\neg x \oplus y) \in P$ or $\neg(\neg y \oplus x) \in P$;

(v) $(A/\theta_P, \leq)$ is a chain;

(vi) the set of all pre-ideals containing P is a chain under inclusion.

In what follows, let $(A, \oplus, \neg, 0)$ be a commutative basic algebra. We say that a pre-ideal is *closed* if it is closed under all existing suprema. By a *value* of $0 \neq a \in A$ we mean a pre-ideal which is maximal with respect to not containing a. If a pre-ideal is the only value of some non-zero element, then we call it a *special value*.

Theorem 3. *Every special value is closed.*

We denote by $D(A)$ the intersection of all closed prime pre-ideals of A and call it the *distributive radical* of A. It easily follows that $D(A)$ equals the intersection of the closed values and contains those elements having no closed values.

A basic algebra is *completely distributive* if

$$\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f: I \to J} \bigwedge_{i \in I} a_{i f(i)}$$

for all $\{a_{ij} \mid i \in I, j \in J\}$ for which the indicated suprema and infima exist.

Theorem 4. *A commutative basic algebra* $(A, \oplus, \neg, 0)$ *is completely distributive if and only if* $D(A) = \{0\}$.

References

[1] M. Botur: *An example of a commutative basic algebra which is not an MV-algebra*, Math. Slovaca, to appear.

[2] M. Botur and R. Halaš: *Finite commutative basic algebras are MV-algebras*, J. Mult.-Val. Log. Soft Comput. **14** (2008), 69–80.

[3] I. Chajda, R. Halaš and J. Kühr: *Many valued quantum algebras*, Algebra Univers. **60** (2009), 63–90.

Antonio Ledda

A duality for quasi ordered structures (I)

(joint work with Hector Freytes and Majid Alizadeh)

Recently, several authors extended Priestley duality for distributive lattices [9] to other classes of algebras, such as, e.g. distributive lattices with operators [7], MV-algebras [8], MTL- and IMTL-algebras [2]. In [5] necessary and sufficient conditions for a normally presented variety to be naturally dualizable, in the sense of [6], i.e. with respect to a discrete topology, have been provided. Under this perspective, also bounded distributive quasi lattices (bdq-lattices), introduced in [4], are naturally dualizable. Nonetheless, quasi lattices, constitute a generalization of lattice ordered structures to quasi ordered ones, i.e. structures in which the ordering relation is reflexive and transitive, but it may fail to be antisymmetric. Consequently, a sensible question arises: is there any "natural" candidate which stands to Priestley spaces as bounded distributive quasi lattices stand to bounded distributive lattices? In this talk, following an idea from [1], where a representation of bdq-lattices has been proposed, we present an alternative form of dualization of bdq-lattices via the notion of *preordered Priestley spaces*. Preordered Priestley spaces are, in our opinion, of interest in that they naturally generalize Priestley spaces to a preordered setting, and, at the same time, they share with Priestley spaces desirable features. We will see that preordered Priestley spaces interpret, with respect to bdq-lattices, exactly the same rôle Priestley spaces play with respect to bounded distributive lattices.

References

[1] M. Alizadeh, A. Ledda, H. Freytes, "Completion and amalgamation of bounded distributive quasi lattices", *Logic Journal of the IGPL*, forthcoming.

[2] L.M. Cabrer, S.A. Celani, "Priestly dualities for some lattice-ordered algebraic structures, including MTL, IMTL, and MV-algebras", *Central European Journal of Mathematics* **4**, (2006), 600-623.

[3] S.A. Celani "Distributive lattice with fusion and implication", *Southeast Asian Bull. Math* **28** (2004), 999-1010.

[4] I. Chajda "Lattices in quasiordered sets", *Acta Univ. Palack Olumuc*, **31** (1992), 6-12.

[5] I. Chajda, R. Halas, A.G. Pinus, I.G. Rosenberg "Duality of normally presented varieties", *International Journal of Algebra and computation*, **10** (2000), 651-664.

[6] D.M. Clark, B.A. Davey, *Natural dualities for the working algebraists*, Cambridge University Press, Cambridge, 1998.

[7] R. Goldblatt "Varieties of complex algebras", *Annals of pure and applied logic*, 44 (3), (1989), 173-242.

[8] N.G. Martinez "The Priestley duality for Wajsberg algebras", *Studia Logica*, **49**, 1, (1989), 31-46.

[9] H.A. Priestley, "Ordered topological spaces and the representation of distributive lattices", *Proc. London Math. Soc.*, **24**, (1972), 507-530

Zhe Lin

Finite Embeddability Property of S4 modal residuated groupoids

We prove FEP of the class of residuated groupoids with an unary (modal) operator $\Diamond$, and its residual $\Box$, satisfying the S4-axioms. T: $a \leq \Diamond a$, 4: $\Diamond\Diamond a \leq a$ and a special axiom K: $\Diamond(a \cdot b) \leq \Diamond a \cdot \Diamond b$ introduced by Moortgat[10]. NL_{S4} denotes Nonassociative Lambek Calculus with S4-modalities $\Diamond$ and its residual $\Box$. The proof also yields FEP for the classes axiomatized by T and 4 or T only.

Blok and van Alten [5] prove FEP of the class of integral residuated groupoids, but they leave it as an open problem (Problem 4.2) whether the integrality assumption can be dropped. It was solved by Farulewski (2006, published RML 2008): the class of residuated groupoids has FEP. Buszkowski and Farulewski[3] and Buszkowski[4] show many other results of this kind, not assuming integrality. This paper continues line of research. Some techniques e.g. interpolation of formula-trees by formulae, are taken from [4]. However we directly prove the extended subformula property for systems with assumptions, using a cut-elimination theorem for these systems. (This theorem also holds for many other substuctural logics, e.g Lambek calculus, Full Lambek Calculus.)

Further, we use nuclear completions of the form elaborated in Okada and Terui [9], Belardinelli, Jipsen and Ono [6], and adapted to nonassociative systems in Buszkowski and Farulewski [3].

In [8] we have shown that categorial grammars based on NL_{S4} with assump- tions generate context-free languages.

The key argument of the papers is the extended subformula property for Gentzen style axiomatization and the interpolation lemma. By using an interpolation of members of the free groupoid, generated by a finite partial algebra by members of this algebra, we established FEP for the class of S4 modal residuated groupoids.

References

[1] Shun'ichi J.Amano, Finite Embeddability Property for Some Modal Algebra, Master thesis, School of information Science Japan Advance Insititue of Science and Techology.

[2] W. Buszkowski, Lambek Calculus with Nonlogical Axioms. In C. Casadio, P. J. Scott, and R. A. G. Seely (eds.), Language and Grammar. Studies in Mathematical Linguistics and Natural Language, CSLI, Lecture Notes 168:77-93, Stanford 2005

[3] W. Buszkowski, and M. Farulewski, Nonassociative Lambek Calculus with Addi- tives and Context-Free Languages. In: O. Grumberg et al. (eds.), Languages: From Formal to Natural, LNCS 5533:45-58, 2009.

[4] W. Buszkowski, Interpolation and FEP for Logic of Residuated Algebras, Logic Journal of the IGPL, To appear (avaiable in an electronic version via Advanced Access).

[5] W. J. Blok, C. J. Van Alten, On the finite embeddability property for residuated ordered groupoids, Transactions of the AMS 357 (10) (2005).

[6] F. Belardinelli, P. Jipsen, and H. Ono, Algebraic Aspects of Cut Elimination, Studia Logica 77:209-240, 2004.

[7] J. Lambek, The mathematics of sentence structure. American Mathematical Monthly 65:154-170, 1958.

[8] Z. Lin, Modal Nonassociative Lambek Calculus with Assumptions: Complexity and Context-freeness, In: A.H. Dediu, H. Fernau, C. Martin-Vide (eds.), Language and Automata Theory and Applications, Fourth International Conference, LATA 2010, LNCS 6031, 414-426.

[9] M. Okada, and K. Terui, The Finite Model Property for Various Fragments of Intuitionistic Linear Logic Journal of Symbolic Logic 64:790-802, 1999.

[10] M. Moortgat, Multimodal linguistic inference. Journal of Logic, Language and Information 5:349-385, 1996.

[11] A.R. Plummer, S4 enriched multimodal categorial grammars are context-free. The- oretical Computer Science 388:173-180, 2007. Corrigendum. Theoretical Computer Scinece 403:406-408, 2008.

[12] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, Residuated Lattices: An Algebraic Glimpse at Substructural Logic. Elsevier, 2007.

George Metcalfe
Craig Interpolation for Semilinear Varieties
(joint work with Enrico Marchioni)

Craig interpolation is a fundamental property of first-order classical logic enjoying close connections with Robinson's joint consistency theorem, Beth definability, quantifier elimination, and amalgamation, and having applications to, e.g., model checking, consistency proofs, and building modular ontologies. In the context of non-classical (intermediate, substructural, modal, fuzzy) logics, however, this property and its neighbours separate into a broad and often quite bewildering range of notions. The aim of this talk is to investigate Craig interpolation and related properties in the context of logics based on semilinear varieties, i.e., logics with algebraic semantics based on chains. As is well known, Łukasiewicz logic does not have Craig interpolation; indeed, it has been shown by Baaz and Veith that the only t-norm based fuzzy logic with Craig interpolation is Gödel logic. The central question that we address in the talk is whether similar classifications of logics based on semilinear varieties admitting and failing Craig interpolation can be obtained in a wider setting. In particular, we investigate whether idempotency is a necessary condition for Craig interpolation to hold for such logics.

Daniele Mundici
Deductive interpolation in Łukasiewicz logic and amalgamation of MV-algebras

There are several proofs of deductive interpolation for Łukasiewicz infinite-valued propositional logic, and amalgamation for MV-algebras. Some of them use the categorical equivalence between MV-algebras and unital ℓ-groups (and then rely on Pierce's amalgamation theorem), others use Panti's classification of prime ideals in MV-algebras, others, like the proof by Kihara and Ono, follow by applying to MV-algebras general results in universal algebra. A short, quite elementary proof can be given using the basic properties of rational polyhedra, i.e., finite unions of simplexes with rational vertices, and noting that the zeroset of every McNaughton function f is a rational polyhedron, and rational polyhedra are preserved under projections onto rational hyperplanes.

Carles Noguera
First-Order Substructural Logics: An Algebraic Approach
(joint work with Petr Cintula)

Substructural logics have been traditionally characterized as those logics such that, when presented by means of a (sequent) proof system, lack one or more of the usual structural rules: weakening, exchange and contraction. Nonetheless, it is well known that they can also be, very usefully, roughly described as the *logics of residuated lattices*. Indeed, from this point of view we have seen, specially in the last decade, a florescence of works on propositional substructural logics, mainly capitalizing on the fact that they can be given an algebraic semantics based on some class of (expansions or (sub)reducts of) residuated lattices, and hence by using the tools and techniques from (Abstract) Algebraic Logic (see e.g. [8]). The same applies, to a lesser extent, to first-order formalisms for substructural logics, inasmuch they can given a semantics which, though not purely algebraic as in the propositional case, it contains an essential algebraic part together with a domain of individuals to interpret first-order variables and terms. Prominent examples of this approach are the following:

- Rasiowa-Sikorski style Intuitionistic first-order predicate logic (see [12])
- Rasiowa implicative first-order predicate logics (see [11])
- Gödel-Dummett first-order predicate logic (see [6])
- First-order fuzzy logics (Hájek, Esteva, Godo, ...) (see e.g. [9],[7],[10],[2])

In all these cases one starts from a propositional logic, which enjoys an equivalent algebraic semantics in the sense of [1], and an implication connective defining an order relation on the algebras. Then, a first-order language is defined in the traditional way (with the usual notions of variables, functionals, predicates, existential and universal quantifiers, bounded and free occurrences of variables, notion of substitutability, atomic formulae, formulae etc.) by using the connectives of the underlying propositional logic to combine first-order atomic formulae and quantifications thereof. Finally, a semantics for this syntax is introduced in such a way that: (1) predicate symbols are interpreted as mappings sending (tuples of) individuals in the domain to truth-values in a member of the algebraic semantics, (2) connectives are interpreted as their corresponding operations in the algebra, (3) the existential (resp. universal) quantification of a formula is interpreted as the supremum (resp. infimum) of the values of its instances w.r.t. the order given by the implication.

In this talk we will generalize this approach to first-order logics based on any propositional substructural logic (algebraizable finitary expansions of the implicational fragment of the logic of non-associative residuated lattices) and obtain the following results:

1. A general uniform proof of completeness theorem for these logics.
2. In particular, we will have (1) a notion of *minimal* first-order logic over a given propositional logic L, which is complete w.r.t. the widest algebraic semantics based on L-algebras and its axiomatization, and (2) a notion of *first-order logic over* L which are complete w.r.t. subclasses of that semantics.
3. By means of a generalized notion of disjunction,(the necessary background will be introduced in the previous talk Cintula, Noguera: *Lattice disjunction is not a disjunction (in many substructural logics)* and it is based on the papers [3],[4]) we obtain a characterization and axiomatization of the first-order logic over a given L which is complete w.r.t. the semantics based on RFSI L-algebras (in particular, we will have the axiomatization of the extension given by linearly ordered algebras).
4. A form of Skolemization for these logics.

The technical details of our results will be available in the forthcoming paper [5].

References

[1] W.J. Blok, D. Pigozzi. Algebraizable logics, *Memoirs of the American Mathematical Society* 396, vol 77, 1989.

[2] P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna, C. Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies. *Annals of Pure and Applied Logic*, 160(1):53–81, 2009.

[3] P. Cintula, C. Noguera. Implicational (semilinear) logics I: a new hierarchy. *Archive for Mathematical Logic*, 49: 417–446, 2010.

[4] P. Cintula, C. Noguera. Implicational (semilinear) logics II: disjunction and completeness properties. In preparation.

[5] P. Cintula, C. Noguera. First-order predicate implicational logics. In preparation.

[6] M. Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 27:97–106, 1959.

[7] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems*, 124: 271–288, 2001.

[8] N. Galatos, P. Jipsen, T. Kowalski, H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[9] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordercht, 1998.

[10] P. Hájek, P. Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.

[11] H. Rasiowa. *An Algebraic Approach to Non-Classical Logics*. North-Holland, Amsterdam, 1974.

[12] H. Rasiowa, R. Sikorski *The Mathematics of Metamathematics*, Panstwowe Wydawnictwo Naukowe, Warsaw, 1963.

Hiroakira Ono

Regular completions of residuated lattices

Completions of lattice-ordered algebras have been studied in many literatures on lattices and order. In the present talk, we will discuss completions of residuated lattices with regular embeddings, i.e. embeddings preserving all existing joins and meets, and will show how these completions can provide us algebraic completeness of substructural predicated logics. Both canonical extensions and MacNeille completions are two important ways of completions. While MacNeille completions are always regular, canonical extensions are never so. On the other hand, it is well-known that MacNeille completions do not always preserve distributivity. We will focus our attention on 'complete ideal completions' of residuated lattices. These completions are always regular and moreover preserve (infinite) distributivity. By using these completions, we will show also some logical results on algebraic completeness of "distributive" substructural predicate logics, and on conservativity of Heyting implication.

Francesco Paoli

Quasi-subtractive varieties

(joint work with Tomasz Kowalski and Matthew Spinks)

Algebras in classical varieties have the property that their lattice of congruences is isomorphic to the lattice of some "special" subsets: e.g. normal subgroups of groups, two-sided ideals of rings, filters (or ideals) of Boolean algebras. Why do such pleasant theorems occur? A rather satisfactory explanation is usually provided along the following lines:

- In every member **A** of a τ-regular variety $\mathbb{V}$ - namely, a variety that arises as the equivalent algebraic semantics of an algebraisable deductive system - the lattice of congruences is isomorphic to the lattice of deductive filters on **A** of the τ-assertional logic of $\mathbb{V}$ ([1]; [2]; [3]).
- In the pointed case, if $\mathbb{V}$ is 1-subtractive [7], then the deductive filters on $\mathbf{A} \in \mathbb{V}$ of the 1-assertional logic of $\mathbb{V}$ coincide with the $\mathbb{V}$-ideals of **A** in the sense of [5], which is even better due to the availability of a manageable concept of *ideal generation*.

However, ideal-congruence isomorphism theorems abound in the literature that are not subsumed by this picture: for example, the correspondence between open filters and congruences in pseudointerior algebras, or between deductive filters and congruences in residuated lattices, or between ideals and MV congruences in quasi-MV algebras. The aim of the present talk is to appropriately generalise the concepts of subtractivity and τ-regularity in such a way as to obtain a general framework encompassing all the previously mentioned results.

A variety $\mathbb{V}$ whose type ν includes a nullary term 1 and a unary term $\Box$ is called *quasi-subtractive* w.r.t. 1 and $\Box$ iff there is a binary term $\rightarrow$ of type ν s.t. $\mathbb{V}$ satisfies the equations

$$\text{(i) } \Box x \rightarrow x \approx 1 \qquad \text{(iii) } \Box(x \rightarrow y) \approx x \rightarrow y$$
$$\text{(ii) } 1 \rightarrow x \approx \Box x \qquad \text{(iv) } \Box(x \rightarrow y) \rightarrow (\Box x \rightarrow \Box y) \approx 1$$

If $\Box$ is the identity, this definition collapses onto the standard definition of subtractive variety. If it is not, we call $\mathbb{V}$ *properly* quasi-subtractive.

Examples include: subtractive varieties (and their nilpotent shifts), pseudointerior algebras, Boolean algebras with operators, residuated lattices, Nelson algebras, subresiduated lattices, basic algebras, quasi-MV algebras. Some of these varieties are not subtractive; some are but can be viewed as properly quasi-subtractive with a different choice of witness terms.

The most important tool for the investigation of quasi-subtractive varieties is the notion of *open filter*, which is to quasi-subtractive varieties as the concept of Gumm-Ursini ideal is to subtractive varieties. A $\mathbb{V}$-*open filter term* in the variables $\overrightarrow{x}$ is is an $n+m$-ary term $p\left(\overrightarrow{x},\overrightarrow{y}\right)$ of type ν s.t.

$$\{\Box x_i \approx 1 : i \leq n\} \vdash_{Eq(\mathbb{V})} \Box p\left(\overrightarrow{x},\overrightarrow{y}\right) \approx 1.$$

A $\mathbb{V}$-*open filter* of $\mathbf{A} \in \mathbb{V}$ is a subset $F \subseteq A$ which is closed w.r.t. all $\mathbb{V}$-open filter terms p (i.e. whenever $a_1, ..., a_n \in F, b_1, ..., b_m \in A$, $p\left(\overrightarrow{a},\overrightarrow{b}\right) \in F$) and such that for every $a \in A$, we have that $a \in F$ iff $\Box a \in F$.

Finally, a variety $\mathbb{V}$ is called *weakly* $(\Box x, 1)$-*regular* iff the $\{\Box x \approx 1\}$-assertional logic $\mathcal{S}(\mathbb{V})$ of $\mathbb{V}$ is strongly and finitely algebraisable.

We have achieved several results, including:

1) A proof that every quasi-subtractive variety $\mathbb{V}$ is such that for all $\mathbf{A} \in \mathbb{V}$, every $\mathbb{V}$-open filter of $\mathbf{A}$ is a τ-class of some $\theta \in \mathrm{Con}(\mathbf{A})$ (for $\tau = \{\Box x \approx 1\}$);

2) A proof that, if $\mathbb{V}$ is quasi-subtractive and $\mathbf{A} \in \mathbb{V}$, $\mathbb{V}$-open filters of $\mathbf{A}$ coincide with deductive filters on $\mathbf{A}$ of $\mathcal{S}(\mathbb{V})$;

3) A manageable description of generated $\mathbb{V}$-open filters;

4) A decomposition of a quasi-subtractive variety as a subdirect product of a subtractive and a "flat" variety (together with conditions under which the decomposition is direct), having as corollaries the Galatos-Tsinakis decomposition theorem for GMV algebras [4] and the Jónsson-Tsinakis decomposition theorem for the join $\mathbb{LG} \vee \mathbb{IRL}$ in the lattice of subvarieties of residuated lattices [6];

5) An investigation of constructions generalising kernel contractions in residuated lattices;

6) A proof that, if $\mathbb{V}$ is quasi-subtractive and weakly $(\Box x, 1)$-regular and $\mathbb{V}'$ is the equivalent algebraic semantics of $\mathcal{S}(\mathbb{V})$, then in any $\mathbf{A} \in \mathbb{V}$ there is a lattice isomorphism between the lattice of $\mathbb{V}$-$\mathbb{V}'$ congruences on $\mathbf{A}$ and the lattice of $\mathbb{V}$-open filters on $\mathbf{A}$.

In view of the examples mentioned above, we believe that the notion of quasi-subtractive variety could provide a common umbrella for the algebraic investigation of several families of logics, including substructural logics, modal logics, quantum logics, logics of constructive mathematics.

References

[1] Blok W.J., Pigozzi D., *Algebraizable Logics*, Memoirs of the AMS, number 396, American Mathematical Society, Providence, RI, 1989.

[2] Blok W.J., Raftery J.G., "Ideals in quasivarieties of algebras", in X. Caicedo and C.H. Montenegro (Eds.), *Models, Algebras and Proofs*, Dekker, New York, 1999, pp. 167-186.

[3] Czelakowski, J., "Equivalential logics I", *Studia Logica*, 45, 1981, pp. 227-236.

[4] Galatos N., Tsinakis C., "Generalized MV algebras", *Journal of Algebra*, 283, 1, 2005, pp. 254-291.

[5] Gumm, H.P., Ursini, A., "Ideals in universal algebra", *Algebra Universalis*, 19, 1984, pp. 45-54.

[6] Jónsson B., Tsinakis C., "Products of classes of residuated structures", *Studia Logica*, 77, 2004, pp. 267-292.

[7] Ursini A., "On subtractive varieties I", *Algebra Universalis*, 31, 1994, pp. 204-222.

Jiří Rachůnek

State operators on bounded residuated ℓ-monoids

States on algebras of many valued and fuzzy logics, such as MV-algebras, GMV-algebras, BL-algebras, pseudo BL-algebras and bounded residuated ℓ-monoids ($R\ell$-monoids, in short), are analogues of probability measures. Flaminio and Montagna (2007, 2009) introduced state MV-algebras as MV-algebras with a unary operation, an internal state, satisfying some basic properties of states. Di Nola and Dvurečenskij (2009) presented a stronger variation of state MV-algebras called state-morphism MV-algebras. Rachůnek and Šalounová (2010) introduced state GMV-algebras and state-morphism GMV-algebras which need not be commutative. Ciungu, Dvurečenskij and Hyčko (2010) recently presented state BL-algebras and state-morphism BL-algebras.

We present state $R\ell$-monoids and state-morphism $R\ell$-monoids as a (non-commutative) generalization of the mentioned algebras.

James Raftery

Finiteness properties for idempotent residuated structures

(joint work Ai-ni Hsieh)

A class K of similar algebras is said to have the *finite embeddability property* (briefly, the *FEP*) if every finite subset of an algebra in K can be extended to a finite algebra in K, with preservation of all partial operations. If a finitely axiomatized variety or quasivariety of finite type has the FEP, then its universal first order theory is decidable, hence its equational and quasi-equational theories are decidable as well. Where the algebras are residuated ordered groupoids, these theories are often interchangeable with logical systems of independent interest. Partly for this reason, there has been much recent investigation of finiteness properties such as the FEP in varieties of residuated structures.

A residuated partially ordered monoid is said to be *idempotent* if its monoid operation is idempotent. In this case, the partial order is equationally definable, so the structures can be treated as pure algebras. Such an algebra is said to be *conic* if each of its elements lies above or below the monoid identity t; it is *semiconic* if it is a subdirect product of conic algebras. We prove that

> *the class* SCIP *of all semiconic idempotent commutative residuated po-monoids is locally finite,*

i.e., every finitely generated member of this class is a finite algebra. It turns out that SCIP is a quasivariety; it is not a variety.

The lattice-ordered members of SCIP form a variety SCIL, provided that we add the lattice operations $\wedge, \vee$ to the similarity type. This variety is not locally finite, but the local finiteness of SCIP facilitates a proof that SCIL has the FEP. In fact, we show that *for every relative subvariety* K *of* SCIP*, the lattice-ordered members of* K *form a variety with the FEP.*

(A *relative subvariety* of SCIP is a subclass axiomatized, relative to SCIP, by some set of equations.) It is also shown that

SCIL *has a continuum of semisimple subvarieties.*

Note that SCIL contains all Brouwerian lattices, i.e., the algebraic models of positive intuitionistic logic. It also includes all positive Sugihara monoids (cf. [3]); these algebras model the positive fragment of the system **R**–mingle. The results here give a unified explanation of the strong finite model property for many extensions of these and other systems. They generalize Diego's Theorem, as well as the main theorem of [5], which showed that the variety generated by all idempotent commutative residuated *chains* is locally finite. Another generalization of the latter result, in a different direction, has been obtained in [4]. Further, we show that

the involutive algebras in SCIL *are subdirect products of chains.*

Although SCIL is finitely axiomatized, it is not clear whether SCIP has a finite basis. Motivated in part by this question, we consider the larger quasivariety IP of *all* idempotent commutative residuated po-monoids. It is proved that

a relative subvariety of IP *consists of semiconic algebras if and only if it satisfies* $x \approx (x \to \mathsf{t}) \to x$.

It follows that SCIP is not itself a relative subvariety of IP. The result also has corollaries for the logical system $\mathbf{RMO}^*$, which adds fusion and the Ackermann truth constant to Anderson and Belnap's $\mathbf{RMO}_{\to}$ in a natural (and conservative) manner. Because the axiomatic extensions of $\mathbf{RMO}^*$ are in one-to-one correspondence with the relative subvarieties of IP, we can infer the following:

If an axiomatic extension of $\mathbf{RMO}^*$ *has* $((p \to \mathsf{t}) \to p) \to p$ *among its theorems, then it is locally tabular*

(i.e., it has only finitely many inequivalent n–variable formulas, for every finite n). In particular, such an extension is strongly decidable, provided that it is finitely axiomatized.

Most of the results reported here have been written up in [1,2].

References

[1] A. Hsieh, *Some locally tabular logics with contraction and mingle*, Rep. Math. Logic **45** (2010), 143–159.

[2] A. Hsieh, J.G. Raftery, *Semiconic idempotent residuated structures*, Algebra Universalis **61** (2009), 413–430.

[3] J.S. Olson, J.G. Raftery, *Positive Sugihara monoids*, Algebra Universalis **57** (2007), 75–99.

[4] J.S. Olson, J.G. Raftery, *Residuated structures, concentric sums and finiteness conditions*, Communications in Algebra **36** (2008), 3632–3670.

[5] J.G. Raftery, *Representable idempotent commutative residuated lattices*, Trans. Amer. Math. Soc. **359** (2007), 4405–4427.

Antonino Salibra
Weird models of linear logic
(joint work with A. Carraro and T. Ehrhard)

The category of sets and relations is a quite standard denotational model of linear logic which underlies most denotational models of this system (coherence spaces, hypercoherence spaces, totality spaces, finiteness spaces…). In this setting, a formula is interpreted as a set, and a proof of that formula is interpreted as a subset of the set interpreting the formula. Logical connectives are interpreted very simply: tensor product, par and linear implication are interpreted as cartesian products, whereas direct products (with) and direct sums (plus) are interpreted as disjoint union. The linear negation of a set is the same set. The exponential is interpreted as the functor which maps a set X to the set of all finite multisets of elements of X. In the co-Kleisli cartesian closed category of the multiset comonad there exist models of the untyped lambda calculus, whose theories are sensible. With this standard multiset-based interpretation of exponentials, the relational model interprets also the differential extensions of linear logic and of the lambda calculus. In this extension of the lambda calculus, terms can be derived (differentiated) as in differential calculus of real analysis. In this talk, we show that, given a semi-ring with unit which satisfies some conditions, we define an exponential functor on the category of sets and relations which allows to define a denotational model of Differential Linear Logic and of the lambda-calculus with resources. We show that, when the semi-ring has an element which is infinite in the sense that it is equal to its successor, this model does not validate the Taylor formula and that it is possible to build, in the associated co-Kleisli cartesian closed category, a model of the pure lambda-calculus which is not sensible. This is a quantitative analogue of the standard graph model construction in the category of Scott domains.

Dana Šalounová
Filter theory of bounded residuated lattice ordered monoids

Bounded residuated lattice ordered monoids ($R\ell$-monoids) are a common generalization of pseudo-BL-algebras, pseudo-MV-algebras and Heyting algebras, i.e. algebras of the non-commutative basic fuzzy logic (and consequently of the basic fuzzy logic), the non-commutative Łukasiewicz logic (and consequently of the Łukasiewicz logic) and the intuitionistic logic, respectively. Various classes of filters of BL-algebras were studied by Turunen (2001), Haveshki, Saeid and Eslami (2006) and Kondo and Dudek (2008). Zhang and Li (2006) generalized some of the results also for pseudo-BL-algebras. Boolean filters of bounded commutative $R\ell$-monoids were investigated by Rachůnek and Šalounová (2005).

We develop the theory of filters of general bounded $R\ell$-monoids and we describe the classes of filters such that the quotient $R\ell$-monoids corresponding to normal filters of them are Heyting algebras, Boolean algebras and pseudo-MV-algebras, respectively.

Takahiro Seki

An Algebraic Proof of the γ-admissibility of Relevant Modal Logics

The admissibility of Ackermann's rule γ is one of the most important problems in relevant logics. The γ-admissibility was first proved by an algebraic method. However, the development of Routley-Meyer semantics and metavaluational techniques makes it possible to prove the γ-admissibility using methods that make use of normal models or metavaluations, and the use of such methods is preferred. We discuss an algebraic proof of the γ-admissibility of relevant modal logics based on modern algebraic models.

Kazushige Terui

The birth of linear logic

Historically, linear logic has arisen as an outcome of deep semantic analysis of computation under the proofs-as-programs (Curry-Howard) paradigm. Although its syntax is frequently referred to in the substructural logic community, its original motivations are rarely discussed.

To bridge a gap between the two communities, linear and substructural, I will give an informal and personal account to the birth of linear logic, not from a historical, but from a conceptual point of view. In particular, I will stress how the key concepts of linear logic, such as control of structural rules, decomposition of intuitionistic implication and involutivity of negation, naturally arise in the context of denotational semantics; although linear logic looks rather artificial as a formalization of reasoning, it is extremely natural in the semantic understanding of computation.

Jiří Velebil

Logical connections in the many-sorted setting

(joint work with Alexander Kurz)

We generalize the concept of dual adjunctions given by a schizophrenic object to those that are induced by schizophrenic modules. This concept allows one to treat various modal coalgebraic calculi in a uniform way. We show applications to classical modal coalgebraic logics, logics over posets, categories of presheaves etc.

William Young

Modal Substructural Logics

(joint work with Hiroakira Ono)

In this talk, we will consider substructural logics with a modal operator. In particular, we will investigate an extension of the original Gödel-McKinsey-Tarski translation to substructural logics. We will examine the differences between the original translation and our substructural one. As this research has only been fairly recent begun, significant attention will be given to prospects for future research. For example, we will begin an analysis of modal substructural logics which are conservative over a given substructural logic L and also contain all of the translations of another (possibly equal) substructural logic K.