Feasibility of program runs in fuzzified Propositional Dynamic Logic

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Propositional dynamic logic

Recall: $\text{PDL} = \text{logical modeling of actions over states}$
   (non-deterministic programs over possible worlds)

Language of $\text{PDL} = \begin{cases} \text{programs } \alpha, \beta, \ldots \\ \text{formulae } \varphi, \psi, \ldots \end{cases}$

defined by simultaneous recursion from atomic ones:

- Boolean combinations of formulae are formulae
- $[\alpha] \varphi \ldots \varphi$ holds after any run of $\alpha$ (dyn. necessity)
- $\langle \alpha \rangle \varphi \ldots \varphi$ holds after some run of $\alpha$ (dyn. possibility)
- $\alpha \cup \beta \ldots \text{do } \alpha \text{ or } \beta$ (non-deterministic choice)
- $\alpha; \beta \ldots \text{do } \alpha \text{ and then } \beta$ (concatenation)
- $\alpha^* \ldots \text{repeat } \alpha \text{ finitely many times}$ (iteration)
- $\varphi? \ldots \text{continue if } \varphi, \text{ else fail}$ (test)
Semantics of PDL = multimodal Kripke models \( \langle W, R, V \rangle \)

- \( W \) = a non-empty set (of states)
- \( R \) = an evaluation of programs by binary relations on \( W \)
- \( V \) = an evaluation of formulae by subsets of \( W \)

such that

\[
V_{\neg \varphi} = W \setminus V_{\varphi} \quad \text{(complement)}
\]
\[
V_{\varphi \land \psi} = V_{\varphi} \cap V_{\psi} \quad \text{(intersection)}
\]
\[
V_{\langle \alpha \rangle \varphi} = R_{\alpha} \leftarrow V_{\varphi} \quad \text{(preimage)}
\]
\[
V_{[\alpha] \varphi} = R_{\alpha} \leftrightarrow V_{\varphi} \quad \text{(dual preimage)}
\]
\[
R_{\alpha; \beta} = R_{\alpha} \circ R_{\beta} \quad \text{(composition)}
\]
\[
R_{\alpha \cup \beta} = R_{\alpha} \cup R_{\beta} \quad \text{(union)}
\]
\[
R_{\alpha^*} = R_{\alpha}^* \quad \text{(transitive closure)}
\]
\[
R_{\varphi^?} = \text{Id} \cap V_{\varphi} \quad \text{(identity on } V_{\varphi} \text{)}
\]

Validity: \( V_{\varphi} = W \), tautologicity = validity in all models
A Kripke frame:
Axioms of PDL (sound and complete wrt the above semantics):

All propositional tautologies, plus

\[
\langle \alpha \rangle \varphi \leftrightarrow \neg[\alpha] \neg \varphi \quad \text{(interdefinability)}
\]

\[
[\alpha](\varphi \to \psi) \to ([\alpha] \varphi \to [\alpha] \psi) \quad \text{(K)}
\]

\[
[\alpha; \beta] \varphi \leftrightarrow [\alpha][\beta] \varphi \quad \text{(concatenation)}
\]

\[
[\alpha \cup \beta] \varphi \leftrightarrow [\alpha] \varphi \land [\beta] \varphi \quad \text{(choice)}
\]

\[
[\alpha^*] \varphi \leftrightarrow \varphi \land [\alpha][\alpha^*] \varphi \quad \text{(iteration)}
\]

\[
[\varphi?] \psi \leftrightarrow (\varphi \to \psi) \quad \text{(test)}
\]

and derivation rules

\[
\varphi, \varphi \to \psi \mid \psi \quad \text{(MP)}
\]

\[
\varphi \mid [\alpha] \varphi \quad \text{(Nec)}
\]

\[
\varphi \to [\alpha] \varphi \mid \varphi \to [\alpha^*] \varphi \quad \text{(induction)}
\]
**PDL and costs of program runs**

Classical PDL models a run of a program $\alpha$ as a transition from a state $w$ to a state $w'$ such that $R_\alpha ww'$

$R_\alpha$ is *bivalent* ($w'$ is either accessible from $w$ by $\alpha$ or not)

In theory, $w'$ might be accessible from $w$ by a program $\alpha$, yet in practice, the run of $\alpha$ may not be *feasible*:

- Is too long—e.g., $10^{100}$ steps
  (frequent in exponentially complex problems)
- Requires too much memory, etc.

Unfeasible runs should *not* play a role in *practical* assessments of $\langle \alpha \rangle \varphi$, $[\alpha] \varphi$, ...
Problem: Classical PDL cannot exclude unfeasible runs, since there is no sharp boundary to feasibility of runs
(feasibility is a fuzzy property)

Solution: Taking costs of program runs into account

First idea: Weighting $R_{\alpha}ww'$ by costs $C_{\alpha}ww' \in \mathbb{R}$?
But again: no sharp boundary to feasible/cheap/… runs

A logically cleaner approach:
Admit that feasibility is a fuzzy modality
$\Rightarrow$ model costs by using a suitable fuzzy logic
Fuzzy logics as logics of costs


Fuzzy logics admit an interpretation in terms of costs:
(the implicit fuzzy modality “ϕ is cheap”)

- 1 . . . “for free”
- 0 . . . unaffordable or maximal cost  (NB: reverse order!)
- ϕ & ψ . . . the cost of ϕ “plus” the cost of ψ
  Various t-norms = different ways of “adding” costs:
    Łukasiewicz . . . bounded sum
    product . . . ordinary sum (via logarithm)
    Gödel . . . maximum
- ϕ → ψ . . . the cost of ψ, “given” the cost of ϕ
  The &-surplus cost of ψ over the cost of ϕ
Recall: Fully true implications in fuzzy logics preserve degrees
\[ \|\varphi \rightarrow \psi\| = 1 \text{ iff } \|\psi\| \geq \|\varphi\| \]

\[\Rightarrow\] Tautologies of fuzzy logics in the form of implication
express transformations of costs that “preserve cheapness”
\[ = \text{ inference salvis expensis} \]

Particular fuzzy logics (of a single t-norm: G, Ł, Π, . . .)
correspond to various ways of combining prices

General fuzzy logics (of classes of t-norms: MTL, BL, . . .)
give the rules independent of the cost fusion
Fuzzified semantics of non-classical logics

Our approach to modeling costs of program runs:
   cost-interpreted fuzzification of PDL models

Fuzzified models for other logics
   (including multimodal Kripke models = semantics of PDL)
   can conveniently be described in higher-order fuzzy logic

Recall: (Henkin-style) higher-order fuzzy logic has
   ● (Fuzzy) predicates of all finite orders and arities
   ● Typed fuzzy membership predicate
   ● Axioms of comprehension and extensionality
   ● Usual mathematical abbreviations
Modeling costs of program runs in fuzzy PDL

To take costs into account, replace bivalent $R_\alpha w w'$ in $\langle W, R, V \rangle$ by a fuzzy relation:

$\tilde{R}_\alpha w w' = \text{the truth degree of } \text{"the run } w \xrightarrow{\alpha} w' \text{ is cheap"}$

Then it is natural to define $\tilde{R}_{\alpha;\beta}$ as

$\tilde{R}_{\alpha;\beta} w w' \equiv (\exists w'')(\tilde{R}_\alpha w w'' \& \tilde{R}_\beta w'' w')$

& . . . summing the costs for $w \xrightarrow{\alpha} w'' \xrightarrow{\beta} w'$ (by some t-norm)

$\exists \ldots \text{ we are interested in the cheapest way}$

$(\text{inf of costs} = \text{sup of degrees} = \exists \text{ in first-order fuzzy logic})$

Thus $\tilde{R}_{\alpha;\beta} = \tilde{R}_\alpha \circ \tilde{R}_\beta \text{ (composition of fuzzy relations)}$

exactly as in classical PDL (where $R_{\alpha;\beta} = R_\alpha \circ R_\beta$)
Similarly it is natural to define

\[ \tilde{R}_{\alpha \cup \beta} = \tilde{R}_{\alpha} \cup \tilde{R}_{\beta} \]  
(fuzzy union)

\[ \tilde{R}_{\alpha^*} = \tilde{R}_{\alpha}^* \]  
(fuzzy transitive closure)

exactly as in classical PDL

⇒ **Costs of complex programs** are calculated by fuzzy versions of classical operations (we postpone the discussion of tests \( \varphi ? \))
The apparatus of fuzzy logic allows us to define fuzzy properties like “feasible run”:

\( \tilde{R}_\alpha ww' \) can be understood as expressing the fuzzy property of feasibility/cheapness/…

\( \langle \alpha \rangle \varphi \) defined by \( V_{\langle \alpha \rangle \varphi} \equiv (\exists w')(\tilde{R}_\alpha ww' \& V\varphi w') \) then means “\( \varphi \) holds after a feasible run of \( \alpha \)”

Notice: \( V_{\langle \alpha \rangle \varphi} \) can be fuzzy even for bivalent \( \varphi \)

\( \Rightarrow V \) has to be fuzzified as well … models \( \langle W, \tilde{R}, \tilde{V} \rangle \)
A fuzzified Kripke frame:
We can derive theorems on costs by derivations in fuzzy logic

E.g.: checking the soundness of the axioms of classical PDL

\[
\begin{align*}
(;) & \quad [\alpha; \beta] \varphi \iff [\alpha][\beta] \varphi & \text{sound} \\
(\cup) & \quad [\alpha \cup \beta] \varphi \iff [\alpha] \varphi \land [\beta] \varphi & \text{sound} \\
([\cdot]) & \quad [\alpha^*] \varphi \iff \varphi \land [\alpha][\alpha^*] \varphi & \text{sound} \\
(MP) & \quad \varphi, \varphi \to \psi \lor \psi & \text{sound} \\
(Nec) & \quad \varphi \lor [\alpha] \varphi & \text{sound} \\
(Ind) & \quad \varphi \to [\alpha] \varphi \lor \varphi \to [\alpha^*] \varphi & \text{sound} \\
(K) & \quad [\alpha] (\varphi \to \psi) \to ([\alpha] \varphi \to [\alpha] \psi) & \text{fails} \\
(Def) & \quad \langle \alpha \rangle \varphi \iff \neg [\alpha] \neg \varphi & \text{fails}
\end{align*}
\]

**NB:** Interdefinability fails \(\Rightarrow\) dual axioms for \(\langle \cdot \rangle\) are needed

(they are sound by Morsi’s duality)

**Open problem:** Complete axiomatization of fuzzy PDL…?
The role of tests

In classical PDL,

\[ w_0 \xrightarrow{\alpha_0} w_1 \xrightarrow{\alpha_1} w_2 \ldots \xrightarrow{\alpha_i} w_i \not\xrightarrow{\varphi?} w_{i+1} \]

Defining tests analogically in fuzzy PDL (by \( \tilde{\varphi}? = \text{Id} \cap \tilde{\varphi} \))

would make the runs costly rather than partially blocked

The modalities \( \langle \alpha \rangle \varphi \) and \([\alpha] \varphi \) would work well, but

the interpretation of \( \tilde{R}_\alpha \) as the cost of execution would be lost:

- \( \varphi \) can be (almost) true, but difficult to test
  (for instance, a complex tautology)
- or (almost) false, but straightforward to test (e.g., \( \bot \))
We need to distinguish between

- **Feasibility** $\bar{R}_\alpha \omega \omega'$ and
- **Admissibility** $\bar{S}_\alpha \omega \omega'$

of the run of $\alpha$ between the states $\Rightarrow$ models $\langle W, \bar{R}, \bar{S}, \bar{V} \rangle$

Admissibility degrees allow:

- **Branching according to fuzzy conditions**
  
  ("if temperature is low, do $\alpha$")

- **Defining modalities for admissible feasible runs**

  $[\alpha] \varphi \ldots \varphi$ holds after all *admissible* runs
  
  $\llbracket \alpha \rrbracket \varphi \ldots \varphi$ holds after all *feasible admissible* runs

Semantics: $V_{\langle \alpha \rangle \varphi} = \bar{S}_\alpha \leftarrow \bar{V}_\varphi$, $V_{\langle \langle \alpha \rangle \rangle \varphi} = (\bar{S}_\alpha \cap \bar{R}_\alpha) \leftarrow \bar{V}_\varphi$

Axioms: those for $[\cdot], \langle \cdot \rangle$ duplicated also for $\llbracket \cdot \rrbracket, \llcorner \cdot \lrcorner$, plus $\llbracket \alpha \rrbracket \varphi \rightarrow [\alpha] \varphi$ (and dual)
Conclusions

- Classical PDL does not take the costs of program runs into account.

- Costs of runs can be captured by fuzzified PDL:
  (tests $\Rightarrow$ cost and admissibility must be distinguished)

- Fuzzy PDL allows modeling feasible runs, small number of iterations, branching by fuzzy conditions, etc
  (not only for programs, but arbitrary actions)

- Most axioms of classical PDL remain valid in fuzzy PDL
  (complete axiomatization = open problem)