Relating fuzzy autoepistemic logic and Łukasiewicz KD45 modal logic

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Answer set programming

An ASP program is a set of rules of the form

$$a_1 \lor \ldots \lor a_n \leftarrow b_1 \land \ldots \land b_m \land \text{not } c_1 \land \ldots \land \text{not } c_k$$

with

- $a_i$, $b_j$, $c_l$ literals, $0$, $1$
- “not” the negation-as-failure operator

Example:

- suitable $\leftarrow$ certificate $\land$ not criminal record
- certificate $\leftarrow \overline{1}$
Answer set programming

$r_1 : \text{suitable} \leftarrow \text{certificate} \land \neg \text{criminal record}$

$r_2 : \text{certificate} \leftarrow \overline{I}$

Guess: \{suitable, certificate\}

\[
\begin{align*}
\text{suitable} & \leftarrow \text{certificate} \\
\text{certificate} & \leftarrow \overline{I}
\end{align*}
\]

minimal model = guess $\Rightarrow$ \{suitable, certificate\} is an answer set

Guess: \{criminal record\}

\[
\text{certificate} \leftarrow \overline{I}
\]

minimal model $\neq$ guess $\Rightarrow$ \{criminal record\} is not an answer set
Autoepistemic logic

Extension of classical propositional logic with a modal operator $\square$: 

$$\square \alpha : \text{“} \alpha \text{ is believed”}$$

Stable expansion

$T$ is a stable expansion of a theory $A$ iff 

$$T = \{ \varphi \mid A \cup \{ \square \psi \mid \psi \in T \} \cup \{ \neg \square \psi \mid \psi \notin T \} \vdash \varphi \} .$$

- $\vdash$ denotes the notion of proof for classical propositional logic
- each formula $\square \alpha$ is considered as a new atom
Embedding ASP in autoepistemic logic

Program $P$

$$r : a_1 \lor \ldots \lor a_n \leftarrow b_1 \land \ldots \land b_m \land \text{not } c_1 \land \ldots \land \text{not } c_k$$

Corresponding autoepistemic theory $\sigma(P)$

$$\sigma(r) : (b_1 \land \Box b_1) \land \ldots \land (b_m \land \Box b_m) \land \neg \Box c_1 \land \ldots \land \neg \Box c_k$$

$$\rightarrow (a_1 \land \Box a_1) \lor \ldots \lor (a_n \land \Box a_n)$$

Embedding

$M$ is an answer set of $P$ iff $M'$ is a stable expansion of $\sigma(P)$.

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Fuzzy answer set programming

A FASP program is a set of rules of the form

\[
g(a_1, \ldots, a_n) \leftarrow f(b_1, \ldots, b_m, \text{not } c_1, \ldots, \text{not } c_k)
\]

with

- \(a_i, b_l, c_j\) literals or constants in \([0, 1] \cap \mathbb{Q}\)
- \text{not}_j\) negation-as-failure operator and corresponding negator \(N\)
- \(f\) and \(g\) applications of connectives from a fuzzy logic
- \(\leftarrow\) residual implicator

\(\alpha \leftarrow \beta\): truth value of \(\alpha\) is greater or equal than the one of \(\beta\)
Fuzzy answer set programming

\[ r_1: \ a \ \leftarrow \ \text{not } b \]
\[ r_2: \ b \ \leftarrow \ \text{not } a \]

Guess \( l(a) = x, \ l(b) = 1 - x \)

\[ r_1: \ a \ \leftarrow \ [\text{not } b]_l \]
\[ r_2: \ b \ \leftarrow \ [\text{not } a]_l \]

\[ r_1: \ a \ \leftarrow \ x \]
\[ r_2: \ b \ \leftarrow \ 1 - x \]

\( \Rightarrow \ l \) is an answer set
Fuzzy autoepistemic logic

Fuzzy logic with truth constants $\mathcal{L}$ expanded with $\square : \mathcal{L}_\square$.

**Fuzzy stable expansion**

$T : \mathcal{L}_\square \rightarrow [0, 1]$ is a fuzzy stable expansions of a theory $A$ iff

$$T(\varphi) = \inf \left\{ v(\varphi) \mid v \text{ model of } A \cup \left\{ \square \psi \leftrightarrow \overline{T(\psi)} \mid \psi \in \mathcal{L}_\square \right\} \right\}.$$  

- $v : \mathcal{L}_\square \rightarrow [0, 1] : v$ models $\varphi$ iff $v(\varphi) = 1$
- each formula $\square \alpha$ is considered as a new atom
Embedding FASP in fuzzy autoepistemic logic

Program $P$

\[
\begin{align*}
  r : g(a_1, \ldots, a_n) & \leftarrow f(b_1, \ldots, b_m, \text{not } c_1, \ldots, \text{not } c_k) \\
\end{align*}
\]

Corresponding autoepistemic theory $\sigma(P)$

\[
\begin{align*}
  \sigma(r) : f(\min(b_1, \square b_1), \ldots, \min(b_m, \square b_m), N(\square c_1), \ldots, N(\square c_k)) & \\
  \rightarrow g(\min(a_1, \square a_1) \lor \ldots \lor \min(a_n, \square a_n))
\end{align*}
\]

Embedding

$M$ is an answer set of $P$ iff $M'$ is a fuzzy stable expansion of $\sigma(P)$.

M. Blondeel, S. Schockaert, M. De Cock, D. Vermeir, Fuzzy autoepistemic logic and its relation to fuzzy answer set programming, to appear in *Fuzzy Sets and Systems*
Fuzzy autoepistemic logic

- $\nu \in \Omega$, $\emptyset \neq E \subseteq \Omega = \{w \mid w : \mathcal{L} \to [0,1]\}$
  - $\varphi$ formula of $\mathcal{L}$: $\|\varphi\|_{\nu, E} = \nu(\varphi)$
  - $\|\Box \psi\|_{\nu, E} = \inf_{w \in E} \|\psi\|_{w, E}$

Fuzzy autoepistemic model

A fuzzy autoepistemic model of a theory $A$ is a set $E \subseteq \Omega$ such that

$$E = \{\nu \in \Omega \mid \|\varphi\|_{\nu, E} = 1 \text{ for each } \varphi \in A\}.$$  

Equivalence fuzzy autoepistemic models and fuzzy stable expansions

$T$ is a fuzzy stable expansion of $A$ iff there exists a fuzzy autoepistemic model $E$ of $A$ such that $T(\varphi) = \inf_{\nu \in E} \|\varphi\|_{\nu, E}$. 
Axiomatization

- $(k+1)$-valued Łukasiewicz logic with truth constants $L^c_k$
- $S_k = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$ set of the truth constants

- sound and complete axiomatization for the structures in fuzzy autoepistemic logic
- algebraic semantics for this axiomatization
Startpoint \( \Lambda(\text{CFr}, \mathbb{L}_k^C) \): Syntax

- **Language:**
  - every formula of \( \mathbb{L}_k^C \) is a formula
  - if \( \varphi \) and \( \psi \) are formulas, then \( \Box \varphi \), \( \varphi \odot \psi \), and \( \varphi \to \psi \) are formulas as well

- **Axioms:**
  - All the axioms from \( \mathbb{L}_k^C \).
  - \((K)\) \( \Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi) \)
  - \((\Box 2)\) \( \Box (\varphi \land \psi) \to \Box (\varphi \land \psi) \)
  - \((\Box 3)\) \( \Box (\bar{r} \to \varphi) \leftrightarrow (\bar{r} \to \Box \varphi) \), for each \( r \in S_k \)
  - \((\Box 4)\) \( \Box (\bar{r} \lor \varphi) \leftrightarrow (\bar{r} \lor \Box \varphi) \)
  - \((\Box 5)\) \( \Box (\varphi \oplus \Box \varphi) \leftrightarrow \Box (\varphi \oplus \varphi) \)

and its inference rules are modus ponens (from \( \varphi \) and \( \varphi \to \psi \), infer \( \psi \)) and necessitation for \( \Box \) (from \( \varphi \) infer \( \Box \varphi \)).

- Necessitation is *local*: if \( \varphi \) is a theorem, then \( \Box \varphi \) is a theorem as well

Startpoint $\Lambda(\text{CFr}, \mathcal{L}_k^C)$: Semantics

- Truth defined relative to Kripke structures $M = (W, e, R)$
  - $W \neq \emptyset$ set of worlds
  - $R : W \times W \to \{0, 1\}$ accessibility relation
  - for every $w \in W$ $e(\cdot, w) : \mathcal{L}_k^C \to S_k$

The truth value of $\varphi$ in a world $w \in W$ is defined as

(a) If $\varphi$ is a formula of $\mathcal{L}_k^C$, then $\|\varphi\|_{M,w} = e(\varphi, w)$.

(b) $\|\Box \psi\|_{M,w} = \inf\{\|\psi\|_{M,v} | v \in W, R(w, v) = 1\}$.

- Consequence relation $\Gamma \models \Box \varphi$ iff for every $M = (W, e, R)$ and for every $w \in W$

  $\|\psi\|_{M,w} = 1$ for every $\psi \in \Gamma \Rightarrow \|\varphi\|_{M,w} = 1$.

**KD45(\text{CFr}, L^C_k): Syntax**

**Axioms:**

- All the axioms from $\Lambda(\text{CFr}, L^C_k)$.
- $(D) \quad \Box 1$
- $(4) \quad \Box \varphi \rightarrow \Box \Box \varphi$
- $(5) \quad \Diamond \Box \varphi \rightarrow \Box \varphi$

with $\Diamond \varphi = \neg \Box \neg \varphi$.

and its inference rules are modus ponens (from $\varphi$ and $\varphi \rightarrow \psi$, infer $\psi$) and necessitation for $\Box$ (from $\varphi$ infer $\Box \varphi$).

- Necessitation is *local*: if $\varphi$ is a theorem, then $\Box \varphi$ is a theorem as well.
Truth defined relative to Kripke structures $M = (W, e, R)$
- $W \neq \emptyset$ set of worlds
- $R : W \times W \to \{0, 1\}$ accessibility relation
- for every $w \in W$ $e(\cdot, w) : L^c_k \to S_k$

The truth value of $\varphi$ in a world $w \in W$ is defined as
(a) If $\varphi$ is a formula of $L^c_k$, then $\|\varphi\|_{M,w} = e(\varphi, w)$.
(b) $\|\Box \psi\|_{M,w} = \inf\{\|\psi\|_{M,v} \mid v \in W, R(w, v) = 1\}$.

Consequence relation $\Gamma |= \Box \varphi$ iff for every $M = (W, e, R)$ and for every $w \in W$

$\|\psi\|_{M,w} = 1$ for every $\psi \in \Gamma \Rightarrow \|\varphi\|_{M,w} = 1$.

**Theorem**

$KD45(CFr, L^c_k)$ is sound and complete with respect to the class $KD45$ of Kripke structures $(W, e, R)$ where $R : W \times W \to \{0, 1\}$ is serial, transitive and euclidean.
Motivation and background

What do we want to do?

Startpoint $\Lambda(CFr, L_k^c)$

$KD45(CFr, L_k^c)$

Algebraic semantics

Idea proof completeness

**Idea proof completeness**

Proof completeness by contraposition. Assume $\not\vdash \Box \alpha$.

- $p^* = p$, $r^* = r$,
- $(\varphi \cdot \psi)^* = \varphi^* \cdot \psi^*$,
- $(\Box \varphi)^* = p\varphi$ with $p\varphi$ a new variable

**Lemma**

Let $T \cup \{\alpha\}$ be a set of formulas in $KD45(CFr, L_k^c)$. Let $T^* = \{\varphi^* | \varphi \in T\}$ and $\Lambda = \{\varphi^* | \varphi \text{ axiom in } KD45(CFr, L_k^c)\} \cup \{(\Box \varphi)^* | \vdash \Box \varphi\}$. Then it holds that

$$T \vdash \Box \alpha \quad \text{iff} \quad T^* \cup \Lambda \vdash \alpha^*$$

where $\vdash \Box$ denotes the notion of proof in $KD45(CFr, L_k^c)$ and $\vdash$ the notion of proof in $L_k^c$.

$\not\vdash \Box \alpha \Rightarrow \Lambda \not\vdash \alpha^*$ (lemma)

$\Rightarrow \Lambda \not\vdash \alpha^*$ (strong completeness of $L_k^c$)

$\Rightarrow \exists L_k^c$-model $\nu$ of $\Lambda$ and $\nu(\alpha^*) < 1$
Define $M_{can} = (W_{can}, R_{can}, e_{can})$

- $W_{can} = \{v \mid v \text{ is a } L^C_k\text{-evaluation model of } \Lambda\}$,
- $R_{can}(v_1, v_2) = 1$ iff for every formula $\varphi$ it holds that $v_1((\Box \varphi)^*) = 1$ implies that $v_2(\varphi^*) = 1$, and $R_{can}(v_1, v_2) = 0$ otherwise,
- $e_{can}(v, p) = v(p)$, for each propositional variable $p$.

We show that $M_{can}$ is in $KD45$ and there exists $v \in W_{can}$ such that $\|\alpha\|_{M_{can}, v} < 1$. 
Motivation and background

What do we want to do?

Startpoint $\Lambda(C_{Fr}, L^c_k)$

$KD_{45}(C_{Fr}, L^c_k)$

Algebraic semantics

Idea proof completeness

Lemma

(Truth-lemma) For each formula $\varphi$ of $KD_{45}(C_{Fr}, L^c_k)$ and every $v \in W_{can}$ we have $v(\varphi^*) = \|\varphi\|_{M_{can}, v}$.

This claim follows from the monotonicity for $\square$ and the meet distribution property, by using Lemma 4.20 from F. Bou, F. Esteva, L. Godo, R. Rodríguez, On the minimum manyvalued modal logic over a finite residuated lattice. *Journal of Logic and Computation* 21, 5(2011), 739-790

1. $R_{can}$ is
   - serial (axiom $(D) \Diamond I$)
   - transitive (axiom (4) $\square \varphi \rightarrow \square \square \varphi$)
   - euclidean (axiom (5) $\Diamond \square \varphi \rightarrow \square \varphi$)

2. $\exists L^c_k$-model $v$ of $\Lambda$ and $v(\alpha^*) < 1$. Hence $v \in W_{can}$ and $\|\alpha\|_{M_{can}, v} = v(\alpha^*) < 1$
Using classical techniques one can prove that $KD45(CFr, L^c_k)$ is also sound and complete with respect to the class of Kripke models $(W, e, R)$ where $R = W \times E$ with $\emptyset \neq E \subseteq W$.

From this, the next corollary follows.

**Corollary**

$KD45(CFr, L^c_k)$ is sound and complete with respect to the class $M^{ae}_k$.

where $M^{ae}_k$ is the class of structures $(v, E)$ with $v \in \Omega_k$, $E \subseteq \Omega_k = \{ w \mid w : L^c_k \rightarrow S_k \}$

- $\varphi$ formula of $\mathcal{L}$: $\|\varphi\|_{v, E} = v(\varphi)$
- $\|\Box \psi\|_{v, E} = \inf_{w \in E} \|\psi\|_{w, E}$
Algebraic semantics

A finite MV-algebra with internal necessity operator (NMV-algebra) is a pair \((A, N)\) with \(A\) a finite MV-algebra of the type \((S_k)^X\) and \(N : A \to A\) satisfies

\[
\begin{align*}
(A1) \quad & N(\bar{r}) = \bar{r} \text{ for every } r \in S_k \\
(A2) \quad & N(f \to g) \leq N(f) \to N(g) \\
(A3) \quad & N(f \land g) = N(f) \land N(g) \\
(A4) \quad & N(f) \oplus N(f) = N(f \oplus f) \\
(A5) \quad & N(\bar{r} \to f) = \bar{r} \to N(f) \\
(A6) \quad & N(N(f)) = N(f)
\end{align*}
\]

Theorem

*The class of tautologies of the class of NMV-algebras is equal to the class of tautologies in the class\( KD45\).*
Future work

- decidability and complexity

- $S_k$-valued relation

- infinitely valued Łukasiewicz logic