Extending Giles’s Game for Lukasiewicz Logic to Fuzzy Quantification

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Motivation

- Quantifiers beyond $\forall$ and $\exists$ are important for applications of degree based models of vague language (fuzzy logic): “many”, “few”, “about half”, . . .

- Modeling vague quantifiers by fuzzy quantifiers suffers from a peculiar problem (diagnosed, e.g., by Glöckner): The space of candidates for possible truth functions is vast and hard to constrain axiomatically.
  $\implies$ “Embarrassment of Riches” (EOR challenge)

- Two moves to meet the EOR-challenge — half ways:
  - focus on Łukasiewicz logic (e.g., Novak)
  - focus on semi-fuzzy quantifiers first (Glöckner)

- These moves are useful, but EOR doesn’t disappear.

- We propose that game semantics answers more fully to EOR: its principles of reasoning naturally lead to a quite specific, plausibly motivated family of (monadic) fuzzy quantifiers.
Robin Giles about reasoning in theories of physics


Principles of Giles’s analysis of reasoning:

- All assertions have to be **tested** with respect to concrete (instances of) **binary experiments**:
  Each atomic assertion \( P(t_1, \ldots, t_n) \) is connected to a parameterized experiment \( E_{P}^{t_1, \ldots, t_n} \) that may fail or succeed.

- Experiments may show **dispersion**: different instances of the same experiment may yield **different results**.

- To provide a **tangible meaning** to sentences one imagines a **dialogue** between **me** and **you**, where we are willing to pay \( 1 \) € to the opponent for each false atomic assertion, i.e., one where the corresponding instance of the experiment fails.
  **Note**: since experiments are dispersive, assertions are **risky**!

- A **tenet** collects all assertions of a **player** (**me** or **you**).
  **Repetita juvant**: Tenets are **multisets** of interpreted sentences.
Important observations:

- I can quantify the expected loss for my tenet \( \{q_1, \ldots, q_n\} \) of atomic assertions by assigning a subjective failure probability \( \langle q_i \rangle \) to the experiment \( E_{q_i} \).

- While these probabilities may have some objective grounds they are still subjective in the sense that I don’t care which values you associate with the same experiments.

- Events are (unrepeatable) instances of (repeatable) elementary experiments. In other words: experiments are event types, such that the same probabilities are assigned to events of the same type.

- Final (or: atomic) game states of are denoted by \([p_1, \ldots, p_n \parallel q_1, \ldots, q_m]\), where \( \{p_1, \ldots, p_n\} \) is your tenet and \( \{q_1, \ldots, q_m\} \) is my tenet of assertions.

My corresponding risk, i.e., my expected loss of money is

\[
\sum_{1\leq i\leq m} \langle q_i \rangle \$ - \sum_{1\leq j\leq n} \langle p_j \rangle \$
\]
What about logically complex statements?

NB: So far, no logic has been involved!

For the reduction of logically complex assertions to atomic states, Giles refers to the logical rules introduced by Paul Lorenzen in his dialogue game for constructive reasoning.

Giles states the rules in the following — old fashioned! — way:

- He who asserts \( A \supset B \) agrees to assert \( B \) if his opponent will assert \( A \).
- He who asserts \( A \lor B \) undertakes to assert either \( A \) or \( B \) at his own choice.
- He who asserts \( A \land B \) undertakes to assert either \( A \) or \( B \) at his opponent’s choice.

Defining \( \neg A = A \supset \bot \) leads to

- He who asserts \( \neg A \) agrees to pay 1\$ to his opponent if he will assert \( A \).
Observations about the dialogue part of Giles’s game

(1) Assertions are attacked at most once: *repetita juvant*.

(2) Principle of limited liability for attacks:
The players may explicitly choose not to attack an assertion.

(3) In contrast to Lorenzen:
- no regulations on the succession of moves!
- no restrictions on who can attack what!

(4) Giles defends the $\land$-rule by reference to the above principle of limited liability: each assertion carries a maximal risk of 1$.

Giles has no rule for strict conjunction ($\&$)!
By extending the principle to defense move we obtain:

- *If a player asserts $A \& B$ she has to assert either both, $A$ and $B$, or else has to assert $\bot$ (i.e., to pay 1€).*
Adequateness of Giles’s game for propositional Ł

Theorem (coarse version):
I always have a strategy for avoiding expected loss for precisely those initial statements that are valid in Łukasiewicz logic.

Theorem (refined version):
Suppose we play the game starting with my assertion of $F$ with respect to given assignment $\langle \cdot \rangle$ of risk values to atomic assertions.
The following are equivalent:

1. $F$ evaluates to $1 - x$ in Łukasiewicz logic under the interpretation that assigns $1 - \langle p \rangle$ to each atom $p$.
2. My best strategy guarantees that the play ends in a state, where my risk is at most $x\mathbb{E}$, while you have a strategy enforcing that my risk is at least $x\mathbb{E}$. 
Remarks on our proof of Giles’s Theorem

We have to show that my risk $\langle \cdot \rangle$ can be extended from elementary to arbitrary states in such a way that

$$
\langle \Gamma \mid A \supset B, \Delta \rangle = \max(\langle \Gamma \mid \Delta \rangle, \langle \Gamma, A \mid B, \Delta \rangle) \quad (1)
$$

$$
\langle \Gamma, A \supset B \mid \Delta \rangle = \min(\langle \Gamma \mid \Delta \rangle, \langle \Gamma, B \mid A, \Delta \rangle) \quad (2)
$$

(analogous conditions have to hold for other connectives)

This can be achieved by defining

$$
\langle \Gamma \mid \Delta \rangle^v =_{\text{def}} |\Delta| - |\Gamma| + \sum_{G \in \Gamma} v(G) - \sum_{F \in \Delta} v(F).
$$

for the valuation $v$ assigning $1 - \langle p \rangle$ to each atom $p$.

The fact that no regulations are needed in Giles’s game falls out from the proof. From a game theoretic point of view it is more natural to assume regulations, and prove that they don’t affect the players’ respective ‘power’. (See [FM, StudLog09])
What about quantifiers?

Giles also considered the following quantifier rules:
(We assume that there are constants for all domain elements)

- If a player asserts $\forall x A(x)$ she has to assert $A(c)$ for some $c$ picked by her opponent.
- If a player asserts $\exists x A(x)$ she has to assert $A(c)$ for some $c$ picked by herself.

A complication:
Infima and suprema (of risk) may have no witnessing elements!

Definition:
A game is $r$-valued for me if, for every $\epsilon > 0$
- I have a strategy to limit my expected loss to at most $(r + \epsilon) \€$
- you have a strategy ensuring an expected gain of $(r - \epsilon) \€$

Theorem:
$F$ evaluates to $x$ iff the game for $F$ is $(1 - x)$-valued for me.
Randomizing choices of witnesses

To go beyond $\forall/\exists$ and nevertheless remain with the spirit of Giles’ game we introduce a third type of challenge to asserting $QxA(x)$:

(D) defender picks the witness ($\Rightarrow \existsxA(x)$)

(A) attacker picks the witness ($\Rightarrow \forallxA(x)$)

(R) the witness is picked randomly (uniform distribution!)

The simplest quantifier rule with type-R challenge:

- *If a player asserts $\Pi xA(x)$ she has to assert $A(c)$ for some randomly picked $c$.*

The truth function for quantifier $\Pi$ amounts to a (Sugeno type) integral of the fuzzy set corresponding to $A(x)$.

Intended application restrict to finite domains and semi-fuzzy quantification, i.e., $A(x)$ corresponds to a crisp set.

\[
\Rightarrow ||\Pi xA(x)||_I = Prop_x A(x) = \frac{\sum_{c \in D} ||A(c)||_I}{|D|}
\]
Bets for and bets against

Bet for $A$: assert $A$, risking $(1 - \|A\|_I)\mathbb{E}$

Bet against $A$: assert $\bot$ in exchange for the opponent’s assertion of $A$, risking $\|A\|_I\mathbb{E}$

Example: proportionality quantifiers $\Pi^k_m$:

- If a player asserts $\Pi^k_m x A(x)$ then $k + m$ constants are picked randomly. The player has to partition those constants into $\{c_1, \ldots, c_k\} \cup \{d_1, \ldots, d_m\}$ and to bet for $A(c_1), \ldots, A(c_k)$ as well as to bet against $A(d_1), \ldots, A(d_m)$.

$$\|\Pi^k_m x A(x)\|_I = \binom{k + m}{k} (\text{Prop}_x A(x))^k (1 - \text{Prop}_x A(x))^m$$
Towards linguistic adequateness

Note: no (semi-)fuzzy quantifier directly corresponds to truth conditions as observed by linguistic studies!

However for certain purposes (e.g., “near NL”-query languages) such fuzzy models may nevertheless be very useful.

Randomized game semantics leaves some “riches”: namely a guided choice of parameters. But the “embarrassment of riches” disappears.

Example: the vague quantifier “about half” can be modeled by \( k \) bets for and \( k \) bets against randomized instances in exchange for a wager by the attacker (assertion of \( \perp \)). (NB: in fact only a binary version of the quantifier is adequate.)

More empirical linguistic research is needed!
Conclusion

- We have introduced a new, very simple concept extending Giles-style game semantics for quantified Łukasiewicz logic: randomly chosen witnesses.
- This allows us to model vague quantifiers, while avoiding the “embarrassment of riches” implied by unrestricted many-valued truth functions.
- Many questions for further research remain:
  - characterization of game-representable quantifiers
  - “semi-fuzzy to fuzzy” transformers (à la Glöckner)
  - quantifiers of higher arities
  - automated reasoning, proof theory
  - relation to “fuzzy events” (Mundici, Kroupa)
  - …