BCK is not structurally complete

Tomasz Kowalski

Department of Mathematics and Statistics
La Trobe University, Melbourne

14 September 2012
Logics and consequence operators

A logic in this talk is a set $L$ of formulas in some language $\mathcal{L}$, closed under a finitary, structural consequence operator $C$, that is a compact closure operator acting on sets of formulas, and closed under substitution.
Logics and consequence operators

A logic in this talk is a set $L$ of formulas in some language $\mathcal{L}$, closed under a finitary, structural consequence operator $C$, that is a compact closure operator acting on sets of formulas, and closed under substitution. Any such logic can be presented in good old way, as a set of axioms and a set of rules of inference of the form

$$\frac{\alpha_1, \ldots, \alpha_n}{\beta}$$
Logics and consequence operators

A logic in this talk is a set $L$ of formulas in some language $\mathcal{L}$, closed under a finitary, structural consequence operator $C$, that is a compact closure operator acting on sets of formulas, and closed under substitution. Any such logic can be presented in good old way, as a set of axioms and a set of rules of inference of the form

$$\frac{\alpha_1, \ldots, \alpha_n}{\beta}$$

If $L$ has a reasonable algebraic semantics, then any such rule has a natural quasi-identity counterpart:

$$\alpha_1 \approx 1 \& \ldots \& \alpha_n \approx 1 \implies \beta \approx 1$$
Logics and consequence operators

A logic in this talk is a set $L$ of formulas in some language $\mathcal{L}$, closed under a finitary, structural consequence operator $C$, that is a compact closure operator acting on sets of formulas, and closed under substitution. Any such logic can be presented in good old way, as a set of axioms and a set of rules of inference of the form

$$\frac{\alpha_1, \ldots, \alpha_n}{\beta}$$

If $L$ has a reasonable algebraic semantics, then any such rule has a natural quasi-identity counterpart:

$$\alpha_1 \approx 1 & \ldots & \alpha_n \approx 1 \implies \beta \approx 1$$

We will write $\mathcal{Q}(L)$ for the quasivariety naturally associated with $L$. 
Derivable and admissible rules

- An inference rule $R$ of the form

\[
\frac{\alpha_1, \ldots, \alpha_n}{\beta}
\]

is derivable in $L$ if $\beta \in C(\{\alpha_1, \ldots, \alpha_n\})$, where $C$ is the consequence operation of $L$. 
Derivable and admissible rules

- An inference rule $R$ of the form

$$
\frac{\alpha_1, \ldots, \alpha_n}{\beta}
$$

is derivable in $L$ if $\beta \in C(\{\alpha_1, \ldots, \alpha_n\})$, where $C$ is the consequence operation of $L$. Algebraically:

$$
A \models \alpha_1 \approx 1 & \ldots & \alpha_n \approx 1 \implies \beta \approx 1
$$

for any algebra $A \in Q(L)$. 
Derivable and admissible rules

- An inference rule $R$ of the form

$$
\frac{\alpha_1, \ldots, \alpha_n}{\beta}
$$

is derivable in $L$ if $\beta \in C(\{\alpha_1, \ldots, \alpha_n\})$, where $C$ is the consequence operation of $L$. Algebraically:

$$
A \models \alpha_1 \approx 1 \& \ldots \& \alpha_n \approx 1 \implies \beta \approx 1
$$

for any algebra $A \in Q(L)$.

- An inference rule is admissible if $\sigma(\alpha_1), \ldots, \sigma(\alpha_n) \in L$ implies $\sigma(\beta) \in L$, for any substitution $\sigma$. 
Derivable and admissible rules

- An inference rule $R$ of the form

\[
\frac{\alpha_1, \ldots, \alpha_n}{\beta}
\]

is derivable in $L$ if $\beta \in C(\{\alpha_1, \ldots, \alpha_n\})$, where $C$ is the consequence operation of $L$. Algebraically:

\[
A \models \alpha_1 \approx 1 \& \ldots \& \alpha_n \approx 1 \implies \beta \approx 1
\]

for any algebra $A \in Q(L)$.

- An inference rule is admissible if $\sigma(\alpha_1), \ldots, \sigma(\alpha_n) \in L$ implies $\sigma(\beta) \in L$, for any substitution $\sigma$. Algebraically:

\[
F(\omega) \models \alpha_1 \approx 1 \& \ldots \& \alpha_n \approx 1 \implies \beta \approx 1
\]

for the free $\omega$-generated algebra in $Q(L)$. 
Structural completeness

Definition

- A logic $L$ is **structurally complete** if every rule admissible in $L$ is also derivable in $L$.
- A quasivariety $Q$ is **structurally complete** if every quasi-identity of $F_Q(\omega)$, holds throughout $Q$. 
Structural completeness

Definition

- A logic $L$ is **structurally complete** if every rule admissible in $L$ is also derivable in $L$.

- A quasivariety $Q$ is **structurally complete** if every quasi-identity of $F_Q(\omega)$, holds throughout $Q$.

For a quasivariety $Q$, let $Q^-$ be $Q \setminus \{\text{trivial algebra}\}$. 
Structural completeness

Definition

- A logic \( L \) is \textit{structurally complete} if every rule admissible in \( L \) is also derivable in \( L \).
- A quasivariety \( Q \) is \textit{structurally complete} if every quasi-identity of \( F_Q(\omega) \), holds throughout \( Q \).

For a quasivariety \( Q \), let \( Q^- \) be \( Q \setminus \{ \text{trivial algebra} \} \).

Theorem (Bergmann)

If \( Q \) is structurally complete, then \( Th(Q^-) \cap \{ \varphi, \neg \varphi \} \neq \emptyset \), for any positive existential sentence \( \varphi \) in the language of \( Q \).
BCK logic

Axioms

(B) \((\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \psi))\)

(C) \((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\phi \rightarrow \chi))\)

(K) \(\phi \rightarrow (\psi \rightarrow \phi)\)

Inference rule

\[
\frac{\phi, \phi \rightarrow \psi}{\psi}
\]
BCK logic

Axioms

(B) \((\phi \to \psi) \to ((\chi \to \phi) \to (\chi \to \psi))\)

(C) \((\phi \to (\psi \to \chi)) \to (\psi \to (\phi \to \chi))\)

(K) \(\phi \to (\psi \to \phi)\)

Inference rule

\[
\begin{align*}
\phi, \phi \to \psi &\quad \Rightarrow \\
\psi
\end{align*}
\]

Algebraically, a quasivariety defined by

1. \(x \to 1 \approx 1\)
2. \(1 \to x \approx x\)
3. \((x \to y) \to ((y \to z) \to (x \to z)) \approx 1\)
4. \(x \to (y \to z) \approx y \to (x \to z)\)
5. \(x \to y \approx 1 \& y \to x \approx 1 \quad \Rightarrow \quad x \approx y\)
Sequent system for BCK

Sequents: $\Gamma \Rightarrow \alpha$, with $\Gamma$ a multiset of terms. Initial sequents:

$$\Gamma, x \Rightarrow x$$

where $x$ is a variable. Logical inference rules:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta \Rightarrow \gamma}{\Gamma, \Delta, \alpha \rightarrow \beta \Rightarrow \gamma} \quad (\rightarrow\Rightarrow) \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \quad (\Rightarrow\rightarrow)$$
Sequent system for BCK

Sequents: $\Gamma \Rightarrow \alpha$, with $\Gamma$ a multiset of terms. Initial sequents:

$$\Gamma, x \Rightarrow x$$

where $x$ is a variable. Logical inference rules:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \beta \Rightarrow \gamma}{\Gamma, \Delta, \alpha \rightarrow \beta \Rightarrow \gamma} \quad (\rightarrow\Rightarrow) \quad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \quad (\Rightarrow\rightarrow)$$

Structural rules of cut and weakening

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta} \quad \frac{\Gamma \Rightarrow \alpha}{\beta, \Gamma \Rightarrow \alpha}$$

can also be used, but are eliminable.
Properties of sequent system for BCK

Lemma

Cut and weakening are eliminable. Rule $\to\to$ is invertible.
Properties of sequent system for BCK

Lemma

Cut and weakening are eliminable. Rule $(\Rightarrow\rightarrow)$ is invertible.

Lemma

The sequent $\Gamma \Rightarrow v$, where $v$ is a variable and $v \notin \Gamma$, is provable if and only if there exists a $\gamma = \beta_n\beta_{n-1}\cdots\beta_1 \rightarrow \beta_0 \in \Gamma$ and a partition $\{\Gamma_i\}_{i=1}^{n}$ of $\Gamma \setminus \{\gamma\}$ such that

1. $\beta_0 = v$,

2. for every $i \in \{1, \ldots, n\}$ there is a $\Gamma_i$ such that $\Gamma_i \Rightarrow \beta_i$ is provable.
Properties of sequent system for BCK

**Lemma**

Cut and weakening are eliminable. Rule $(\Rightarrow \rightarrow)$ is invertible.

**Lemma**

The sequent $\Gamma \Rightarrow v$, where $v$ is a variable and $v \notin \Gamma$, is provable if and only if there exists a $\gamma = \beta_n \beta_{n-1} \cdots \beta_1 \rightarrow \beta_0 \in \Gamma$ and a partition $\{\Gamma_i\}_{i=1}^n$ of $\Gamma \setminus \{\gamma\}$ such that

1. $\beta_0 = v$,
2. for every $i \in \{1, \ldots, n\}$ there is a $\Gamma_i$ such that $\Gamma_i \Rightarrow \beta_i$ is provable.

Such a term $\gamma \in \Gamma$ will be called a *split term*. 
Two simplification lemmas

**Lemma**

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a multiset of terms. Suppose the sequent $\Gamma, \alpha \Gamma \rightarrow x \Rightarrow x$ is provable with split term $\gamma_1 = \beta_1 \ldots \beta_k \rightarrow x$. Then, for some $\Gamma' \subseteq \Gamma \setminus \{\gamma_1\}$ and some $i \in \{1, \ldots, n\}$, the sequent $\alpha \Gamma' \rightarrow \beta_i \Rightarrow \Gamma' \rightarrow \beta_i$ is provable. Moreover, the sequent $\Gamma' \rightarrow \beta_i \Rightarrow \Gamma \rightarrow x$ is also provable.
Two simplification lemmas

**Lemma**

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a multiset of terms. Suppose the sequent $\Gamma, \alpha \Gamma \rightarrow x \Rightarrow x$ is provable with split term $\gamma_1 = \beta_1 \ldots \beta_k \rightarrow x$. Then, for some $\Gamma' \subseteq \Gamma \setminus \{\gamma_1\}$ and some $i \in \{1, \ldots, n\}$, the sequent $\alpha \Gamma' \rightarrow \beta_i \Rightarrow \Gamma' \rightarrow \beta_i$ is provable. Moreover, the sequent $\Gamma' \rightarrow \beta_i \Rightarrow \Gamma \rightarrow x$ is also provable.

**Lemma**

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a multiset of terms. Suppose the sequent $\Gamma, \alpha \Gamma \rightarrow x \Rightarrow x$ is provable with split term $\alpha \Gamma \rightarrow x \Rightarrow x$ and $\alpha$ is not a theorem. Then, some $\gamma \in \Gamma$ is a theorem. Therefore, the sequent $\Gamma', \alpha \Gamma' \rightarrow x \Rightarrow x$ is provable, where $\Gamma' = \Gamma \setminus \{\gamma\}$. 
A disjunction property, and main result

**Lemma**

*Suppose the sequent $\alpha \rightarrow \beta \Rightarrow \beta$ is provable. Then $\alpha$ is a theorem or $\beta$ is a theorem.*
A disjunction property, and main result

**Lemma**

*Suppose the sequent* $\alpha \rightarrow \beta \Rightarrow \beta$ *is provable. Then* $\alpha$ *is a theorem or* $\beta$ *is a theorem.*

**Theorem**

*The inference rule*

\[
\frac{(\alpha \rightarrow \beta) \rightarrow \beta}{(\beta \rightarrow \alpha) \rightarrow \alpha}
\]

*is admissible, but not derivable in BCK. Therefore BCK is not structurally complete.*
Proof

To show admissibility, suppose $(\alpha \rightarrow \beta) \rightarrow \beta$ is a theorem, so the sequent $\Rightarrow (\alpha \rightarrow \beta) \rightarrow \beta$ is provable. Thus, the sequent $(\alpha \rightarrow \beta) \Rightarrow \beta$ is provable. By disjunction property then $\alpha$ is a theorem or $\beta$ is a theorem. In either case $(\beta \rightarrow \alpha) \rightarrow \alpha$ is a theorem.
Proof

- To show admissibility, suppose \((\alpha \rightarrow \beta) \rightarrow \beta\) is a theorem, so the sequent \(\Rightarrow (\alpha \rightarrow \beta) \rightarrow \beta\) is provable. Thus, the sequent \((\alpha \rightarrow \beta) \Rightarrow \beta\) is provable. By disjunction property then \(\alpha\) is a theorem or \(\beta\) is a theorem. In either case \((\beta \rightarrow \alpha) \rightarrow \alpha\) is a theorem.

- To prove non-derivability consider the \(\{\rightarrow, 1\}\)-reduct of the totally ordered three-element Heyting algebra \(\{0, a, 1\}\). Take \(\alpha\) and \(\beta\) to be variables, define a valuation \(v\) putting \(v(\alpha) = a\) and \(v(\beta) = 0\). Then, \(v(\alpha \rightarrow \beta) = 0\) and so \(v((\alpha \rightarrow \beta) \rightarrow \beta) = 1\). But \(v(\beta \rightarrow \alpha) = 1\) and therefore of \(v((\beta \rightarrow \alpha) \rightarrow \alpha) = 0\).
Overflow completeness

In 2005, Wroński considered the following weakening of structural completeness.

**Definition (Wroński)**

Let $Q$ be a quasivariety. An overflow rule, is a quasi-identity of the form

$$\alpha_1 = \beta_1 \& \ldots \& \alpha_n = \beta_n \implies x = y$$

where $x$ and $y$ are variables not occurring in any $\alpha_i, \beta_i$. 
Overflow completeness

In 2005, Wroński considered the following weakening of structural completeness.

**Definition (Wroński)**

Let $Q$ be a quasivariety. An overflow rule, is a quasi-identity of the form

$$\alpha_1 = \beta_1 \& \ldots \& \alpha_n = \beta_n \implies x = y$$

where $x$ and $y$ are variables not occurring in any $\alpha_i, \beta_i$.

**Definition (Wroński)**

A quasivariety $Q$ is overflow complete if every admissible overflow rule of $Q$ is derivable in $Q$. 
Overflow completeness = positive existential completeness

Overflow completeness characterises precisely the “positive existential completeness” property isolated by Bergmann.
Overflow completeness = positive existential completeness

Overflow completeness characterises precisely the “positive existential completeness” property isolated by Bergmann.

**Theorem (Wroński)**

Let $Q$ be a quasivariety. The following are equivalent:

- $Q$ is overflow complete.
- $Th(Q^-) \cap \{ \varphi, \neg \varphi \} \neq \emptyset$, for any positive existential sentence $\varphi$ in the language of $Q$. 
Overflow completeness $= \text{positive existential completeness}$

Overflow completeness characterises precisely the “positive existential completeness” property isolated by Bergmann.

**Theorem (Wroński)**

Let $Q$ be a quasivariety. The following are equivalent:

- $Q$ is overflow complete.
- $\text{Th}(Q^-) \cap \{\varphi, \neg \varphi\} \neq \emptyset$, for any positive existential sentence $\varphi$ in the language of $Q$.

But overflow completeness is often trivial, as noticed by James Raftery during my presentation. Raftery’s observation amounts to:
Overflow completeness = positive existential completeness

Overflow completeness characterises precisely the “positive existential completeness” property isolated by Bergmann.

**Theorem (Wroński)**

Let $Q$ be a quasivariety. The following are equivalent:

- $Q$ is overflow complete.
- $Th(Q^-) \cap \{\varphi, \neg \varphi\} \neq \emptyset$, for any positive existential sentence $\varphi$ in the language of $Q$.

But overflow completeness is often trivial, as noticed by James Raftery during my presentation. Raftery’s observation amounts to:

**Lemma (Raftery)**

*If $Q$ has a definable constant $c$ such that $\{c\}$ is a subalgebra of every $A \in Q$, then $Q$ is (vacuously) overflow complete.*
Overflow completeness in intuitionistic world

Let $\mathbb{H}$ be the variety of Heyting algebras, with basic operations \{\lor, \land, \rightarrow, \neg\}, and \(\tau\) be a set of term operations of Heyting algebras. For any \(K \subseteq \mathbb{H}\), let \(K_\tau\) be the variety of type \(\tau\) generated by \(K\).
Overflow completeness in intuitionistic world

Let $\mathbb{H}$ be the variety of Heyting algebras, with basic operations \{\lor, \land, \rightarrow, \neg\}, and $\tau$ be a set of term operations of Heyting algebras. For any $K \subseteq \mathbb{H}$, let $K_{\tau}$ be the variety of type $\tau$ generated by $K$.

**Theorem (Wroński)**

*If $\top$ and $\bot$ are both $\tau$-definable and distinct in all non-trivial algebras of $K_{\tau}$, then $K_{\tau}$ is overflow complete. The distinctness assumption is necessary.*
Overflow completeness in intuitionistic world

Let $\mathbb{H}$ be the variety of Heyting algebras, with basic operations \{\lor, \land, \to, \neg\}, and $\tau$ be a set of term operations of Heyting algebras. For any $K \subseteq \mathbb{H}$, let $K_\tau$ be the variety of type $\tau$ generated by $K$.

**Theorem (Wroński)**

If $\top$ and $\bot$ are both $\tau$-definable and distinct in all non-trivial algebras of $K_\tau$, then $K_\tau$ is overflow complete. The distinctness assumption is necessary.

The varieties $\mathbb{H}\{\to\}$, $\mathbb{H}\{\to, \land\}$, $\mathbb{H}\{\to, \land, \neg\}$ are structurally complete, and so are all varieties of linear Heyting algebras. On the other hand, adding $\lor$ to any of the above makes it structurally incomplete. Also $\mathbb{H}\{\to, \neg\}$ is not structurally complete. But, they are all overflow complete.
Some remarks and a question

Some remarks and a question


- The splitting term technique was invented for BCI, to answer a question of Humberstone and Meyer. See T.K. “Self-implications in BCI”, NDJFL 49, no. 3 (2008), 295–305.
Some remarks and a question

- The splitting term technique was invented for BCI, to answer a question of Humberstone and Meyer. See T.K. “Self-implications in BCI”, NDJFL 49, no. 3 (2008), 295–305.
- Recently, Humberstone and T.K. used it to show BCI-admissibility of a certain Abelian rule.
Some remarks and a question


- The splitting term technique was invented for BCI, to answer a question of Humberstone and Meyer. See T.K. “Self-implications in BCI”, NDJFL 49, no. 3 (2008), 295–305.

- Recently, Humberstone and T.K. used it to show BCI-admissibility of a certain Abelian rule.

- Wronski’s proof makes essential use of Glivenko theorem, and the fact that Glivenko embedding takes intuitionism to classical logic.
Some remarks and a question

- The splitting term technique was invented for BCI, to answer a question of Humberstone and Meyer. See T.K. “Self-implications in BCI”, NDJFL 49, no. 3 (2008), 295–305.
- Recently, Humberstone and T.K. used it to show BCI-admissibility of a certain Abelian rule.
- Wroński’s proof makes essential use of Glivenko theorem, and the fact that Glivenko embedding takes intuitionism to classical logic.

Question

Is bounded BCK overflow complete?