Representation theorem for interlaced q-bilattices

Yu.M.Movsisyan, D.S.Davidova

Department of mathematics and mechanics
Yerevan State University, Yerevan, Armenia
E-mail: yurimovsisyan@yahoo.com

Annotation. It is proved in this work that every interlaced q-bilattice is isomorphic to the superproduct of two q-lattices.

Introduction and preliminaries. Bilattices are the algebraic structures that were introduced by Ginsberg [1, 2] as a general and uniform framework for a diversity of applications in artificial intelligence. In a series of papers it was shown that these structures can serve as a foundation for many areas, such as logic programming [3-5], computational linguistics, distributive knowledge processing and reasoning with imprecise information. Bilattices are useful in the context of fuzzy logics as well.

A bilattice is the algebra, \((L; \wedge, \vee, *, \triangle)\), with four binary operations such that the following two reducts, \(L_1 = (L; \wedge, \vee)\) and \(L_2 = (L; *, \triangle)\), are lattices.

A bilattice is called interlaced if all the basic bilattice operations are order preserving with respect to the both corresponding orders.

In papers [1, 3, 5-9] bounded distributive or bounded interlaced bilattices are characterized (note that distributive lattices with third additional operation are studied in [10]-[13]). In [14], interlaced bilattices without bounds are characterized (see also [15]).

Definition 1. The algebra, \((L; \wedge)\), is called a q-semilattice, if it satisfies the following identities:
1. \(a \land b = b \land a\) (commutativity);
2. \(a \land (b \land c) = (a \land b) \land c\) (associativity);
3. \(a \land (b \land c) = a \land b\) (weak idempotency).

Definition 2. The algebra, \((L; \land, \lor)\), is called a q-lattice (see [16]), if the reducts, \((L; \wedge)\) and \((L; \lor)\), are q-semilattices and the following identities, \(a \land (b \lor a) = a \land a\), \(a \lor (b \land a) = a \lor a\) (weak absorption), \(a \land a = a \lor a\) (equality) are valid.

For each q-semilattice, \((L; \land)\), there exists a quasiorder, \(Q\) (i.e. a reflexive and transitive relation), which is defined in the following manner: \(aQb \iff a \land b = a \land a \iff a \lor b = b \lor b\).

For example, \((Z \setminus \{0\}; \land, \lor)\), where \(x \land y = \gcd(x, y)\) and \(x \lor y = \lcm(x, y)\) (here \((x, y)\) and \([x, y]\) are the greatest common division (gcd) and the least common multiple (lcm) of \(x\) and \(y\), respectively), is a q-lattice, since \(x \land x \neq x\) and \(x \lor x \neq x\).

Definition 3. A q-bilattice is an algebraic structure, \((L; \land, \lor, *, \triangle)\), with the two q-lattice reducts, \(L_1 = (L; \land, \lor)\) and \(L_2 = (L; *, \triangle)\), which also satisfies the following identity: \(a \ast a = a \land a\) (the quasiorder of the first reduct, \((L; \land, \lor)\), is denoted by \(\leq_L\), and that of the second reduct - by \(\leq_*\)).

Definition 4. The operation, \(*\), of the q-semilattice, \((L; \ast)\), is called interlaced with the operations, \(\land\) and \(\lor\), of the q-lattice, \((L; \land, \lor)\), if the q-semilattice operation, \(*\), preserves the q-lattice quasiorder \(\leq_L\), and q-lattice operations, \(\land\) and \(\lor\), preserve the q-semilattice quasiorder \(\leq_*\).
**Definition 5.** The q-bilattice, \((L; \wedge, \lor, *, \triangle, \perp)\), is called interlaced if all the basic q-bilattice operations are quasiorder preserving with respect to the both corresponding quasiorders.

Let us recall that a hyperidentity is a second order formula of the following type:

\[
\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2),
\]

where \(X_1, \ldots, X_m\) are functional variables, and \(x_1, \ldots, x_n\) are objective variables in the words (terms) of \(w_1, w_2\). Hyperidentities are usually written without the quantifiers, that is to say, \(w_1 = w_2\). We say that in the algebra, \((Q; F)\), the hyperidentity, \(w_1 = w_2\), is satisfied if this equality is valid, when every objective variable and every functional variable in it is replaced by any element of \(Q\) and by any operation of the corresponding arity from \(F\) respectively (supposing the possibility of such replacement) [17-18].

On characterization of hyperidentities of varieties of lattices, modular lattices, distributive lattices, Boolean and De Morgan algebras see [19-22]. About hyperidentities in term (polynomial) algebras of lattices see [23].

For example, the q-bilattice, \(L = (L; \wedge, \lor, *, \triangle)\), is interlaced iff it satisfies the following hyperidentity:

\[
X(Y(X(x, y), z), Y(y, z)) = Y(X(x, y), z).
\]

For a categorical definition of hyperidentities, in [17] the (bi)homomorphisms between the two algebras, \((Q; F)\) and \((Q'; F')\), are defined as the pair, \((\varphi, \tilde{\psi})\), of the maps:

\[
\varphi : Q \rightarrow Q', \quad \tilde{\psi} : F \rightarrow F', |A| = |\tilde{\psi}A|,
\]

with the following condition:

\[
\varphi A(a_1, \ldots, a_n) = (\tilde{\psi}A)(\varphi a_1, \ldots, \varphi a_n),
\]

for any \(A \in F, |A| = n, a_1, \ldots, a_n \in Q\).

Algebras with their (bi)homomorphisms, \((\varphi, \tilde{\psi})\), (as morphisms) form a category with products. The product in this category is called superproduct of algebras and is denoted by \(Q \bowtie Q'\) for the two algebras, \(Q\) and \(Q'\). For example, a superproduct of the two q-lattices, \(Q(+, -)\) and \(Q'(+, -)\), is the binary algebra, \(Q \times Q' ((+, +), (\cdot, \cdot), (\cdot, +), (\cdot, +))\), with four binary operations, where the pairs of the operations operate componentwise, i.e.

\[
(A, B)((x, y), (u, v)) = (A(x, u), B(y, v)),
\]

and \(Q \bowtie Q'\) is an interlaced q-bilattice.

**Definition 6.** The subset, \(F \subseteq L\), is called a filter of the q-bilattice, \((L; \wedge, \lor, *, \triangle, \perp)\), if \(F\) satisfies the following conditions:

\(f f 1\) if \(x, y \in F\), then \(x \land y \in F\);
\(f f 2\) if \(x \in F\), \(y \in L\) and \(x \leq \land y\), then \(y \lor y \in F\);
\(f f 3\) if \(x, y \in F\), then \(x \lor y \in F\);
\(f f 4\) if \(x \in F\), \(y \in L\) and \(x \leq \land y\), then \(y \lor y \in F\).

Denote the set of all filters of the q-bilattice, \(L\), by \(FF(L)\).

**Definition 7.** The subset, \(I \subseteq L\), is called an ideal of the q-bilattice, \((L; \wedge, \lor, *, \triangle, \perp)\), if \(I\) satisfies the following conditions:

\(i i 1\) if \(x, y \in I\), then \(x \land y \in I\);
\(i i 2\) if \(x \in I\), \(y \in L\) and \(x \leq \land y\), then \(y \lor y \in I\);
\(i i 3\) if \(x, y \in I\), then \(x \lor y \in I\);
\(i i 4\) if \(y \in I\), \(x \in L\) and \(x \leq \land y\), then \(x \not \in \frac{1}{2} \in I\).
Denote the set of all ideals of the q-bilattice, $L$, by $FI(L)$.

Let for each $a \in L$, $B_f(a) = \{ X \in FF(L) | a \in X \}$, $B_i(a) = \{ X \in FI(L) | a \in X \}$. Let us define on the sets, $B_f(L) = \{ B_f(a) | a \in L \}$ and $B_i(L) = \{ B_i(a) | a \in L \}$, the binary operations, $\cap^*$ and $\cup^*$, in the following manner:

\[
\begin{align*}
B_f(a) \cap^* B_f(b) &= \{ F \in FF(L) | (\exists X \in B_f(a \land a)) (\exists Y \in B_f(b \land b)) X \cup Y \subseteq F \}; \\
B_f(a) \cup^* B_f(b) &= \{ F \in FF(L) | (\exists X \in B_f(a \land a)) (\exists Y \in B_f(b \land b)) X \cap Y \subseteq F \}; \\
B_i(a) \cap^* B_i(b) &= \{ F \in FI(L) | (\exists X \in B_i(a \land a)) (\exists Y \in B_i(b \land b)) X \cup Y \subseteq F \}; \\
B_i(a) \cup^* B_i(b) &= \{ F \in FI(L) | (\exists X \in B_i(a \land a)) (\exists Y \in B_i(b \land b)) X \cap Y \subseteq F \}.
\end{align*}
\]

It is easy to show that $(B_f(L); \cap^*, \cup^*)$ and $(B_i(L); \cap^*, \cup^*)$ are q-lattices.

Main result

Theorem. Every interlaced q-bilattice $L = (L; \land, \lor, *, \triangle)$ is isomorphic to the super-product of the following two q-lattices $B_f(L) = (B_f(L); \cap^*, \cup^*)$ and $B_i(L) = (B_i(L); \cap^*, \cup^*)$, i.e.

$L \cong B_f(L) \Join B_i(L)$.

References


