When every principal congruence is an intersection of maximal congruences

Daniele Mundici
Department of Mathematics “Ulisse Dini”
University of Florence, Florence, Italy
mundici@math.unifi.it
• maximal congruence of $A$ is one which is maximal among those $\neq A^2$
• the principal congruence generated by two elements $a$ and $b$ of $A$ is the smallest congruence $\approx$ of $A$ such that $a \approx b$
• $A$ is strongly semisimple if every principal congruence of $A$ is an intersection of maximal congruences of $A$
• Dubuc and Poveda introduce this notion in 2010 (Ann. Pure. Appl. Logic, vol. 161): an MV-algebra $A$ is strongly semisimple if every principal ideal of $A$ is an intersection of maximal ideals of $A$
the case of boolean algebras

• in every boolean algebra $A$, every prime ideal is maximal
• every ideal of $A$ is an intersection of prime ideals
• so in particular every principal ideal is an intersection of maximals
• and $A$ is strongly semisimple
• a very general problem: which groups, lattices, Heyting algebras, semigroups, lattice-ordered groups, rings, vector lattices, Banach algebras, etc., are strongly semisimple?
Severi-Bouligand tangents


H. Bouligand, Ann. Soc. Polonaïse Math. 9 (1930) 32-41
u is a tangent unit vector of $X$ in $\mathbb{R}^2$ at $x$

Any triangle with vertex $x$ containing $[x, x+u]$ in its interior, contains $\infty$ many points of $X$
u is a tangent unit vector of $X$ in $\mathbb{R}^2$ at $x$

Any triangle with vertex $x$ containing $[x, x+u]$ in its interior, contains $\infty$ many points of $X$
u is a tangent unit vector of $X$ in $\mathbb{R}^2$ at $x$

Any triangle with vertex $x$ containing $[x,x+u]$ in its interior, contains $\infty$ many points of $X$
u is a tangent unit vector of $X$ in $\mathbb{R}^2$ at $x$

Any triangle with vertex $x$ containing $[x, x+u]$ in its interior, contains $\infty$ many points of $X$
in all small cones, \([x,x+u]\) intersects \(X\) only at \(x\)
definition of tangent vector $u$ of a closed set $X$ in euclidean space $\mathbb{R}^n$ at a point $x$

For all small $\partial, h > 0$, the cone $C$ with vertex $x$, axis parallel to $u$, angle $\partial$, and height $h$, contains infinitely many points of $X$, but none of them except $x$ lies in the segment $[x, x+u]$

NOTE: $X$ is an *arbitrary* closed set in euclidean space
classical consequence (B. Bolzano) deals with models=valuations=interpretations

we write \( f(m) = 1 \) instead of “\( m \) is a model of \( f \)”

A formula \( f \) is a consequence (in the sense of Bolzano) of a set \( P \) of formulas if every model \( m \) of every \( p \) in \( P \) is also a model of \( f \):

If \( p(m) = 1 \) for all \( p \) in \( P \) then \( f(m) = 1 \)

A completeness theorem often gratifies this definition by showing that all the consequences of \( P \) can be computed by a logical calculus of tautologies and Modus Ponens
For instance, let $P$ be a set of boolean formulas in the variables $X_1, \ldots, X_n$. Every valuation $m$ is uniquely determined by the tuple $X_1(m), \ldots, X_n(m)$.

Thus a valuation is a point in the space $\{0, 1\}^{\{X_1, \ldots, X_n\}} = \{0, 1\}^n$.

This space of models has the most rudimentary structure: it is the finite discrete topological space with $2^n$ elements. All finite-valued logics have this zerodimensional structure.
particular case: boolean consequence

given a set $S = \{X_1, \ldots, X_n\}$ of propositional variables,
the set of all possible interpretations of these symbols
= the set of boolean functions on the vertex of the $n$-cube
= the set of **models** of formulas in the variables of $S$
= the set of valuations of these formulas
Valuations are functions from $\{X_1,...X_n\}$ into $[0,1]$

Writing $[0,1]^{\{X_1,...X_n\}} = [0,1]^n$ the space of models inherits the rich topological, algebraic, linear, differential structure of the $n$-cube $[0,1]^n$

Any $[0,1]$-valued logic enjoys this structure
given a set \( S = \{X_1, \ldots, X_n\} \) of propositional variables, the set of all possible interpretations of these symbols
= [0,1]-valued functions on the vertex of the n-cube
= [0,1]-models of formulas in the variables of \( S \)
= [0,1]-valuations of these formulas
an example

the difference between
\{0,1\}-valuations
and \[0,1\]-valuations
Call for projects date:  February 15, 2012, hrs 15.18

Application submission deadline:  March 9, 2012, 17.00 hrs

Co-fund rule:  Applicants must provide 30% of the required funding from sources ≠ the Ministry of Scientific Research.

Co-funding deadline:  BEFORE  March 9, 2012, 17.00 hrs
Applicant A’s co-funding record

% = certified cofunding percentage

t = normalized time

0 = February 15, 15.18
1 = March 9, 17.00
Applicant B’s cofunding record

% = certified cofunding percentage

t = normalized time

0 = February 15, 15.18

1 = March 9, 17.00
Applicant C’s cofunding record

Too late!
The system didn’t accept this application.

% = certified cofunding percentage

0 = February 15, 15.18
1 = March 9, 17.00

t = normalized time
A formula $f$ is a consequence of a set $P$ of premises if every model $m$ of every $p$ in $P$ is also a model of $P$.
A formula $f$ is a consequence of a set $P$ of premises if every model $m$ of every $p$ in $P$ is also a model of $P$

but now any “model” looks around all directions $d$

$p(m) = 1$ for all $p$ in $P$  \textbf{and}  $\partial p(m)/\partial d = 0$ for all $p$ in $P$

$f(m) = 1$  \textbf{and}  $\partial f(m)/\partial d = 0$
telling the difference between $<$ and $\leq$

• Applicant C got the full percentage of cofunding at the very instant when the deadline expired.

• in Lukasiewicz logic $\mathcal{L}_\infty$ we can write a set $P$ of formulas such that a formula $f$ is a consequence of $P$ iff $f$ describes a cofunding record where the required cofunding is obtained before ($<$) the deadline.

• $\mathcal{L}_\infty$ distinguishes between $<$ and $\leq$
Lukasiewicz-Chang axioms (logic-algebra)

\[ A \rightarrow (B \rightarrow A) \]
\[ (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)) \]
\[ (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A) \]
\[ ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A) \]

**MV-algebras** are involutive abelian monoids with 1, satisfying

\[ x + 1 = 1 \] and \[ \neg(\neg x + y) + y = \neg(\neg y + x) + x \]

where \( a + b \) stands for \( \neg a \rightarrow b \)
the prototypical MV-algebra

the unit real interval \([0,1]\)
equipped with the distinguished constant 0
with the unary operation \(-x = 1-x\)
with the binary operation \(x \oplus y = \min(1, x+y)\)

THEOREM (Chang) \(\text{MV} = \text{HSP}[0,1]\)
the free MV-algebra on 1 generator $\text{FREEMV}_1$ is the set of functions $f: [0,1] \rightarrow [0,1]$ obtained from the identity function $x$ by pointwise application of the operations of the prototypical MV-algebra.

A typical element of $\text{FREEMV}_1$
the free MV-algebra over one generator has enough expressive power to describe the Italian co-fund system

let $P$ be a set of all functions in $\text{FREEMV}_1$ which have value 1 arbitrarily close to the deadline

then $f$ is accepted by the Italian co-fund system iff it complies with the conditions in $P$. These conditions ask that $f$ must get value 1 at the deadline 1, and must also keep value 1 over some left neighbourhood of 1.

$$f(1) = 1 \quad \text{and} \quad \frac{\partial f(1)}{\partial (x^-)} = 0$$
The expressive power of Lukasiewicz logic goes beyond the Bolzano paradigm.

\[ \frac{\partial f(1)}{\partial x} = 0 \]

\( f \) is accepted by the Italian co-fund system iff it complies with the conditions in \( P \).

These conditions ask that \( f \) must get value 1 at the deadline 1, and must also keep value 1 over some left neighbourhood of 1.

**Yes**

\[ f \]

**No**

\[ g \]
For a formula \( f \) to be a **stable consequence** of a set \( P \) of premises the following conditions are necessary:

**(Bolzano condition for \( \{0,1\}\)-logics)** every model \( m \) of every \( p \) in \( P \) is also a model of \( f \),

**(Stability condition for \([0,1]\)-logics)** if every \( p \) in \( P \) is stably true along some direction \( d \), then so must be \( f \)

\[
p(m) = 1 \quad \text{and} \quad \frac{\partial p(m)}{\partial d} = 0 \quad \text{for all } p \text{ in } P
\]

\[
f(m) = 1 \quad \text{and} \quad \frac{\partial f(m)}{\partial d} = 0
\]
stable consequence is gratified by a **strong completeness theorem**

which is not the case of so called “semantic” consequence

THEOREM. A formula $F$ is a stable consequence of $P$ iff Modus Ponens derives $F$ from $P$ and the tautologies iff $F$ is a syntactic consequence of $P$.

stable consequence = syntactic consequence = consequence

THEOREM. The consequence relation $G \vdash F$ is coNP-complete.
upon defining two formulas $F$ and $G$ to be $T$-equivalent iff $T$ proves both $F \Rightarrow G$ and $G \Rightarrow F$, we get the **Lindenbaum algebra** $\text{Lind}(T)$ of $T$. We then have a correspondence between deductively closed sets $T$ of sentences and ideals $I_T$ in the free MV-algebra $F$ of all formulas.
a set $T$ of formulas in $L_\infty$ contains a wealth of information not only on the set $\text{Mod}(T)$ of models of $T$

but also on the **tangent space** of $\text{Mod}(T)$ as a subset of the set $[0,1]^n$ of all possible models

when $T$ is finitely axiomatizable, $\text{Mod}(T)$ is a rational polyhedron

**unification and admissibility** involve a lot of geometry

(Cabrer, Ciabattoni, Jerabek, Marra, Metcalfe, Spada,...)
**Problem 1**

**Notation**

\[ T'^{-} = \text{the set of consequences of } T \]
\[ T'^{=} = \text{the set of Bolzano consequences} \]

**Problem 1:**

*When does \( T'^{-} \) coincide with \( T'^{=} \)?*
**Definition**
Given a set $T$ of formulas we say that $T \models$ *strongly* coincides with $T$ if for any formula $b$ we have $(T + b) \models = (T + b)\vdash$.

**Problem 2:**
When does $T \vdash$ *strongly* coincide with $T \models$?
Let $T$ be a set of formulas, with its set $T^|=\mid$ of Bolzano consequences, and its set $T^|\leftarrow$ of consequences. The following conditions are equivalent:

$T^|=\mid$ coincides with $T^|\leftarrow$.

The Lindenbaum algebra of $T$ is semisimple

(i.e., the 0 ideal is intersection of maximal ideals)

particular case: when $T$ is finitely axiomatizable
(this is the Hay-Wójcicki theorem)
Theorem

For any set \( T \) of formulas the following conditions are equivalent:

\[ T \models \text{strongly coincides with } T^\perp, \text{ i.e., } (T + b) \models = (T + b)^\perp \]

whenever a new axiom \( b \) is added to \( T \)

The Lindenbaum algebra of \( T \) is strongly semisimple

(not only \( 0 \), but any principal ideal is an intersection of maximals)
**Strongly semisimple → semisimple**

- **A is semisimple:**
  - The zero ideal is intersection of maximal ideals:
    - A is semisimple
    - A is archimedean
    - A is algebra of real-valued functions
    - A does not have infinitesimals

- **A is strong semisimple:**
  - Every principal ideal is an intersection of maximals
semisimple → strongly semisimple

**semisimple:**
the zero ideal is an intersection of maximal ideals

**strongly semisimple:**
every principal ideal is an intersection of maximals
semisimple → ? strongly semisimple

**semisimple:**
the zero ideal is an intersection of maximal ideals

**example**
the subalgebra $A$ of $C([0,1])$ generated by $x$ and $x^2$ is semisimple; the ideal $P$ generated by $x^2$ is principal, but differs from the only maximal ideal $M$ above $P$: $x$ belongs to $M$ but not to $P$; no multiple of $x^2$ dominates $x$ near 0

**strongly semisimple:**
every principal ideal is an intersection of maximals
the subalgebra $A$ of $C([0,1])$ generated by $x$ and $x^2$ is semisimple; the ideal $P$ generated by $x^2$ is principal, but differs from the only maximal ideal $M$ above $P$: $x$ belongs to $M$ but not to $P$; no multiple of $x^2$ dominates $x$ near 0

example

no multiple of $x^2$ dominates $x$
no multiple of $x^2$ dominates $x$

**example**

the subalgebra $A$ of $C([0,1])$ generated by $x$ and $x^2$ is semisimple; the ideal $P$ generated by $x^2$ is principal, but differs from the only maximal ideal $M$ above $P$: $x$ belongs to $M$ but not to $P$; no multiple of $x^2$ dominates $x$ near 0
the subalgebra $A$ of $C([0,1])$ generated by $x$ and $x^2$ is semisimple; the ideal $P$ generated by $x^2$ is principal, but differs from the only maximal ideal $M$ above $P$: $x$ belongs to $M$ but not to $P$; no multiple of $x^2$ dominates $x$ near $0$.
understanding failure of strong semisimplicity in semisimple algebras

joint work with Manuela Busaniche
we need at least two dimensions, because

for one-generator MV-algebras, semisimplicity coincides with strong semisimplicity
the free MV-algebra on 2 generators is the set of functions \( f:[0,1]^2 \rightarrow [0,1] \) obtained from the identity functions \( x \) and \( y \) by pointwise application of the operations of the prototypical MV-algebra.

A typical element of \( \text{FREEMV}_2 \) and its density plot.
the semisimple quotient operation on $\text{FREEMV}_2$

let $\mathbf{X}$ be a closed set in $[0,1]^2$. Restrict every function $f$ of $\text{FREEMV}_2$ to $\mathbf{X}$. Then the MV-algebra of restrictions to $\mathbf{X}$ is the most general semisimple two-generator MV-algebra
By mapping $X \rightarrow x$, $Y \rightarrow x^2$ we get a parabola $P$. We let $t$ be the tangent of $P$ at the point $(1/2, 1/4)$.

Further, we let $M|_P$ denote the algebra of all restrictions to $P$ of the functions of the free algebra $\text{FREEMV}_2$. The MV-algebra of $x$ and $x^2$.
FREEMV$_2$ contains a function $g : [0,1]^2 \rightarrow [0,1]$ only vanishing along $t$.
we now let $<g>$ be the ideal of $\text{FREEMV}_2$ given by all functions $f$ which are dominated on $P$ by a multiple of $g$. 
any function $f$ in $\langle g \rangle$ will satisfy $\frac{\partial f(r)}{\partial u} \to 0$ as $u \to t$, because so does $g$, and directional derivatives of McNaughton functions are continuous.
but the restriction to $P$ of the function $j$ does not satisfy this condition: as we see, $\frac{\partial j(r)}{\partial u} > 0$
Thus \(<g>\) is a principal ideal of \(M|P\) different from the only maximal ideal \(<j>\) above \(<g>\). For, \(j\) belongs to \(<j>\) and does not belong to \(<g>\).
M|P is not strongly semisimple: P has a rational B-S tangent t

we MUST use B-S tangents, for P can be a very general closed set
Theorem (M. Busaniche, D.M. 2012)

If an MV-algebra is semisimple but not strongly semisimple then its maximal spectral space has a Bouligand-Severi (B-S) tangent
Theorem

For any closed nonempty subset $X$ of the unit square, the MV-algebra $M/X$ has the Dubuc-Poveda property if and only if $X$ has no rational tangents.
what happens when $T^\vdash \neq T^\models$ ?

there is a formula $g$ such that the set of models of $T+g$ has a B-S tangent $t$ at some model $r$

every model $v$ of $T+g$ satisfies each formula $f$ of $T+g$, and $v$ satisfies $f$ along direction $t$, $\partial f(v)/\partial t = 0$

but some formula $j$ satisfied by all models of $T+g$, has $\partial j(v)/\partial t \neq 0$
byproduct: a concrete representation of \textit{infinitesimals} as directional derivatives

Let $A$ be the quotient of $M|P$ by the ideal $<g>$ generated by $g|P$

$j|P / <g>$ is \textit{infinitesimal} in $A$

$j|P / <g>$ has value 0, but is not the zero element of $A$
let $A$ be the quotient of $M|P$ by the ideal
$\langle g \rangle$ generated by $g|P$

$A$ is the MV-algebra of all possible
behaviours (=germs) of McNaughton
functions $f$ at point $r$ along direction $t$

these germs are determined by the
value $f(r)$ and its derivative $\frac{\partial f(r)}{\partial t}$
elements of $A$ are GERMS, i.e., values of functions $f(r)$ together with their directional derivatives $\partial f(r)/\partial t$
j is an infinitesimal in A: j has value zero but nonzero derivative.
j is an MV-algebraic infinitesimal

the consequences of S form a smaller set than the set of consequences in the sense of Bolzano

1-j represents a formula with \( j(r)=1 \) but \( \frac{\partial j(r)}{\partial x} < 0 \),

while for every formula \( f \) of S, \( f(r)=1 \) and \( \frac{\partial f(r)}{\partial x} = 0 \)

The deduction rules are so that every consequence of S must have the same stability properties which are common to all formulas of S.

1-j is not a consequence of S because it is unstable
the Dubuc-Poveda property in MV-algebras is decisive in understanding the semantics of $\infty$-valued Lukasiewicz logic

the classical notion of “semantic (Bolzano-Tarski) consequence” does not coincide with provability, because it is insensitive to small perturbations of the models of a theory

a strongly complete semantics can be obtained by using valuations that take into account the differential behaviour of formulas, and the tangent spaces of their model sets
Contemporary Lukasiewicz logic is a key tool to handle non-boolean information, e.g., in error-correcting codes with feedback (Rényi-Ulam game). Being precisely the zero sets of McNaughton functions, rational polyhedra are the key tool for the analysis of Lukasiewicz consequence relation, projective MV-algebras, and unification.

In 1958 Chang introduced MV-algebras as equivalence classes of Lukasiewicz formulas to prove the completeness of the Lukasiewicz axioms. Today MV-algebras deeply interact with many mathematical areas, e.g., ordered groups, toric varieties, algebraic topology, and approximately finite-dimensional C*-algebras.
thank you