On the Decidability of Modal Lambek Calculus

Dongning Liu\textsuperscript{1}  Zhe Lin\textsuperscript{2}

Faculty of Computer Science GuangDong University Of Technology, Guangzhou China\textsuperscript{1}
Faculty of Mathematics and Computer Science Adam Mickiewicz University, Poznań, Poland\textsuperscript{2}
Institute of Logic and Cognition, Sun Yat-sen University, Guangzhou, China\textsuperscript{2}

12-09-2012, LATD2012
Outline

- Background and Introduction
- Sequent systems, Cut-elimination, Decidability
- Interpolation
- Decidability of Consequence relations
Modal (Nonassociative) Lambek Calculus ($L\diamond$, $NL\diamond$), Multi-modal Lambek Calculus ($NL\diamond_R$): Kripke semantic Sequent systems, Cut-elimination, Decidability.. (Moortgat).

Complexity, Generative Capacity, Proof net, . . . (Moot, Jäger, Carpenter, . . .)

Multimodal Lambek Calculus with unrestricted set of $R$ recognizes recursively enumerable languages: Undecidable (Carpenter)

Multimodal Lambek Calculus with unrestricted set of $R^*$ recognizes tree-adjioning languages (Decidable) (Moot)

Modal extension of FNL, FL?
Good proof theory properties

- Cut-elimination
- Generality
- Dosěn principle
FL\diamond

1) \diamond 0 \Rightarrow 0. 2) if \ A \Rightarrow B \ then \ \diamond A \Rightarrow \diamond B. 3) \diamond \bot \iff \bot

(\diamond L) \quad \frac{\Gamma_1, \langle A \rangle, \Gamma_2 \Rightarrow B}{\Gamma_1, \diamond A, \Gamma_2 \Rightarrow B} \quad (\diamond R) \quad \frac{\Gamma \Rightarrow A}{\langle \Gamma \rangle \Rightarrow \diamond A} \quad (\diamond) \quad \frac{\Gamma \Rightarrow}{\langle \Gamma \rangle \Rightarrow
• FL\(\Diamond\)
  1) \(\diamond 0 \Rightarrow 0\). 2) if \(A \Rightarrow B\) then \(\diamond A \Rightarrow \diamond B\). 3) \(\bot \Leftrightarrow \bot\)

\[
\begin{align*}
(\Diamond L) & \quad \frac{\Gamma_1, \langle A \rangle, \Gamma_2 \Rightarrow B}{\Gamma_1, \diamond A, \Gamma_2 \Rightarrow B} \\
(\Diamond R) & \quad \frac{\Gamma \Rightarrow A}{\langle \Gamma \rangle \Rightarrow \diamond A} \\
(\diamond) & \quad \frac{\Gamma \Rightarrow}{\langle \Gamma \rangle \Rightarrow}
\end{align*}
\]

• FL\(\Box\)
  1) \(1 \Rightarrow \Box 1\). 2) if \(A \Rightarrow B\) then \(\Box A \Rightarrow \Box B\). 3) \(\Box \top \Leftrightarrow \top\).

\[
\begin{align*}
(\Box L) & \quad \frac{\Gamma_1, A, \Gamma_2 \Rightarrow B}{\Gamma_1, \langle \Box A \rangle, \Gamma_2 \Rightarrow B} \\
(\Box R) & \quad \frac{\langle \Gamma \rangle \Rightarrow A}{\Gamma \Rightarrow \Box A} \\
\Box & \quad \frac{1 \Rightarrow}{\Box 1 \Rightarrow} \\
(\diamond) & \quad \frac{\Gamma \Rightarrow}{\langle \Gamma \rangle \Rightarrow}
\end{align*}
\]
\[ \text{FL}_K \diamond (\text{FL}_K \Box) \]

\[ \diamond (A \cdot B) \Rightarrow \diamond A \cdot \diamond B \ (\Box A \cdot \Box B \Rightarrow \Box (A \cdot B)) \]

\[ (rK) \quad \frac{\Gamma_1, \langle \Delta_1 \rangle, \langle \Delta_2 \rangle, \Gamma_2 \Rightarrow A}{\Gamma_1, \langle \Delta_1, \Delta_2 \rangle, \Gamma_2 \Rightarrow A} \]
\( \text{FL}_K \diamond (\text{FL}_K \Box) \)

\[ \diamond (A \cdot B) \Rightarrow \diamond A \cdot \diamond B (\Box A \cdot \Box B \Rightarrow \Box (A \cdot B)) \]

(rK) \[ \frac{\Gamma_1, \langle \Delta_1 \rangle, \langle \Delta_2 \rangle, \Gamma_2 \Rightarrow A}{\Gamma_1, \langle \Delta_1, \Delta_2 \rangle, \Gamma_2 \Rightarrow A} \]

\( \text{FL}_T \diamond (\text{FL}_T \Box) \)

\[ A \Rightarrow \diamond A (\Box A \Rightarrow A) \]

(rT) \[ \frac{\Gamma_1, \langle \Delta \rangle, \Gamma_2 \Rightarrow A}{\Gamma_1, \Delta, \Gamma_2 \Rightarrow A} \]
- $\text{FL}_K \Diamond (\text{FL}_K \Box)$
  - $\Diamond (A \cdot B) \Rightarrow \Diamond A \cdot \Diamond B (\Box A \cdot \Box B \Rightarrow \Box (A \cdot B))$

  \[
  \text{(rK)} \quad \frac{\Gamma_1, \langle \Delta_1 \rangle, \langle \Delta_2 \rangle, \Gamma_2 \Rightarrow A}{\Gamma_1, \langle \Delta_1, \Delta_2 \rangle, \Gamma_2 \Rightarrow A}
  \]

- $\text{FL}_T \Diamond (\text{FL}_T \Box)$
  - $A \Rightarrow \Diamond A (\Box A \Rightarrow A)$

  \[
  \text{(rT)} \quad \frac{\Gamma_1, \langle \Delta \rangle, \Gamma_2 \Rightarrow A}{\Gamma_1, \Delta, \Gamma_2 \Rightarrow A}
  \]

- $\text{FL}_{S_4} \Diamond (\text{FL}_{S_4} \Box)$
  - $A \Rightarrow \Diamond A (\Box A \Rightarrow A)$ and $\Diamond \Diamond A \Rightarrow \Diamond A (\Box A \Rightarrow \Box \Box A)$

  \[
  \text{(r4)} \quad \frac{\Gamma_1, \langle \Delta \rangle, \Gamma_2 \Rightarrow A}{\Gamma_1, \langle \langle \Delta \rangle \rangle, \Gamma_2 \Rightarrow A}
  \]
Theorem

Cut elimination holds for $S$

Theorem

$S$ is decidable
(5) \( \Diamond A \rightarrow \Box \Diamond A \)

\( \Diamond A \Rightarrow B \) iff \( A \Rightarrow \Box B \) (using (4), (T))
(5) ◊A → □◊A

◊A ⇒ B iff A ⇒ □B (using (4), (T))

◊ and □ connection in Classical Modal Logic:

◊ ↔ ¬□¬.
(5) $\Diamond A \to \Box \Diamond A$

$\Diamond A \Rightarrow B$ iff $A \Rightarrow \Box B$ (using (4), (T))

$\Diamond$ and $\Box$ connection in Classical Modal Logic:

$\Diamond \leftrightarrow \neg \Box \neg$

$\Diamond$ and $\Box$ connection in Intuitionistic Modal Logic:

$\Diamond (A \to B) \to (\Box A \to \Diamond B)$ and $(\Diamond A \to \Box B) \to \Box (A \to B)$
(5) \( \Diamond A \rightarrow \Box \Box A \)

\( \Diamond A \Rightarrow B \) iff \( A \Rightarrow \Box B \) (using (4), (T))

\( \Diamond \) and \( \Box \) connection in Classical Modal Logic:

\( \Diamond \leftrightarrow \neg \Box \neg \).

\( \Diamond \) and \( \Box \) connection in Intuitionistic Modal Logic:

\( (A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B) \) and \( (\Diamond A \rightarrow \Box B) \rightarrow \Box (A \rightarrow B) \)

\( \Diamond \) and \( \Box \) connection in Modal extension of FL

\( (A/B) \Rightarrow (\Box A/\Diamond B) \) and \( (\Diamond A/\Box B) \Rightarrow \Box (A/\Box B) \) (Lacking of Exchange rule and Deduction Theorem)

(\( \Diamond \Box \)): \( \Diamond A \cdot \Diamond B \Rightarrow \Diamond (\Diamond A \cdot B) \)
(5) $\Diamond A \rightarrow \Box \Diamond A$

$\Diamond A \Rightarrow B$ iff $A \Rightarrow \Box B$ (using (4), (T))

$\Diamond$ and $\Box$ connection in Classical Modal Logic:

$\Diamond \leftrightarrow \neg \Box \neg$

$\Diamond$ and $\Box$ connection in Intuitionistic Modal Logic:

$\Diamond (A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B)$ and $(\Diamond A \rightarrow \Box B) \rightarrow \Box (A \rightarrow B)$

$\Diamond$ and $\Box$ connection in Modal extension of FL

$\Diamond (A/B) \Rightarrow (\Box A/\Diamond B)$ and $(\Diamond A/\Box B) \Rightarrow \Box (A \setminus B)$ (Lacking of Exchange rule and Deduction Theorem)

$(\Diamond \Box): \Diamond A \cdot \Diamond B \Rightarrow \Diamond (\Diamond A \cdot B)$

$(\Diamond \Box r) \quad \begin{array}{c}
\Gamma_1, \langle \Delta_1 \rangle, \Delta_2 \rangle, \Gamma_2 \Rightarrow A \\
\Gamma_1, \langle \Delta_1 \rangle, \Delta_2 \rangle, \Gamma_2 \Rightarrow A
\end{array}$
Theorem

Cut elimination holds for $\text{FL}_{S5}$

Theorem

$\text{FL}_{S5}$ is decidable
Let $\mathcal{T}(\Gamma \Rightarrow A)$ be a set of all subformulae of formulae appearing in sequent $\Gamma \Rightarrow A$. $\mathcal{T}^*$ denotes the closure of $\mathcal{T}$ under $\cdot$, $\lor$ and $\land$. 

Remarks:
W. Craig proved the interpolation lemma for classical logic. Craig's interpolation lemma for FL and its various extensions are obtained in Ono (1998). Variants for NL, DFNL, DGL, are obtained in Buszkowski (2009) which are used to prove the complexity and the FEP for the corresponding logics.
Let $T(\Gamma \Rightarrow A)$ be a set of all subformulae of formulae appearing in sequent $\Gamma \Rightarrow A$. $T^*$ denotes the closure of $T$ under $\cdot$, $\lor$ and $\land$.

The following interpolation lemma for systems $S$, where $S=FLK\Box$, $FLT\Box$, $FLK4\Box$, $FLS4\Box$, $FLS5$, . . .

**Lemma**

If $\vdash S \Gamma_1, \Delta, \Gamma_2 \Rightarrow A$ then there exists a formula $D \in T^*(\Gamma \Rightarrow A)$ such that $\vdash S \Delta \Rightarrow A$ and $\vdash S \Gamma_1, D, \Gamma_2 \Rightarrow A$.

**Remarks:**

W. Craig proved the interpolation lemma for classical logic. Craig’s interpolation lemma for FL and its various extensions are obtained in Ono (1998). Variants for NL, DFN, DGL, are obtained in Buszkowski(2009) which are used to prove the complexity and the FEP for the corresponding logics.
Consequence relations

Hereafter, we discuss the decidability problem of the consequence relations on modal logics over $FL_{ew}$.

$$
\begin{align*}
(E) & \quad \frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow C}{\Gamma_1, B, A, \Gamma_2 \Rightarrow C} \\
(W) & \quad \frac{\Gamma_1, \Gamma_2 \Rightarrow C}{\Gamma_1, A, \Gamma_2 \Rightarrow C}
\end{align*}
$$

Another characterization of the sequent system

$$
\begin{align*}
(\square) & \quad \frac{\Gamma \Rightarrow A}{\square \Gamma \Rightarrow \square A} \\
(\square \rightarrow) & \quad \frac{\square \Gamma \Rightarrow A}{\square \Gamma \Rightarrow \square A} \\
(\rightarrow \square 1) & \quad \frac{B, \Gamma \Rightarrow A}{\square B, \Gamma \Rightarrow A}
\end{align*}
$$

Remarks:
The cognate sequent (sequent of form $A_{k_1}, \ldots A_{k_n} \rightarrow D$ is essential in the proof.
Subformula property

Let \( \Phi \) be a set of sequents of the form \( A \Rightarrow B \). The sequents from \( \Phi \) will be treated as assumptions. We introduce a restricted cut-rule, \( \Phi \)-restricted cut.

\[
(\Phi - \text{CUT}) \quad \frac{\Gamma_2 \Rightarrow A}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow C} \quad \frac{\Gamma_1, B, \Gamma_3 \Rightarrow C}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow C}
\]

where \( A \Rightarrow B \in \Phi \).
Consequence relations

- Subformula property

Let \( \Phi \) be a set of sequents of the form \( A \Rightarrow B \). The sequents from \( \Phi \) will be treated as assumptions. We introduce a restricted cut-rule, \( \Phi\)-restricted cut.

\[
(\Phi - \text{CUT}) \quad \frac{\Gamma_2 \Rightarrow A \quad \Gamma_1, B, \Gamma_3 \Rightarrow C}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow C}
\]

where \( A \Rightarrow B \in \Phi \).

- Finite derivation tree

The number of cognate sequent (sequent of form \( A_{k_1}, \ldots A_{k_n} \rightarrow D \) is finite. (see Ono 1998)
Consequence relations

- **Subformula property**

Let $\Phi$ be a set of sequents of the form $A \Rightarrow B$. The sequents from $\Phi$ will be treated as assumptions. We introduce a restricted cut-rule, $\Phi$-restricted cut.

\[
(\Phi - \text{CUT}) \quad \frac{\Gamma_2 \Rightarrow A \quad \Gamma_1, B, \Gamma_3 \Rightarrow C}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow C}
\]

where $A \Rightarrow B \in \Phi$.

- **Finite derivation tree**

The number of cognate sequent (sequent of form $A_{k_1}, \ldots A_{k_n} \rightarrow D$ is finite. (see Ono 1998)

**Theorem**

*The consequence relations of, $FL_{ew} \Box - K, FL_{ew} \Box - T, FL_{ew} \Box - S4$ is decidable.*


Thank you for your attention