A parametric representation of linguistic hedges in Zadeh's fuzzy logic

V.N. Huynh [a,b,*], T.B. Ho [b], Y. Nakamori [b]

[a] *Department of Computer Science, University of Quinhon, 170 An Duong Vuong, Quinhon, Vietnam*
[b] *School of Knowledge Science, Japan Advanced Institute of Science and Technology, Tatsunokuchi, Ishikawa 923-1292, Japan*

Abstract

This paper proposes a model for the parametric representation of linguistic hedges in Zadeh's fuzzy logic. In this model each linguistic truth-value, which is generated from a primary term of the linguistic truth variable, is identified by a real number r depending on the primary term. It is shown that the model yields a method of efficiently computing linguistic truth expressions accompanied with a rich algebraic structure of the linguistic truth domain, namely De Morgan algebra. Also, a fuzzy logic based on the parametric representation of linguistic truth-values is introduced.
© 2002 Elsevier Science Inc. All rights reserved.

Keywords: Linguistic hedges; Linguistic variable; Distributive lattice; De Morgan algebra; Fuzzy logic; Approximate reasoning

1. Introduction

In 1970s, Zadeh [19–23] introduced and developed the theory of approximate reasoning based on the notions of linguistic variable and fuzzy logic. Informally, by a linguistic variable we mean a variable whose values are words in a natural or artificial language. For example, *Age* is a linguistic variable

---

* Corresponding author. Tel.: +845-681-8429; fax: +845-684-6089.
E-mail address: huynh@jaist.ac.jp (V.N. Huynh).

whose values are linguistic such as *young*, *old*, *very young*, *very old*, *quite young*, *more or less young*, *not very young and not very old*, etc. As is well-known, the values of a linguistic variable are generated from primary terms (e.g., *young* and *old* in the case of linguistic variable *Age*) by various linguistic hedges (e.g., *very*, *more or less*, etc.) and connectives (e.g., *and*, *or*, *not*).

In Zadeh's view of fuzzy logic, the truth-values are linguistic, e.g., of the form "*true*", "*very true*", "*more or less true*", "*false*", "*possible false*", etc., which are expressible as values of the linguistic variable *Truth*, and the rules of inference are approximate rather than exact. In this sense, approximate reasoning (also called fuzzy reasoning) is, for the most part, qualitative rather than quantitative in nature, and almost all of it falls outside of the domain of applicability of classical logic (see [2,22,23]). The primary aim of the theory of approximate reasoning is to mimic human linguistic reasoning particularly in describing the behaviour of human-centered systems.

Throughout this paper, by a fuzzy logic we mean a fuzzy logic in the sense of Zadeh, that is, its truth-values are linguistic values of the linguistic truth variable, which are represented by fuzzy sets in the interval $[0, 1]$.

According to Zadeh's rule for truth qualification [23], a proposition such as "Lucia is very young" is considered as being *semantically equivalent* with the proposition "Lucia is young is very true". This semantic equivalence relation plays an important role in approximate reasoning. In fuzzy set based approaches to fuzzy reasoning [2,7,22,23], the primary linguistic truth-values such as *true* and *false* are correspondingly assigned to fuzzy sets defined over the interval $[0, 1]$, which are designed to interpret the meaning of these primary terms. The composite linguistic truth-values are then computed by using the following procedure:

- Linguistic hedges,[1] for example *very* and *more or less*, are defined as unary operators on fuzzy sets, for example CON, DIL, respectively;
- The logic connectives such as *and*, *or*, *not* and *if...then* are defined generally as operators such as *t-norm*, *t-conorm*, *negation*, and *implication*, respectively.

As is well known, one of inherent problems in a model of fuzzy reasoning is that of linguistic approximation, i.e., how to name by a linguistic term a resulted fuzzy set of the deduction process. This depends on the shape of the resulted fuzzy set in relation with the primary fuzzy sets and the operators.

Based on two characteristics of linguistic variables introduced by Zadeh (namely, the context-independent meaning of linguistic hedges and connectives, and the universality of linguistic domains), and the meaning of linguistic hedges in natural language, Nguyen and Wechler [15,16] proposed an algebraic approach to the structure of linguistic domains (term-sets) of linguistic vari-

---

[1] Also called linguistic modifiers [6].

ables. It is shown in [12–14] that the obtained structure is rich enough for the investigation of some kinds of fuzzy logic. Furthermore, the approach also provides a possibility for introducing methods of linguistic reasoning that allow us to handle linguistic terms directly, and hence, to avoid the problem of linguistic approximation [10,11].

It is of interest that in [6], Di Lascio et al. have proposed a model for representation of linguistic terms satisfying the hypotheses imposed on linguistic hedges introduced by Nguyen and Wechler [15]. In their model, each linguistic term of a linguistic variable is characterized by three parameters and can be identified by only a positive real number. It is shown that the set of linguistic terms of the linguistic truth variable in Di Lascio's model exhibited interesting semantic properties justified by intuitive meaning of linguistic hedges, which were axiomatically formulated in the terms of hedge algebras [15]. However, going back to the membership function representation, Di Lascio's model does not give a good interpretation at the intuitive level on logical basis behind the shape of membership functions of linguistic truth-values (see Fig. 1).

It is important to note that in the conventional approach to fuzzy reasoning, fuzzy logic, which a method of fuzzy reasoning bases on, can be viewed as a fuzzy extension of a underlying multi-valued logic (i.e., *base logic*), in which the truth-values are fuzzy sets of the truth-value set of the base logic (see, e.g., [2,22,23]). Although membership functions of primary terms such as *true* or *false* are defined subjectively, it will be natural to hope that a fuzzy logic should meet the base logic at the limited cases. For example, for membership function of the **unitary** truth-value *u-true* [23], that is $\mu_{u\text{-}true}(v) = v$ for $v \in [0, 1]$, and the linguistic hedge *very* defined by CON operation, we have $very^n true$ tends to *Absolutely true* as n tends to infinity, where *Absolutely true* is identified with 1 as a nonfuzzy truth-value, see Fig. 2. Unfortunately, this is not the case for Di Lascio's model, again see Fig. 1.
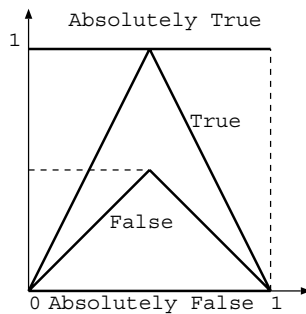

Fig. 1. Membership functions of truth-values by Di Lascio's model.
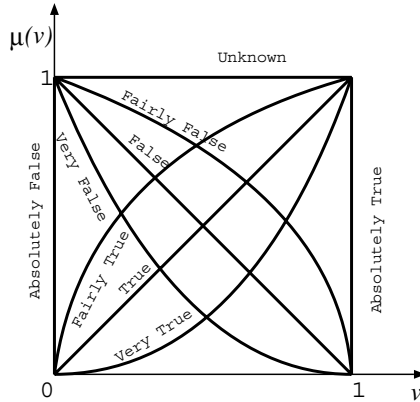
Fig. 2. Membership functions of unitary truth-values [4,23].

In this paper, we introduce a new representation model for linguistic terms of the linguistic truth variable in fuzzy logic. In this model, each linguistic truth-value generated from a primary term of the linguistic truth variable is identified by a real number r depending on the primary term. It will be shown that the model not only satisfies the interesting semantic properties justified by intuitively meaning of linguistic hedges as Di Lascio's model, but also meets in the special cases the well-known models in the literature.

The paper is organized as follows. In Section 2, we briefly present some preliminary notions on linguistic variables, the fuzzy set based interpretation of linguistic hedges, as well as the related work in the literature. A new representation model for linguistic terms of the linguistic truth variable will be introduced in Section 3. The model allows to represent two ordered sets of linguistic terms generated from two primary terms *true* and *false*; each linguistic truth-value is associated with a real number depending on the primary term from which it is generated. Section 4 introduces a fuzzy logic based on this model in comparison with the models already known in literature. Finally, some concluding remarks will be given in Section 5.

2. Preliminaries

2.1. Linguistic variables

In this section, we briefly recall the notion of linguistic variables and the fuzzy set theoretic interpretation of linguistic hedges introduced by Zadeh in 1970s. More details can be referred to [5,19–21,23].

Formally, a *linguistic variable* is characterized by a quintuple $(\mathcal{X}, T(\mathcal{X}), U, R, M)$, where:

$\mathcal{X}$ is the name of the variable such as age variable *Age*, truth variable *Truth*, etc.;

$T(\mathcal{X})$ denotes the **term-set** of $\mathcal{X}$, that is, the set of linguistic values of the linguistic variable $\mathcal{X}$;

U is a universe of discourse of the base variable;

R is a syntactic rule for generating linguistic terms of $T(\mathcal{X})$;

M is a semantic rule assigning to each linguistic term a fuzzy set on U.

As an illustration, we consider an example of a linguistic variable *Age*, i.e., $\mathcal{X} = Age$, taken from [20]. The term-set $T(\mathcal{X})$ is represented as follows:

$$T(Age) = \{young, very\ young, not\ young, very\ very\ young, not\ very\ young,$$
$$\ldots, old, very\ old, not\ old, \ldots, not\ very\ young\ and\ not\ very\ old,$$
$$\ldots, extremely\ young, \ldots, more\ or\ less\ young, \ldots\}.$$

The universe of discourse for *Age* may be taken to be the interval $U = [0, 100]$, with the base variable u ranging over U. Then, a linguistic value of *Age*, for example, *young* is viewed as a name of a fuzzy set of U which is designed to define the meaning of *young*. That is, the meaning of the linguistic value *young* is characterized by its compatibility function $c : U \to [0, 1]$, with $c(u)$ representing the compatibility of a numerical age u with the label *young*. For example, the compatibilities of the numerical ages 20, 25, 30, and 35 with *young* may be 1, 0.9, 0.8, and 0.6, respectively. As such, the meaning of a linguistic value can be regarded as the membership function of a *fuzzy restriction* on the values of the base variable u. Fig. 3 sketches the above mentioned relationships [24,25].

Typically, the values of a linguistic variable such as *Age* are built up of one or more primary terms such as *young* and *old*, with one being an antonym of the other, together with a set of linguistic hedges, such as *very*, *more or less*, *quite*, *extremely*, etc., and connectives which allow a composite linguistic value to be generated from primary terms.

Assume that the meaning of a linguistic value X is defined by the membership function $\mu_X(u)$ of U, then linguistic hedges *very*, *more or less*, *slightly* are constructed by mathematical representations as follows [18].

Concentration: *very* $X = \mathrm{CON}(X)$, where $\mu_{\mathrm{CON}(X)}(u) = (\mu_X(u))^2$;

Dilation: *more or less* $X = \mathrm{DIL}(X)$, where $\mu_{\mathrm{DIL}(X)}(u) = (\mu_X(u))^{0.5}$;

Intensification: denote by $\mathrm{INT}(X)$, and

$$\mu_{\mathrm{INT}(X)}(u) = \begin{cases} 2(\mu_X(u))^2 & \text{if } 0 \leqslant \mu_X(u) \leqslant 0.5, \\ 1 - 2(1 - \mu_X(u))^2 & \text{if } 0.5 \leqslant \mu_X(u) \leqslant 1. \end{cases}$$
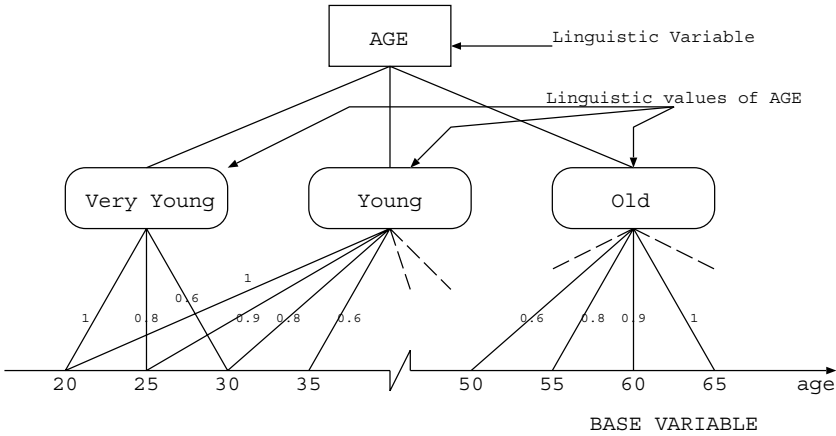
Fig. 3. Hierarchical structure of the linguistic variable *Age*.

And the hedge *slightly* may be defined by one of the following expressions:

slightly X = NORM(*X and not very X*),

slightly X = INT(NORM(*plus X and not very X*)),

slightly X = INT(NORM(*plus X and not plus very X*)),

where NORM is the operation of normalization and *plus* is an artificial hedge defined by

$$\mu_{\text{NORM}(X)}(u) = \left(\sup_U \mu_X(u) \right)^{-1} \mu_X(u), \quad \text{and} \quad \mu_{plus\,X}(u) = (\mu_X(u))^{1.25}.$$

A more detailed discussion of linguistic hedges from a fuzzy set theoretic point of view can be found in [3,5,18].

A linguistic variable is called to be a Boolean linguistic variable provided that its values are Boolean expressions in variables of the form X_p, hX_p, X or hX, where h is a linguistic hedge or a string of linguistic hedges, X_p is a primary term and hX is the name of a fuzzy set resulting from acting with h on X. For example, in the case of the linguistic variable *Age* whose term-set is defined previously, the term *not very young and not very old* is of this form with $h = very$, $X_p = young$ and $X_{p'} = old$. Similarly, it is also the case for the term *very very young*, here $h = very\ very$ and $X_p = young$. It was shown in [20] that we can construct a context-free grammar for generating the term-set of a Boolean linguistic variable.

2.2. Mathematical representation of linguistic hedges in fuzzy logic

In the conventional approach to fuzzy logic, each primary linguistic truth-value such as *true* or *false* is semantically assigned by a fuzzy set in the interval

$[0, 1]$. A well known form of membership function of *true* is defined by $\mu_{true}(u) = u$ for $u \in [0, 1]$, and the membership function of *false* is defined by $\mu_{false}(u) = \mu_{true}(1 - u)$ for $u \in [0, 1]$. Linguistic hedges are then defined as operators on these primary fuzzy sets to form fuzzy sets for composite linguistic truth-values. For example, linguistic hedges such as *very* and *more or less* (or, *fairly* [1]) are mainly modeled as CON and DIL operators, respectively, [1,8,22,25]. However, the definition of a linguistic hedge as order of the power of a truth-value *true* or *false* as in [1,8,22] suffers from an intuitive criterion when applied infinitely to linguistic hedges [6,15]. For example, it is intuitively agreed that *true* is more true than $(very)^n approximately$ *true*, for any natural number n. Then it should be intuitively suitable if $(very)^n approximately$ *true* tends to *true* as n tends to infinity. However, when we interpret *very* as the CON operator, we get both $(very)^n approximately$ *true* and $(very)^n true$ tend to *Abs. true* as n tends to infinity. This causes a discrepancy between the intuitive utilization made in natural language of linguistic truth-values and the mathematical representation obtained using CON and DIL operators.

To cancel the above mentioned discrepancy, Di Lascio et al. have proposed in [6] a model for representation of linguistic hedges, within which each linguistic value of the truth linguistic variable is characterized by three parameters and can be identified by a positive real number n. It was shown that the set of linguistic terms of the linguistic truth variable in this model exhibited interesting semantic properties justified by intuitively meaning of linguistic hedges, which were axiomatically formulated by Nguyen and Wechler in [15] in the terms of hedge algebras.

To represent the meaning of linguistic values of the linguistic truth variable, Di Lascio et al. introduced the following characteristic function, for $n \in R^+$,

$$\mu_n(u) = \begin{cases} \min(1, nu) & \text{for } 0 \leqslant u \leqslant 0.5, \\ \min(1, -n(u - 1)) & \text{for } 0.5 \leqslant u \leqslant 1. \end{cases}$$

Note that they utilized only this function for a generic linguistic term of the linguistic truth variable irrespective of a linguistic value generating from *true* or *false*. This is essentially different from conventional approaches to fuzzy logic in the literature. For $n \to \infty$, and $n = 0$ the model yields the values *Absolutely true* and *Absolutely false*, respectively. Consequently, *Absolutely true* and *Absolutely false* are identified by the following membership functions (see Fig. 1):

$$\mu_{Abs. true}(u) = 1 \quad \text{for any } u \in [0, 1],$$
$$\mu_{Abs. false}(u) = 0 \quad \text{for any } u \in [0, 1],$$

which are designed to interpret as the truth-values *unknown* and *undefined*, respectively, in [20].

It should be emphasized that in this model, it is impossible to define the special value *unknown* (also called *undecided*) which has been considered to be important in fuzzy logic [1,20]. To overcome these drawbacks while still

maintaining interesting semantic properties of linguistic hedges, an alternative model for the representation of linguistic values of the Boolean linguistic truth variable is introduced in the next section.

3. A new model for the representation of the linguistic truth-values

In this section, we first define two families of parametric membership functions of linguistic truth-values generated from two primary terms *true* and *false*, respectively. Then we examine an algebraic structure of the obtained linguistic truth space via the so-called semantically ordering relation. Also, we introduce a concept of the converse of a given linguistic hedge based on the specific relation introduced in [8]. As we will see in Section 4, this concept can be used in defining another kind of negation in a fuzzy logic.

3.1. Parametric membership function of linguistic truth-values

In our model, each linguistic truth-value is represented by a parametric membership function defined in the interval $[0,1]$. This parameter depends on the primary term from which the linguistic truth-value is generated by applying linguistic hedges.

Consider the Boolean linguistic truth variable *Truth* with two primary terms *true* and *false*. Let us denote by σ a linguistic hedge or a string of linguistic hedges. We now define the membership function of a linguistic value $\sigma \, true$ as follows:

$$\mu_{\sigma \, true} : [0,1] \rightarrow [0,1]$$
$$u \mapsto \mu_{\sigma \, true}(u) = \max(0, (1-n)^{-1}(u-n)) \tag{1}$$

for $n \in (-\infty, 1)$. Similarly, we further define the membership function of a linguistic value $\sigma \, false$ by

$$\mu_{\sigma \, false} : [0,1] \rightarrow [0,1]$$
$$u \mapsto \mu_{\sigma \, false}(u) = \max(0, m^{-1}(m-u)) \tag{2}$$

for $m \in (0, \infty)$.

It is of interest that with these definitions, we obtain the membership functions of some special linguistic truth-values as follows (see Fig. 4):

- *true*, with $\mu_{true}(u) = u$, when $n = 0$;
- *Absolutely true*, when $n \rightarrow 1$;
- *false*, with $\mu_{false}(u) = 1 - u$, when $m = 1$;
- *Absolutely false*, when $m \rightarrow 0$;
- *unknown*, when $n \rightarrow -\infty$ and $m \rightarrow \infty$,

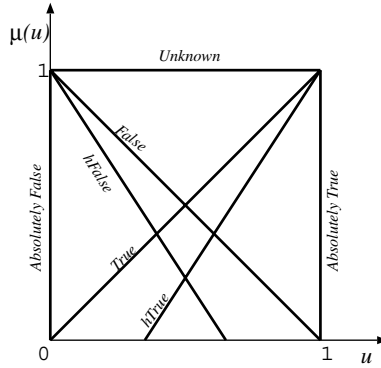which are the same as considered in [1,2,4,8,20].

Fig. 4. A space of parametric membership functions.

Formulation (1) (respectively, (2)) states that an infinite number of hedges can be generated for the linguistic truth-value *true* (respectively, *false*) by a parametric family of membership functions. Let us denote by V the set of all linguistic truth-values generated from (1) and (2) including the limited elements *Absolutely true*, *unknown*, *Absolutely false*.

3.2. An algebraic structure of the linguistic truth space

To analyze the meaning characteristic of the linguistic truth space, we consider the specific relationship between linguistic truth-values as considered in [8]. We note that in our model, when $0 < n < 1$ (respectively, $-\infty < n < 0$), the linguistic value $\sigma\,true$ is more (respectively, less) specific than the truth-value *true*. This is because of when $0 < n < 1$, $\mu_{\sigma\,true}(u) < \mu_{true}(u)$, and when $-\infty < n < 0$, $\mu_{\sigma\,true}(u) > \mu_{true}(u)$. Similarly, when $0 < m < 1$ (respectively, $1 < m < \infty$), the linguistic value $\sigma\,false$ is more (respectively, less) specific than the truth-value *false*. It can be seen that when n approaches 1 (respectively, $-\infty$), the linguistic value $\sigma\,true$ is the most (respectively, the least) specific case with respect to the truth-value *true*. A similar situation is also for the parameter m. That is, the more truth (or falsity) a linguistic value is, the more specific a linguistic value becomes.

This specific relation can be determined through the areas under the membership functions defined as follows:

$$S_{\sigma\,true} = \int_0^1 \mu_{\sigma\,true}(u)\,\mathrm{d}u, \quad \text{and} \quad S_{\sigma\,false} = \int_0^1 \mu_{\sigma\,false}(u)\,\mathrm{d}u.$$

Then, $\sigma\,true$ (respectively, $\sigma\,false$) is more specific than $\sigma'\,true$ (respectively, $\sigma'\,false$) if $S_{\sigma\,true} < S_{\sigma'\,true}$ (respectively, $S_{\sigma\,false} < S_{\sigma'\,false}$). It should be worthwhile now to note that the specific relation defines an ordered relation, denoted by

$\leqslant_s$, on the linguistic truth space, which is completely compatible with the so-called semantically ordering relation defined in [13], as follows:

$$\sigma' \, true \leqslant_s \sigma \, true \iff S_{\sigma \, true} \leqslant S_{\sigma' \, true},$$
$$\sigma' \, false \leqslant_s \sigma \, false \iff S_{\sigma' \, false} \leqslant S_{\sigma \, false}.$$

We also note that due to the semantic characteristic of *true* and *false*, we define $\sigma' \, false \leqslant_s \sigma \, true$ for any σ and σ'. Moreover, this order is fully characterized by the natural order defined on the spaces of parameters as depicted in Fig. 5.

At this point, it is easily seen that the following holds.

Theorem 1. *The structure $(V, \leqslant_s)$ is a completely distributive lattice with Abs. true and Abs. false as the unit and zero elements, respectively. More particularly, for any $x, y \in V$, we have*

$$x \vee y = \begin{cases} x & \text{if } x = \sigma \, true \text{ and } y = \sigma' \, false, \\ y & \text{if } x = \sigma \, false \text{ and } y = \sigma' \, true, \\ \arg\min\{S_x, S_y\} & \text{if both } x \text{ and } y \text{ are generated from true}, \\ \arg\max\{S_x, S_y\} & \text{if both } x \text{ and } y \text{ are generated from false}, \end{cases}$$

$$x \wedge y = \begin{cases} y & \text{if } x = \sigma \, true \text{ and } y = \sigma' \, false, \\ x & \text{if } x = \sigma \, false \text{ and } y = \sigma' \, true, \\ \arg\max\{S_x, S_y\} & \text{if both } x \text{ and } y \text{ are generated from true}, \\ \arg\min\{S_x, S_y\} & \text{if both } x \text{ and } y \text{ are generated from false} \end{cases}$$

for any strings of hedges σ, σ', and where $\vee$ and $\wedge$ stand for the operations join and meet, respectively.

Let us denote $\mathscr{V} = (V, \vee, \wedge, \leqslant_s)$.

For special cases, we have $S_{true} = S_{false} = 0.5$, $S_{Abs.\, true} = S_{Abs.\, false} = 0$, and $S_{unknown} = 1$. Moreover, for the linguistic values generated from *true*, we have
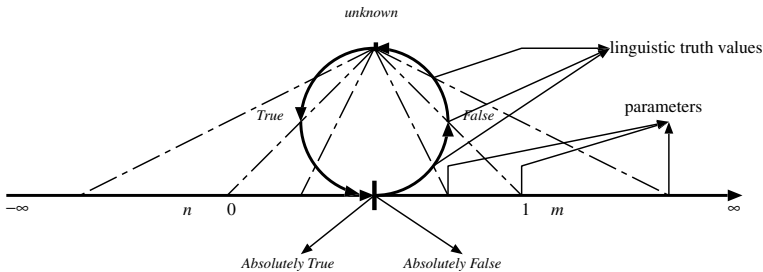

Fig. 5. The ordered relation in the linguistic truth space.

$$S_{\sigma\,true} = \begin{cases} \frac{1-n}{2} & \text{for } 0 < n < 1, \\ 1 - \frac{1}{2(1-n)} & \text{for } -\infty < n < 0. \end{cases} \tag{3}$$

With the same calculation for the linguistic values generated from *false*, we obtain

$$S_{\sigma\,false} = \begin{cases} \frac{m}{2} & \text{for } 0 < m < 1, \\ 1 - \frac{1}{2m} & \text{for } 1 < m < \infty. \end{cases} \tag{4}$$

It should be emphasized that Nafarieh and Keller in [8] also proposed a similar calculation but they defined the parameter n as an order of the power of a linguistic truth-value *true*.

Now, we discuss the problem of how to define the parameter of the antonymous label of a given linguistic truth-value in our model. Without loss of generality, consider a linguistic truth-value $\sigma\,true$ with its parameter $n_{\sigma\,true}$. The antonymous label of $\sigma\,true$ is the value $\sigma\,false$, which is called the contradictory element in [16], and the parameter $m_{\sigma\,false}$ may be defined such that the following holds:

$$S_{\sigma\,true} = S_{\sigma\,false}. \tag{5}$$

Under such a condition, it follows directly from (3) and (4) that

$$n_{\sigma\,true} = 1 - m_{\sigma\,false}. \tag{6}$$

That is, we have an interesting one-to-one correspondence between the parameter of a linguistic truth-value with that of its antonym. Consequently, we have

$$\mu_{\sigma\,false}(u) = \mu_{\sigma\,true}(1 - u) \tag{6'}$$

which may be suitable to intuitive meaning of an antonymous label. For example, let us define[2] $n_{true} = 0, n_{very\,true} = 0.5$, and $n_{fairly\,true} = -1$. Then we obtain $m_{false} = 1$, $m_{very\,false} = 0.5$, and $m_{fairly\,false} = 2$, and the membership functions of these linguistic truth-values are illustrated in Fig. 6.

We now define a negation operation, denoted by $\neg$, in V via (6) and (6'). This means that the negation of a linguistic truth-value is defined by its antonymous linguistic truth-value. This negation operation can be derived in $\mathscr{V}$, and so we write

$$\mathscr{V} = (V, \vee, \wedge, \neg, \leqslant_s).$$

Some fundamental properties of this operation are listed in the following theorem. The proof is easily followed.

---

[2] The hedge *fairly* is considered to have the same meaning as *more or less* in [1].
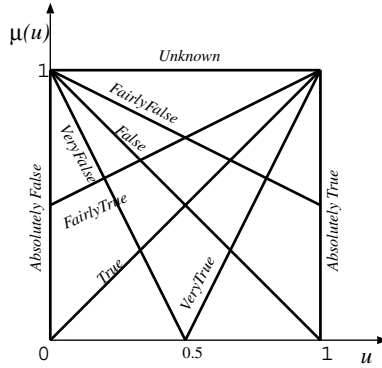
Fig. 6. A linguistic truth space.

Theorem 2. *The following statements hold in $\mathscr{V}$:*
 (i) $\neg\neg x = x$ *for any $x \in V$;*
 (ii) $x \leqslant_s y$ *iff* $\neg y \leqslant_s \neg x$ *for any $x, y \in V$;*
(iii) $\neg\, true = false,\ \neg\, false = true$;
(iv) $\neg\, unknown = unknown$;
 (v) $\neg Abs.\, true = Abs.\, false,\ \neg Abs.\, false = Abs.\, true$.

Furthermore, we have the following:

Theorem 3. $\mathscr{V}$ *is a De Morgan algebra.*[3]

Proof. By Theorem 1 and (i) of Theorem 2, it is sufficient to prove the triple $(\vee, \wedge, \neg)$ forms a De Morgan triple [9]. Indeed, for any $x, y \in V$, we have the following possibilities:
(a) both x and y are generated from *true*, with the associated parameters n_x and n_y, respectively;
(b) both x and y are generated from *false*, with the associated parameters m_x and m_y, respectively;
(c) x is generated from *true* and y is generated from *false*, with the associated parameters n_x and m_y, respectively;
(d) x is generated from *false* and y is generated from *true*, with the associated parameters m_x and n_y, respectively.
   For the case (a), we have

$$m_{\neg(x \vee y)} = 1 - \max(n_x, n_y) = \min(1 - n_x, 1 - n_y). \tag{7}$$

---

[3] Also named as *Soft algebra* [9]

On the other hand, we also have

$$m_{(\neg x \wedge \neg y)} = \min(m_{\neg x}, m_{\neg y}) = \min(1 - n_x, 1 - n_y). \tag{7'}$$

It implies by (7) and (7') that $m_{\neg(x \vee y)} = m_{(\neg x \wedge \neg y)}$, and hence

$$\neg(x \vee y) = (\neg x \wedge \neg y) \tag{8}$$

that we desire. By an analogous argument, we also obtain the equality (8) for the case (b). The remain cases follow directly from the definitions of the relation $\leqslant_s$ and the negation $\neg$.

By duality, we also obtain the equality

$$\neg(x \wedge y) = (\neg x \vee \neg y). \tag{9}$$

The qualities (8) and (9) mean that the triple $(\vee, \wedge, \neg)$ is a De Morgan triple. This completes the proof. $\square$

It is worth to mention that the algebra $\mathscr{V}$ includes the 3-valued Łukasiewicz algebra $\{Abs. false, unknown, Abs. true\}$ as its subalgebra.

3.3. A concept of converse of linguistic hedges

Firstly, we recall that in [1,8] linguistic hedges are identified by orders of powers of the primary linguistic truth-value. For example,

$$
\begin{aligned}
\mu_{(very)^k true}(u) &= [\mu_{true}(u)]^{2^k}, \\
\mu_{(very)^k false}(u) &= [\mu_{false}(u)]^{2^k}, \\
\mu_{(fairly)^k true}(u) &= [\mu_{true}(u)]^{\frac{1}{2^k}}, \\
\mu_{(fairly)^k false}(u) &= [\mu_{false}(u)]^{\frac{1}{2^k}}
\end{aligned}
\tag{10}
$$

for any $k = 0, 1, \ldots, \infty$.

Although it was not presented explicitly in [1], we easily see that the following holds

$$
\begin{aligned}
S_{(very)^k true} &= S_{(very)^k false}, \\
S_{(fairly)^k true} &= S_{(fairly)^k false}, \\
S_{(very)^k true} &= 1 - S_{(fairly)^k true}, \\
S_{(very)^k false} &= 1 - S_{(fairly)^k false}
\end{aligned}
\tag{11}
$$

for any $k = 0, 1, \ldots, \infty$. The first two equations in (11) are consistent with (5) that is used to define the parameter of the antonymous label of a given linguistic truth-value. By (10) we mean that there is an one-to-one correspondence between values $(very)^k true$ and $(fairly)^k true$ (and also, $(very)^k false$

and $(fairly)^k false$) as that between parameters 2^k and $\frac{1}{2^k}$, as well as equations in (11) are satisfied.

Under such an observation, we now introduce a concept of the converse of a given linguistic hedge via the specific relation mentioned above.

Given a linguistic hedge σ, and σX_p is a linguistic truth-value generated from X_p by means of σ, where X_p is *true* or *false*. Then another linguistic hedge σ' is said to be *converse* to σ and vice versa if and only if the following holds:

$$S_{\sigma X_p} = 1 - S_{\sigma' X_p}. \tag{12}$$

For example, in Baldwin's model [1], $(very)^k$ is converse to $(fairly)^k$ and vice versa for $k = 1, 2 \ldots$. We also note that this concept of converse is a special case of that introduced by Nguyen and Wechler in [15].

It should be of interest that the relationship defined by (12) gives an intuitive meaning of the concepts of *positive* and *negative* [6,15,16] of linguistic hedges with respect to a linguistic truth-value to which they are applied directly. For example, *very* strengthens the *positive meaning* of *true*, while *fairly* weakens its *positive meaning*.

We are now ready to establish one-to-one correspondence between parameters of linguistic truth-values exhibited the above property of hedges. For this purpose, we define the following mappings:

$$\psi : (-\infty, 1) \rightarrow (-\infty, 1)$$
$$n \mapsto \psi(n) = \frac{n}{n-1},$$
$$\chi : (0, \infty) \rightarrow (0, \infty)$$
$$m \mapsto \chi(m) = \frac{1}{m}$$

which establish, respectively, one-to-one correspondence between $(0, 1)$ and $(-\infty, 0)$ (for parameter n), and between $(0, 1)$ and $(1, \infty)$ (for parameter m). With this notation, we easily obtain

$$S_{E[n]} = 1 - S_{E[\psi(n)]}, \quad \text{and} \quad S_{E[m]} = 1 - S_{E[\chi(m)]}, \tag{12'}$$

where $E[p]$ stands for the linguistic truth-value associated with parameter p. That is, for $u \in [0, 1]$, the membership function of the linguistic truth-value $E[p]$ is defined by

$$\mu_{E[p]}(u) = \begin{cases} \max(0, (1-p)^{-1}(u-p)) & \text{if } p = n \text{ or } p = \psi(n), \\ \max(0, p^{-1}(p-u)) & \text{if } p = m \text{ or } p = \chi(m). \end{cases}$$

As an illustration, let us define

$$n_{(very)^k true} = \sum_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k}, \quad \text{and} \quad m_{(very)^k false} = 1 - n_{(very)^k true} = \frac{1}{2^k}.$$

Table 1
Different values of parameters n, m and respective linguistic truth-values

n	Linguistic value	m	Linguistic value
1	*Absolutely true*	0	*Absolutely false*
$\frac{3}{4}$	*very very true*	$\frac{1}{4}$	*very very false*
$\frac{1}{2}$	*very true*	$\frac{1}{2}$	*very false*
0	*true*	1	*false*
-1	*fairly true*	2	*fairly false*
-3	*fairly fairly true*	4	*fairly fairly false*
$-\infty$	*unknown*	∞	*unknown*

Then we have the parameters associated respectively with linguistic truth-values $(fairly)^k true$ and $(fairly)^k false$ as follows:

$$n_{(fairly)^k true} = \frac{\sum_{i=1}^{k} \frac{1}{2^i}}{\sum_{i=1}^{k} \frac{1}{2^i} - 1} = 1 - 2^k, \quad \text{and} \quad m_{(fairly)^k false} = 2^k.$$

It follows by formulae (3) and (4) that $S_{(very)^k true} = 1 - S_{(fairly)^k true}$ and $S_{(very)^k false} = 1 - S_{(fairly)^k false}$. Thus, $(very)^k$ is converse to $(fairly)^k$ and vice versa for $k = 1, 2 \ldots$. Table 1 shows some special cases for different values of n and m for truth-values generated from *true* and *false*, respectively, as well as the accepted linguistic translations of these parametric values.

In the next section, we utilize this reverse property of linguistic hedges in defining another kind of negation in a fuzzy logic.

4. A fuzzy logic based on the parametric representation of linguistic truth-values

In this section, we introduce a fuzzy logic based on the parametric representation of linguistic truth-values proposed in the preceding section.

For simplicity of notation, let us denote $N = (-\infty, 1)$, and $M = (0, \infty)$, which are designed as domains of parameters n and m, respectively. Denote V_t (respectively, V_f) the set of linguistic truth-values generated from *true* (respectively, *false*) by means of linguistic hedges.

Let us define logical operations in the linguistic truth space. If we let $\mathbf{v}(P)$ represent the linguistic truth-value of a proposition P. Then for propositions P and Q, the definitions of conjunction, disjunction, negation, implication are given by

$$\mathbf{v}(P \text{ and } Q) = \begin{cases} E[m_{\mathbf{v}(Q)}] & \text{if } \mathbf{v}(P) \in V_t \quad \text{and} \quad \mathbf{v}(Q) \in V_f, \\ E[m_{\mathbf{v}(P)}] & \text{if } \mathbf{v}(Q) \in V_t \quad \text{and} \quad \mathbf{v}(P) \in V_f, \\ E[\min(n_{\mathbf{v}(P)}, n_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_t, \\ E[\min(m_{\mathbf{v}(P)}, m_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_f, \end{cases}$$

(13)

$$\mathbf{v}(P \text{ or } Q) = \begin{cases} E[n_{\mathbf{v}(P)}] & \text{if } \mathbf{v}(P) \in V_t \quad \text{and} \quad \mathbf{v}(Q) \in V_f, \\ E[n_{\mathbf{v}(Q)}] & \text{if } \mathbf{v}(Q) \in V_t \quad \text{and} \quad \mathbf{v}(P) \in V_f, \\ E[\max(n_{\mathbf{v}(P)}, n_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_t, \\ E[\max(m_{\mathbf{v}(P)}, m_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_f, \end{cases} \tag{14}$$

$$\mathbf{v}(\text{not } P) = \begin{cases} E[1 - n_{\mathbf{v}(P)}] & \text{if } \mathbf{v}(P) \in V_t, \\ E[1 - m_{\mathbf{v}(P)}] & \text{if } \mathbf{v}(P) \in V_f, \end{cases} \tag{15}$$

$$\mathbf{v}(P \to Q)$$
$$= \begin{cases} E[\max(m_{\mathbf{v}(\text{not } P)}, m_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(P) \in V_t \quad \text{and} \quad \mathbf{v}(Q) \in V_f, \\ E[\max(n_{\mathbf{v}(\text{not } P)}, n_{\mathbf{v}(Q)})] & \text{if } \mathbf{v}(Q) \in V_t \quad \text{and} \quad \mathbf{v}(P) \in V_f, \\ E[n_{\mathbf{v}(Q)}] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_t, \\ E[n_{\mathbf{v}(\text{not } P)}] & \text{if } \mathbf{v}(P), \mathbf{v}(Q) \in V_f. \end{cases} \tag{16}$$

We note that the parameters $1 - n_{\mathbf{v}(P)}$ and $1 - m_{\mathbf{v}(P)}$ in Eq. (15) are strictly in relation with (6), i.e., that $1 - n_{\mathbf{v}(P)} \in M$, and $1 - m_{\mathbf{v}(P)} \in N$. As a consequence of above definitions of logical connectives and operators defined in the algebra $\mathscr{V}$, we have the following:

Theorem 4. *The operators $\wedge, \vee$, and $\neg$ in $\mathscr{V}$ model exactly logical connectives conjunction, disjunction, and negation, respectively, in the fuzzy logic defined above. More particularly,*

$$\mathbf{v}(P \text{ and } Q) = \mathbf{v}(P) \wedge \mathbf{v}(Q), \quad \mathbf{v}(P \text{ or } Q) = \mathbf{v}(P) \vee \mathbf{v}(Q), \quad \mathbf{v}(\text{not } P) = \neg \mathbf{v}(P).$$

For example, let $\mathbf{v}(P) = \textit{very true}$, $n_{\mathbf{v}(P)} = 0.5$, and $\mathbf{v}(Q) = \textit{fairly false}$, $m_{\mathbf{v}(Q)} = 2$ as defined in the previous section, moreover, by (6) we obtain $m_{\mathbf{v}(\text{not } P)} = 0.5$, and $n_{\mathbf{v}(\text{not } Q)} = -1$. Then, we have

$$\mathbf{v}(P \text{ and } Q) = E[m_{\mathbf{v}(Q)}] = \textit{fairly false} \quad \text{by (13)},$$

$$\mathbf{v}(P \text{ or } Q) = E[n_{\mathbf{v}(P)}] = \textit{very true} \quad \text{by (14)},$$

$$\mathbf{v}(\text{not } P) = E[1 - n_{\mathbf{v}(P)}] = E[m_{\mathbf{v}(\text{not } P)}] = \textit{very false} \quad \text{by (15) and (6)},$$

$$\mathbf{v}(\text{not } Q) = E[1 - m_{\mathbf{v}(Q)}] = E[n_{\mathbf{v}(\text{not } Q)}] = \textit{fairly true} \quad \text{by (15) and (6)},$$

$$\mathbf{v}(P \to Q) = E[\max(m_{\mathbf{v}(\text{not } P)}, m_{\mathbf{v}(Q)})] = E[m_{\mathbf{v}(Q)}] = \textit{fairly false} \quad \text{by (16)}$$

$$\mathbf{v}(\text{not } P \to Q) = E[n_{\mathbf{v}(\text{not not } P)}] = E[1 - m_{\mathbf{v}(\text{not } P)}] = E[n_{\mathbf{v}(P)}]$$
$$= \textit{very true} \quad \text{by (6) and (16)}.$$

This example shows the same result as those obtained in [6]. Now, to compare with Baldwin's model proposed in [1], as in previous section, let us define

$$n_{(very)^k true} = \sum_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k}, \quad \text{and} \quad m_{(very)^k false} = 1 - n_{(very)^k true} = \frac{1}{2^k}$$

for $k = 1, 2, \ldots$. By correspondences ψ and χ at the end of Section 3, we obtain the parameters associated respectively with linguistic truth-values $(fairly)^k true$ and $(fairly)^k false$ as follows:

$$n_{(fairly)^k true} = 1 - 2^k, \quad \text{and} \quad m_{(fairly)^k false} = 2^k.$$

Hence, it follows that

$$n_{(very)^k true} = 1 - \frac{1}{2^k} \to 1 \quad \text{as } k \to \infty,$$

$$m_{(very)^k false} = \frac{1}{2^k} \to 0 \quad \text{as } k \to \infty,$$

$$n_{(fairly)^k true} = 1 - 2^k \to -\infty \quad \text{as } k \to \infty,$$

$$m_{(fairly)^k false} = 2^k \to \infty \quad \text{as } k \to \infty.$$

Consequently, we obtain

$$(very)^k true \to Abs.\, true \quad \text{as } k \to \infty,$$

$$(very)^k false \to Abs.\, false \quad \text{as } k \to \infty,$$

$$(fairly)^k true \to unknown \quad \text{as } k \to \infty,$$

$$(fairly)^k false \to unknown \quad \text{as } k \to \infty.$$

Further, Table 2 is followed easily by using the definitions of conjunction and disjunction, and is easily extended to include other linguistic truth-values.

Table 2
A reduced linguistic truth table for conjunction and disjunction

$\mathbf{v}(P)$	$\mathbf{v}(Q)$	$\mathbf{v}(P \text{ and } Q)$	$\mathbf{v}(P \text{ or } Q)$
false	*false*	*false*	*false*
false	*true*	*false*	*true*
true	*false*	*false*	*true*
true	*true*	*true*	*true*
unknown	*true*	*unknown*	*true*
unknown	*false*	*false*	*unknown*
unknown	*unknown*	*unknown*	*unknown*
unknown	*Abs. false*	*Abs. false*	*unknown*
unknown	*Abs. true*	*unknown*	*Abs. true*
true	*very true*	*true*	*very true*
true	*fairly true*	*fairly true*	*true*
false	*very true*	*false*	*very true*
false	*fairly true*	*false*	*fairly true*
Abs. true	*false*	*false*	*Abs. true*
Abs. false	*true*	*Abs. false*	*true*
Abs. true	*Abs. false*	*Abs. false*	*Abs. true*

The above limited expressions and Table 2 show that our model is compatible with that proposed by Baldwin in [1].

We now establish basic linguistic truth expressions associated with respective parameters as follows:

(i) $n_{(very)^k true} = \sum_{i=1}^{k} \frac{1}{2^i} = 1 - \frac{1}{2^k}$ for $k = 1, \ldots, \infty$.

k	1	2	3	4
Truth value	*very true*	$(very)^2 true$	$(very)^3 true$	$(very)^4 true$
$n_{(very)^k true}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$

(ii) $m_{(very)^k false} = \frac{1}{2^k}$ for $k = 1, \ldots, \infty$.

k	1	2	3	4
Truth value	*very false*	$(very)^2 false$	$(very)^3 false$	$(very)^4 false$
$m_{(very)^k false}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

(iii) $n_{(fairly)^k true} = 1 - 2^k$ for $k = 1, \ldots, \infty$.

k	1	2	3	4
Truth value	*fairly true*	$(fairly)^2 true$	$(fairly)^3 true$	$(fairly)^4 true$
$n_{(fairly)^k true}$	-1	-3	-7	-15

(iv) $m_{(fairly)^k false} = 2^k$ for $k = 1, \ldots, \infty$.

k	1	2	3	4
Truth value	*fairly false*	$(fairly)^2 false$	$(fairly)^3 false$	$(fairly)^4 false$
$m_{(fairly)^k false}$	2	4	8	16

To close this section, as mentioned in the previous section, we now discuss how to use the relationship established by (12) to define a further operation, denoted by $\sim$, via correspondences ψ and χ. Let $\mathbf{v}(P)$ be the linguistic truth-value of a proposition P, we define

$$\sim \mathbf{v}(P) = \begin{cases} E[\chi(m_{\neg \mathbf{v}(P)})] & \text{if } \mathbf{v}(P) \in V_t, \\ E[\psi(n_{\neg \mathbf{v}(P)})] & \text{if } \mathbf{v}(P) \in V_f. \end{cases} \tag{17}$$

Table 3
A reduced linguistic truth table for $\neg$ and $\sim$

$\mathbf{v}(P)$	$\neg\mathbf{v}(P)$	$\sim\mathbf{v}(P)$
true	false	false
fairly true	fairly false	very false
very true	very false	fairly false
$(very)^2$ true	$(very)^2$ false	$(fairly)^2$ false
Abs. true	Abs. false	
Abs. false	Abs. true	
unknown	unknown	
false	true	true
fairly false	fairly true	very true
very false	very true	fairly true
$(very)^2$ false	$(very)^2$ true	$(fairly)^2$ true

With this definition, we have

$$\sim\sim\mathbf{v}(P) = \mathbf{v}(P), \quad \sim true = false, \quad \sim false = true.$$

Recall that in the conventional approach to fuzzy logic [1,22], there are also possible two forms of negation. Particularly, the truth-value of the proposition $(\text{not } P)$ is defined by

$$\mu_{\mathbf{v}(\text{not } P)}(u) = \mu_{\mathbf{v}(P)}(1 - u) \quad \text{for any } u \in [0, 1] \tag{18}$$

while the truth-value $(\text{not } \mathbf{v}(P))$ is given by

$$\mu_{\text{not } \mathbf{v}(P)}(u) = 1 - \mu_{\mathbf{v}(P)}(u) \quad \text{for any } u \in [0, 1]. \tag{19}$$

It is easily seen that the operator $\neg$ in our model is fully compatible with that defined by (18). We now show that the operator $\sim$ defined by (17) gives the same result as that computed by (19) in [1]. As a simple illustration, using computed results in (i)–(iv) and the definition of mappings ψ, χ, we easily establish the result as shown in Table 3.

Comparison of the obtained result in Table 3 with that given in [1, Table 4] may allow us to use $\sim$ as another kind of negation in our model. Note that the computed result in [1] is only obtained after a step of linguistic approximation, while our model gives directly the result without any step of linguistic approximation.

5. Conclusions

A new model for parametric representation of linguistic truth-values has been proposed in this paper. It has been shown that our model is superior to the existing models under several intuitive criteria both algebraically and

computationally. We know that every deductive system in classical or non-classical logic always determines an algebra in a certain class of abstract algebras of the same category of the corresponding algebra of truth-values [17]. An interesting point is that the proposed model not only yields an efficient method for computing linguistic truth expressions without a step of linguistic approximation, but also accompanies with a rich algebraic structure of the linguistic truth domain, namely De Morgan algebra. This may allow us to examine some characteristics of fuzzy linguistic logic through the algebraic structure of the linguistic truth domain. Furthermore, the model proposed in this paper can be also extended to an arbitrary linguistic variable with the shape of triangular and trapezoidal membership functions of primary fuzzy sets. These problems as well as a method of approximate reasoning based on this approach are being the subject of our further work.

Acknowledgements

References

[1] J.F. Baldwin, A new approach to approximate reasoning using a fuzzy logic, Fuzzy Sets and Systems 2 (1979) 309–325.

[2] R.E. Bellman, L.A. Zadeh, Local and fuzzy logics, in: G.J. Klir, B. Yuan (Eds.), Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected papers by L.A. Zadeh, World Scientific, Singapore, 1996, pp. 283–335.

[3] D. Dubois, H. Prade, Fuzzy sets in approximate reasoning, Part 1: Inference with possibility distributions, Fuzzy Sets and Systems 40 (1991) 143–202.

[4] G.J. Klir, T. Folger, Fuzzy Sets, Uncertainty and Information, Prentice-Hall, Englewood Cliffs, NJ, 1988.

[5] G. Lakoff, Hedges: a study in meaning criteria and the logic of fuzzy concepts, Journal of Philosophical Logic 2 (1973) 458–508.

[6] L. Di Lascio, A. Gisolfi, V. Loia, A new model for linguistic modifiers, International Journal of Approximate Reasoning 15 (1996) 25–47.

[7] M. Mizumoto, Comparison of various fuzzy reasoning methods, Preprints of Second IFSA Congress, Tokyo, July 20–25, 1987.

[8] A. Nafarieh, M.J. Keller, A new approach to inference in approximate reasoning, Fuzzy Sets and Systems 41 (1991) 17–37.

[9] C.V. Negoita, D.A. Ralescu, Application of Fuzzy Sets to System Analysis, ISR, vol. 11, Birkhauser, Basel–Boston, 1975.

[10] C.H. Nguyen, A method in linguistic reasoning on a knowledge base representing by sentences with linguistic belief degree, Fundamenta Informaticae 28 (1996) 247–259.

[11] C.H. Nguyen, D.K. Tran, V.N. Huynh, H.C. Nguyen, Hedge algebras, linguistic-valued logic and their application to fuzzy reasoning, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 7 (4) (1999) 347–361.

[12] C.H. Nguyen, V.N. Huynh, A theory of refinement structure of hedge algebras and its application to linguistic-valued fuzzy logic, in: D. Niwinski, M. Zawadowski (Eds.), Logic Algebra and Computer Science, vol. 46, Banach Center Publications, 1999, pp. 63–91.

[13] C.H. Nguyen, V.N. Huynh, An algebraic approach to linguistic hedges in Zadeh's fuzzy logic, Fuzzy Sets and Systems 129 (2002) 229–254.

[14] C.H. Nguyen, V.N. Huynh, Ordered structure-based semantics of linguistic terms of linguistic variables and approximate reasoning, in: D.M. Dubois (Ed.), AIP Conference Proceedings: Computing Anticipatory Systems, vol. 517, American Institute of Physics, New York, 2000, pp. 98–116.

[15] C.H. Nguyen, W. Wechler, Hedge algebras: an algebraic approach to structure of sets of linguistic truth values, Fuzzy Sets and Systems 35 (1990) 281–293.

[16] C.H. Nguyen, W. Wechler, Extended hedge algebras and their application to fuzzy logic, Fuzzy Sets and Systems 52 (1992) 259–281.

[17] H. Rasiowa, R. Sikorski, The Mathematics of Metamathematics, second ed., Polish Scientific Publishers, Warszawa, 1968.

[18] L.A. Zadeh, Fuzzy-set-theoretic interpretation of linguistic hedges, Journal of Cybernetics 2 (1972) 4–34.

[19] L.A. Zadeh, The concept of linguistic variable and its application to approximate reasoning (I), Information Sciences 8 (1975) 199–249.

[20] L.A. Zadeh, The concept of linguistic variable and its application to approximate reasoning (II), Information Sciences 8 (1975) 310–357.

[21] L.A. Zadeh, The concept of linguistic variable and its application to approximate reasoning (III), Information Sciences 9 (1975) 43–80.

[22] L.A. Zadeh, Fuzzy logic and approximate reasoning, Synthese 30 (1975) 407–428.

[23] L.A. Zadeh, A theory of approximate reasoning, in: R.R. Yager, S. Ovchinnikov, R.M. Tong, H.T. Nguyen (Eds.), Fuzzy Sets and Applications: Selected papers by L.A. Zadeh, Wiley, New York, 1987, pp. 367–411.

[24] L.A. Zadeh, On the analysis of large-scale systems, in: H. Gottinger (Ed.), Systems Approaches and Environment Problems, Vandenhoeck and Ruprecht, Gottingen, 1974, pp. 23–37.

[25] H.-J. Zimmermann, Fuzzy Set Theory and Its Applications, second ed., Kluwer Academic Publishers, Boston/Dordrecht/London, 1991.