Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Information Sciences 160 (2004) 111–129
INFORMATION SCIENCES
AN INTERNATIONAL JOURNAL
www.elsevier.com/locate/ins

A context model for fuzzy concept analysis based upon modal logic

V.N. Huynh [a,*,1], Y. Nakamori [a], T.B. Ho [a], G. Resconi [b]

[a] *School of Knowledge Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Tatsunokuchi, Ishikawa 923-1292, Japan*
[b] *Catholic University, Trieste 17, 25128 Brescia, Italy*

Received 23 November 2002; received in revised form 20 June 2003; accepted 4 August 2003

Abstract

In this paper we present interesting relationships between the context model, modal logic and fuzzy concept analysis. It has been shown that the context model proposed by Gebhardt and Kruse [Int. J. Approx. Reason. 9 (1993) 283] can be semantically extended and considered as a data model for fuzzy concept analysis within the framework of the meta-theory developed by Resconi et al. in 1990s. Consequently, the context model provides a practical framework for constructing membership functions of fuzzy concepts and gives the basis for a theoretical justification of suitably use of well-known *t*-norm based connectives such as min–max and product–sum rules in applications. Furthermore, an interpretation of mass assignments of fuzzy concepts within the context model is also established.

Keywords: Context model; Modal logic; Fuzzy concept; Membership function

1. Introduction

While the real world consists of a very large number of instances of events and continuous numeric values, people only represent and process their

* Corresponding author.
E-mail addresses: huynh@jaist.ac.jp, nv_huynh@yahoo.com (V.N. Huynh).
[1] On leave from University of Quinhon, Viet Nam.

0020-0255/$ - see front matter
doi:10.1016/j.ins.2003.08.010

knowledge in terms of abstracted concepts derived from generalization of these instances and numeric values. The fundamental elements in human reasoning are sentences normally containing vague concepts.

The mathematical model of vague concepts was firstly introduced by Zadeh [37] by using the notion of partial degrees of membership, in connection with the representation and manipulation of human knowledge automatically. Since then mathematical foundations as well as successful applications of fuzzy set theory have already been developed [23]. However, concerning the semantics of fuzzy sets, at present there is no uniformity in the interpretation of what a membership grade means. Dubois and Prade [7] have explored three main semantics for membership functions in which each semantics underlies a particular class of applications. As such, fuzzy set-based applications became feasible only when the methods of constructing membership functions of relevant fuzzy sets were efficiently developed in given application contexts.

During the last decade, Resconi et al. [32–34] have developed a hierarchical uncertainty meta-theory based upon modal logic. In particular, they established the usual semantics of propositional modal logic as a unifying framework within which various theories of uncertainty, including the fuzzy set theory, Dempster–Shafer theory of evidence, possibility theory, and Sugeno's λ-measures, can be conceptualized, compared, and organized hierarchically. Although Resconi's theory has shown to be very fruitful and potentially important as a unifying approach in the study of uncertainty, it is also a rather abstract one and, hence, ones need to relate it semantically to data models in particular application situations.

At the same time, Gebhardt and Kruse [9] have also developed a model of vagueness and uncertainty—called the context model—that provides a formal framework for the comparison and semantic foundation of several theories of uncertainty such as Bayes theory, Dempster–Shafer theory, and the possibility theory. For a point of view of formal concept analysis, in [18] we have proposed an approach to the problem of mathematical modeling of fuzzy concepts based on the theory of formal concept analysis [8] and the notion of context model [9,26]. Particularly, we introduced the notion of fuzzy concepts within a context model and the membership functions associated with these fuzzy concepts. It is shown that fuzzy concepts can be interpreted exactly as the collections of α-cuts of their membership functions. While this approach may be suitable for forming fuzzy concepts which are verbal descriptions imposed on quantitative individual characteristics of objects such as *tall*, *short*, *very tall*, etc. It makes it difficult to form complex fuzzy concepts which may be imposed on a combination of individual characteristics of objects, as well as in defining composed fuzzy concepts from fuzzy concepts in different domains such as *tall and heavy*. In [19], based on the meta-theory developed by Resconi et al. in 1990s [32–34], we have proposed a model of modal logic for fuzzy concept analysis from a context model. By this approach, we can integrate context models by using a model of modal logic, and then develop a method of calculating the expression for the membership functions of composed and/or complex fuzzy concepts based on values $\{0, 1\}$ corresponding to the truth values $\{F, T\}$ assigned to a given sentence as the response of a context considered as a possible world. It is of interest to note that fuzzy intersection and fuzzy union operators by this model are truth-functional and, moreover, they are a well-known dual pair of *product t-norm* and *probabilistic sum t-conorm* [20].

In this paper we first explore interesting relationships between the context model, modal logic and fuzzy concept analysis. Then we establish a mass assignment interpretation of fuzzy concepts proposed by Baldwin et al. [1,2] within the context model.

The rest of this paper is organized as follows. In the next section, we briefly present some preliminary concepts: context model, modal logic, and meta-theory (with a short introduction to the modal logic interpretation of various uncertainty theories). Section 3 introduces a context model for fuzzy concept analysis and propose a model of modal logic for formulating fuzzy sets within a context model. The mass assignment interpretation of fuzzy concepts is given in Section 4. Finally, Section 5 presents some concluding remarks.

2. Preliminaries: context model, modal logic, and meta-theory

2.1. Context model

In the framework of fuzzy data analysis, Gebhardt and Kruse [9] have introduced the context model as an approach to the representation, interpretation, and analysis of imperfect data. The short motivation of this approach stems from the observation that the origin of imperfect data is due to situations, where we are not able to specify an object by an original tuple of elementary characteristics because of incomplete information available.

A context model is defined as a triple $\langle D, C, A_C(D)\rangle$, where D is a non-empty *universe of discourse*, C is a non-empty *finite set of contexts*, and the set $A_C(D) = \{a | a : C \to 2^D\}$ which is called the set of all vague characteristics of D with respect to C. Let $a \in A_C(D)$, a is said to be *contradictory* (respectively, *consistent*) if and only if $\exists c \in C$, $a(c) = \emptyset$ (respectively, $\cap_{c\in C}\, a(c) \neq \emptyset$). For a_1, $a_2 \in A_C(D)$, then a_1 is said to be *more specific* than a_2 iff for any $c \in C$, $a_1(c) \subseteq a_2(c)$.

If there is a finite measure P_C on the measurable space $(C, 2^C)$, then $a \in A_C(D)$ is called a *valuated vague characteristic* of D w.r.t. P_C. Then we call a quadruple $\langle D, C, A_C(D), P_C\rangle$ a valuated context model. Formally, if $P_C(C) = 1$ the mapping $a : C \to 2^D$ is a random set but obviously with a different interpretation within the context model. We should mention that a formal connection between fuzzy sets and covering functions of random sets was established in [12,13].

In Gebhardt and Kruse's approach, each characteristic of an observed object is described by a fuzzy quantity formed by context model [26]. It should be emphasized that the forming of a fuzzy quantity by this approach is essentially comparable with the creation of a membership function in fuzzy set theory [22] and a possibility distribution in possibility theory [6], respectively. More refinements of the context model as well as its applications could be referred to Gebhardt and Kruse [10] and Gebhardt [11].

In the connection with formal concept analysis, it is interesting to note that in the case where C is a single-element set, say $C = \{c\}$, a context model formally becomes a formal context in the sense of Ganter and Wille (see [8]) as follows. Let $\langle D, C, A_C(D)\rangle$ be a context model such that $|C| = 1$. Then the triple (O, A, R), where $O = D$, $A = A_C(D)$ and $R \subseteq O \times A$ such that $(o, a) \in R$ iff $o \in a(c)$, is a formal context. Thus, a context model can be considered as a collection of formal contexts. Under such an observation, we have introduced in [18] an approach to the problem of mathematical modeling of fuzzy concepts based on the theory of formal concept analysis and the notion of context model. Particularly, we introduced the notion of fuzzy concepts within a context model and the membership functions associated with these fuzzy concepts. It is of interest to note that this approach to fuzzy concepts provides a unified interpretation for both notions of LT-fuzzy sets in the sense of Rasiowa and Nguyen [31] as well as of fuzzy sets in the sense of Zadeh [37].

2.2. Modal logic

In this subsection, we briefly review the basic concepts of modal logic. Propositional modal logic [5] is an extension of classical propositional logic that adds to the propositional logic two unary modal operators, an operator of necessity, $\Box$, and an operator of possibility, $\Diamond$. Given a proposition p, $\Box p$ stands for the proposition "it is necessary that p", and similarly, $\Diamond p$ represents the proposition "it is possible that p". Modal logic is well developed syntactically [5].

In [32–34], the modal logic interpretation of various uncertainty theories is based on the fundamental semantics of modal logic using Kripke models. A model, M, of modal logic is a triple

$$M = \langle W, R, V\rangle,$$

where W, R, V denote, respectively, a set of possible worlds, a binary relation on W, and a value assignment function, by which truth (T) or falsity (F) is assigned to each atom in each possible world, i.e.

$$V : W \times \mathcal{Q} \to \{T, F\},$$

where $\mathscr{Q}$ is the set of all atoms. The value assignment function is inductively extended to all formulas in the usual way, the only interesting cases being

$$V(w, \Box p) = T \iff \forall w' \in W, (wRw') \Rightarrow V(w', p) = T \iff \mathscr{R}_s(w) \subseteq \|p\|^M$$

$$V(w, \Diamond p) = T \iff \exists w' \in W, (wRw') \quad \text{and}$$
$$V(w', p) = T \iff \mathscr{R}_s(w) \cap \|p\|^M \neq \emptyset.$$

where $\mathscr{R}_s(w) = \{w' \in W | wRw'\}$, and $\|p\|^M = \{w | V(w, p) = T\}$.

Relation R is usually called an *accessibility relation*; we say that world u is accessible to world w when $(w, u) \in R$. If not specified otherwise, we always assume that W is finite. It is convenient to denote $W = \{w_1, w_2, \ldots, w_n\}$ and to represent relation R by a matrix $R = [r_{ij}]$, where

$$r_{ij} = \begin{cases} 1 & \text{if } (w_i, w_j) \in R \\ 0 & \text{if } (w_i, w_j) \notin R \end{cases}$$

Different systems of modal logic are characterised by different additional requirements on accessibility relation R [5]. Some systems of modal logic are depicted as shown in Table 1 (see [5]).

2.3. Meta-theory based upon modal logic

In the context of a research program initiated by Resconi and his colleagues [14–17,24,32–34], the authors have developed a hierarchical uncertainty meta-theory based upon modal logic. In particular, they established the usual semantics of propositional modal logic as a unifying framework within which various theories of uncertainty can be conceptualized, compared, and organized hierarchically. Within this framework, modal logic interpretations for several theories, including the Dempster–Shafer theory, fuzzy set theory, possibility theory, and Sugeno's λ-measures have been already proposed. These interpretations are based on Kripke model of modal logic.

A Kripke model is given by a triple $M = \langle W, R, V \rangle$. Moreover, Resconi et al. have suggested to add a weighting function $\Omega : W \to [0, 1]$ such that

Table 1
Accessibility relation and axiom schemas

No condition	**Df**$\Diamond$. $\Diamond p \leftrightarrow \neg\Box\neg p$
No condition	**K**. $\Box(p \to q) \to (\Box p \to \Box q)$
Serial: $\forall w \exists w' (wRw')$	**D**. $\Box p \to \Diamond p$
Reflexive: $\forall w (wRw)$	**T**. $\Box p \to p$
Symmetric: $\forall w \forall w' (wRw' \Rightarrow w'Rw)$	**B**. $p \to \Box\Diamond p$
Transitive: $\forall w \forall w' \forall w'' (wRw'$ and $w'Rw'' \Rightarrow wRw'')$	**4**. $\Box p \to \Box\Box p$
Connected: $\forall w \forall w' (wRw'$ or $w'Rw)$	**4.3**. $\Box(\Diamond p \vee \Diamond q) \to (\Box\Diamond p \vee \Box\Diamond q)$
Euclidean: $\forall w \forall w' \forall w'' (w'Rw$ and $w'Rw'' \Rightarrow wRw'')$	**5**. $\Diamond p \to \Box\Diamond p$

$$\sum_{i=1}^{n} \Omega(w_i) = 1$$

as a component of the model M. In such a way we obtain a new model of modal logic, namely $M_1 = \langle W, R, V, \Omega \rangle$.

With the model M_1, given a universe of discourse X we can consider propositions that are relevant to fuzzy sets have the following form:

a_x : "x belongs to a given set A"

where $x \in X$ and A denotes a subset of X that is based on a vague concept. Set A is then viewed as an ordinary fuzzy set whose membership function μ_A is defined, for all $x \in X$, by the following formula:

$$\mu_A(x) = \sum_{i=1}^{n} \Omega(w_i)^i a_x$$

where

$$^i a_x = \begin{cases} 1 & \text{if } V(w_i, a_x) = T \\ 0 & \text{otherwise} \end{cases}$$

The set-theoretic operations such as complement, intersection and union defined on fuzzy sets are then formulated within the model M_1 based on logical connectives NOT, AND, OR respectively (see [32,34]).

To model the interpretation of Dempster–Shafer theory of evidence in terms of modal logic, the authors in [14,32] employed propositions of the form

e_A : "A given incompletely characterized element $\epsilon$is classified in set A"

where X denotes a frame of discernment, $A \in 2^X$ and $\epsilon \in X$. Due to the inner structure of these propositions, it is sufficient to consider as atomic propositions only propositions $e_{\{x\}}$, where $x \in X$. Propositions e_A are then defined as $e_A = \vee_{x \in A} e_{\{x\}}$ for $A \neq \emptyset$ and $e_\emptyset = \wedge_{x \in X} \neg e_{\{x\}}$.

Furthermore, for each world $w_i \in W$, it is assumed that $V(w_i, e_{\{x\}}) = T$ for one and only one $x \in X$ and that the accessibility relation R is serial (see Table 1). Then the model M_1 yields the following equations for the four basic functions in the Dempster–Shafer theory:

$$\mathrm{Bel}(A) = \sum_{i=1}^{n} \Omega(w_i)^i(\Box e_A), \quad \mathrm{Pl}(A) = \sum_{i=1}^{n} \Omega(w_i)^i(\Diamond e_A)$$

$$m(A) = \sum_{i=1}^{n} \Omega(w_i)^i \left[\Box e_A \wedge \left(\bigwedge_{x \in A} \Diamond e_{\{x\}} \right) \right], \quad Q(A) = \sum_{i=1}^{n} \Omega(w_i)^i \left(\bigwedge_{x \in A} \Diamond e_{\{x\}} \right)$$

where Bel, Pl, m and Q denote the belief function, plausibility function, basic probability assignment, and commonality function in the Dempster–Shafer theory, respectively.

In the case where a basic probability assignment m in the Dempster–Shafer theory induces a nested family of focal elements, we obtain a special belief function called a *necessity measure*, along with a corresponding special plausibility function called a *possibility measure*. Possibility theory is based on these two special measures [6]. It has been shown in [24] that the accessibility relation R of models associated with possibility theory are transitive and connected, i.e. these models formally correspond to the modal system $S\mathbf{4.3}$ (see Table 1). The authors also showed the completeness of modal logic interpretation for possibility theory.

3. Fuzzy concepts by context model based on modal logic

In this section we propose a context model for fuzzy concept analysis based on modal logic. Firstly, we consider a context model for a single domain of an attribute which can be applied to a set of objects of concern.

3.1. Single domain case

Fuzzy set was introduced as a mathematical modeling of vague concepts in natural language. Obviously, the usefulness of a fuzzy set for modeling a linguistic label depends on the appropriateness of its membership function. Therefore, the practical determination of an accurate and justifiable function for any particular situation is of major concern.

Notice that the specific meaning of a vague concept in a proposition is usually evaluated in different ways for different assessments of an entity by different agents, contexts, etc. [35]. Let us consider the following example.

Consider a sentence such as: "John is tall", where "tall" is a linguistic term of a linguistic variable, the height of people [38]. Assume that the domain $D = [0, 3m]$ which is associated with the base variable of the linguistic variable *height*. Note that in the terms of fuzzy sets, we may know John's height but must determine to what degree he is considered "tall". Next consider a set of worlds W in the sense of the Kripke model in which each world evaluates the sentence as either *true* or *false*. That is each world in W responds either as true or false when presented with the sentence "John is tall". Notice that these worlds may be contexts, agents, persons, etc. This implicitly shows that each world w_i in W determines a subset of D given as being compatible with the linguistic term *tall*. That is this subset represents w_i's view of the vague concept "tall" [21]. At this point we see that the context model introduced by Gebhardt and Kruse [9] can be semantically extended and considered as a data model for constructing membership functions of vague concepts based on modal logic. An important principle mentioned in [35] is "we can not separate the assessments of the entity without some loss property in the representation of the entity itself".

Let us consider a context model $\mathscr{C} = \langle D, C, A_C(D) \rangle$, where D is a domain of an attribute 'at' which is applied to objects of concern, C is a non-empty finite set of contexts, and $A_C(D)$ is a set of linguistic terms associated with the domain D considered now as vague characteristics in the context model. For example, consider $D = [0, 3m]$ which is interpreted as the domain of the attribute *height* for people, C is a set of contexts such as Japanese, American, Swede, etc., and $A_C(D) = \{\textit{very short}, \textit{short}, \textit{medium}, \textit{tall}, \textit{more or less tall}, \ldots\}$. Each context determines a subset of D given as being compatible with a given linguistic term. Formally, each linguistic term can be considered as a mapping from C to 2^D. For linguistic terms such as *tall* and *very tall*, there are two interpretations possible: it may either be meant that *very tall* implies *tall*, i.e. that every very tall person is also tall. Or *tall* is an abbreviation for "tall, but not very tall". These two interpretations have been used in the literature depending on the shape of membership functions of relevant fuzzy sets. The linguistic term *very tall* is more specific than *tall* in the first interpretation, but not in the second one.

Furthermore, we can also associate with the context model a weighting function or a probability distribution Ω defined on C. As such we obtain a valuated context model

$$\mathscr{C} = \langle D, C, A_C(D), \Omega \rangle$$

By this context model, each linguistic term $a \in A_C(D)$ may be semantically represented by the fuzzy set A as follows:

$$\mu_A(x) = \sum_{c \in C} \Omega(c) \mu_{a(c)}(x)$$

where $\mu_{a(c)}$ is the characteristic function of $a(c)$. Intuitively, while each subset $a(c)$, for $c \in C$, represents the c's view of the vague concept a, the fuzzy set A is the result of a weighted combinated view of the vague concept. Now, we can formulate further for the set-theoretic operations on fuzzy sets by a straightforward manner in this model. However, for the sake of a further development in the next subsection, in the sequent we will formulate the problem in the terms of modal logic. To this end, we now consider propositions that are relevant to a linguistic term have the following form:

$$a_x : \text{“}x \text{ belongs to a given set } A\text{”}$$

where $x \in D$ and A denotes a subset of D that is based on a linguistic term a in $A_C(D)$. Assume that $C = \{c_1, \ldots, c_n\}$, we now define a model of modal logic

$$M = \langle W, R, V_D, \Omega \rangle$$

where $W = C$, that is each context c_i is associated with a possible world w_i; R is a binary relation on W, in this case R is the identity, i.e. each world w_i only itself is accessible; and V_D is the value assignment function such that for each world in W, by which truth (T) or falsity (F) is assigned to each atomic proposition a_x by

$$V_D(w_i, a_x) = \begin{cases} 1 & \text{if } x \in a(c_i) \\ 0 & \text{otherwise} \end{cases}$$

With this background, we now define the compatible degree of any value x in the domain D to the linguistic term a (and the set A is then viewed as an ordinary fuzzy set) as the membership expression of truthood of the atomic sentence a_x in M as follows:

$$\mu_A(x) = \sum_{i=1}^{n} \Omega(w_i) V_D(w_i, a_x) \tag{1}$$

In the case without the weighting function Ω, the expression $\mu_A(x)$ can be defined as

$$\mu_A(x) = \frac{|W_{a_x}|}{|W|} = \frac{1}{n} \sum_{i=1}^{n} V_D(w_i, a_x) \tag{2}$$

where $W_{a_x} = \{w \in W | V_D(w, a_x) = 1\}$, and $|\cdot|$ denotes the cardinality of a set.

Similar as in [34], it is straightforward to define the set-theoretic operations such as complement, intersection, union on fuzzy sets induced from linguistic terms in $A_C(D)$ by the model M using logical connectives NOT, AND, and OR respectively. Apply Eq. (1) to the complement A^c of fuzzy set A we have

$$\mu_{A^c}(x) = \sum_{i=1}^{n} \Omega(w_i) V_D(w_i, \neg a_x) = \sum_{i=1}^{n} \Omega(w_i)(1 - V_D(w_i, a_x)) = 1 - \mu_A(x)$$

In addition to propositions a_x, let us also consider propositions

b_x : "x belongs to a given set B"

where $x \in D$ and B denotes a subset of D that is based on another linguistic term b in $A_C(D)$. Then we also have

$$\mu_B(x) = \sum_{i=1}^{n} \Omega(w_i) V_D(w_i, b_x) \tag{3}$$

To define composed fuzzy sets $A \cap B$ and $A \cup B$, we now apply logical connectives AND, OR to propositions a_x and b_x as follows:

$$\mu_{A\cap B}(x) = \sum_{i=1}^{n} \Omega(w_i)V_D(w_i, a_x \wedge b_x) \tag{4}$$

$$\mu_{A\cup B}(x) = \sum_{i=1}^{n} \Omega(w_i)V_D(w_i, a_x \vee b_x) \tag{5}$$

It is easily seen that if a is more specific than b, we have

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

$$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

for any $x \in D$. This interpretation of linguistic hedges such as *very*, *less*, etc., is in accordance with that considered in [38]. This also justifies for the observation that linguistic terms with positive semantic consistency, the min–max rule is more correct in applications.

Following properties of the operations $\vee$, $\wedge$ in classical logic, we easily obtain

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_{A\cap B}(x) \tag{6}$$

Furthermore, it follows directly by (4)–(6) that

$$\max(0, \mu_A(x) + \mu_B(x) - 1) \leqslant \mu_{A\cap B}(x) \leqslant \min(\mu_A(x), \mu_B(x))$$

$$\max(\mu_A(x), \mu_B(x)) \leqslant \mu_{A\cup B}(x) \leqslant \min(1, \mu_A(x) + \mu_B(x))$$

It should be noticed that under the constructive formulation of fuzzy sets by this context model, fuzzy intersection and fuzzy union operations are no longer truth-functional. Furthermore, if there is a non-trivial relationship between contexts, we should take the relation R into account in defining of the fuzzy set A. A solution for this is by using modal operators $\Box$ and $\Diamond$, and results in an interval-valued fuzzy set defined as follows:

$$\mu_A(x) = \left[\sum_{i=1}^{n} \Omega(w_i)V_D(w_i, \Box a_x),\ \sum_{i=1}^{n} \Omega(w_i)V_D(w_i, \Diamond a_x)\right]$$

In the next subsection we deal with the general case where composed fuzzy sets which represent linguistic combinations of linguistic terms of several context models are considered.

3.2. General case

Let us consider a pair of variables x and y which may be interpreted as the values of two attributes at_1 and at_2 for objects of concern, ranging on domains D_1 and D_2, respectively. Let $\mathscr{C}_i = \langle D_i, C_i, A_{C_i}(D_i), \Omega_i\rangle$, for $i = 1, 2$ be context models defined on D_1 and D_2, respectively.

It should be emphasized that in the framework of fuzzy data analysis, characteristics (attributes) of observed objects can be considered simultaneously in the same contexts. However, this situation may not be longer suitable for fuzzy concept analysis. For example, let us consider two attributes *height* and *income* of a set of people. Then, a set of contexts used for formulating of vague concepts of the attribute *height* may be given as in the preceding subsection; while another set of contexts for formulating of vague concepts of the attribute *income* (like *high*, *low*, etc.), may be given as a set of kinds of employees or a set of residential areas of employees.

In this subsection we define a unified model of modal logic for combining these context models in order to formulate composed fuzzy sets which represent linguistic combinations of linguistic terms from different domains.

Given two context models $\mathscr{C}_i = \langle D_i, C_i, A_{C_i}(D_i), \Omega_i\rangle$ defined on D_i, for $i = 1, 2$, respectively. A pair $(x, y) \in D_1 \times D_2$ is then interpreted as the pair of values of two attributes at_1 and at_2 for objects of concern. Recall that each element in $A_{C_i}(D_i)$ is a linguistic term understood as a mapping from $C_i \rightarrow 2^{D_i}$. Assume that $|C_i| = n_i$, for $i = 1, 2$.

We now define a unified Kripke model as follows:

$$M = \langle W, R, V, \Omega\rangle$$

where $W = C_1 \times C_2$, R is the identity relation on W, and

$$\begin{aligned}\Omega : C_1 \times C_2 &\rightarrow [0, 1]\\ (c_i^1, c_j^2) &\mapsto \omega_{ij} = \omega_i\omega_j\end{aligned}$$

where the simplified notations $\Omega(c_i^1, c_j^2) = \omega_{ij}, \Omega_1(c_i^1) = \omega_i, \Omega_2(c_j^2) = \omega_j$ are used.

We should emphasize that the assumption imposed on this definition of Ω is that each individual context model is independent to the other as the example about attributes *height* and *income* just mentioned above.

For $a_i \in A_{C_i}(D_i)$, for $i = 1, 2$, we now formulate composed fuzzy sets, which represent combinated linguistic terms like "a_1 and a_2" and "a_1 or a_2" within model M.

For simplicity of notation, let us denote O a set of objects of concern which we may apply for two attributes at_1, at_2 those values range on domains D_1 and D_2, respectively. Then instead of considering fuzzy sets defined on different domains, we can consider fuzzy sets defined only on a universal set, the set of objects O. As such, we now consider atomic propositions of the form

a_o : "An object o is in relation to a linguistic term a"

where $a \in A_{C_1}(D_1) \cup A_{C_2}(D_2)$ or a is a linguistic combination of linguistic terms in $A_{C_1}(D_1) \cup A_{C_2}(D_2)$.

Notice that this constructive formulation of composed fuzzy sets is comparable with the notion of the translation of a proposition a_o into a *relational assignment equation* introduced in [39].

3.2.1. a is a single term

Firstly we consider the case where $a \in A_{C_1}(D_1)$. For this case, we define the valuation function V in M for atomic propositions a_o by

$$V((c_i^1, c_j^2), a_o) = \begin{cases} 1 & \text{if } \mathrm{at}_1(o) \in a(c_i^1) \\ 0 & \text{otherwise} \end{cases}$$

Then the fuzzy set A which represents the meaning of the linguistic term a is defined in the model M as follows:

$$\mu_A^M(o) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \omega_{ij} V((c_i^1, c_j^2), a_o) \tag{7}$$

Set $W' = \{(c_i^1, c_j^2) \in C_1 \times C_2 \mid V((c_i^1, c_j^2), a_o) = 1\}$. It follows by definition of V that $W' = C_1' \times C_2$, where $C_1' = \{c_i^1 \in C_1 | \mathrm{at}_1(o) \in a(c_i^1)\}$. The following proposition is implied directly.

Proposition 1. *We have*

$$\mu_A^M(o) = \mu_A^{M_1}(o)$$

where $\mu_A^{M_1}(o)$ *is represented by* $\mu_A^{M_1}(\mathrm{at}_1(o))$ *as in preceding subsection, here* $\mathrm{at}_1(o) \in D_1$ *denotes the value of attribute* at_1 *for object* o.

Similar for the case where $a \in A_{C_2}(D_2)$, we define the valuation function V in M for atomic propositions a_o by

$$V((c_i^1, c_j^2), a_o) = \begin{cases} 1 & \text{if } \mathrm{at}_2(o) \in a(c_j^2) \\ 0 & \text{otherwise} \end{cases}$$

Obviously, we also have

Proposition 2

$$\mu_A^M(o) = \mu_A^{M_2}(o)$$

where $\mu_A^{M_2}(o)$ *is represented by* $\mu_A^{M_2}(\mathrm{at}_2(o))$ *as in preceding subsection, here* $\mathrm{at}_2(o) \in D_2$ *denotes the value of attribute* at_2 *for object* o.

3.2.2. a is a composed linguistic term

We now consider for the case where a is a composed linguistic term which is of the form like "a_1 and a_2" and "a_1 or a_2", where $a_i \in A_{C_i}(D_i)$, for $i = 1, 2$. To formulate the composed fuzzy set A corresponding to the term a in the model

M, we need to define the valuation function V for propositions a_o. It is natural to express a_o by

$$a_o = \begin{cases} a_{1,o} \vee a_{2,o} & \text{if } a \text{ is ``} a_1 \text{ or } a_2\text{''} \\ a_{1,o} \wedge a_{2,o} & \text{if } a \text{ is ``} a_1 \text{ and } a_2\text{''} \end{cases}$$

where $a_{i,o}$, for $i = 1, 2$, are propositions of the form

$a_{i,o}$: "An object o is in relation to a linguistic term a_i".

Consider the case where a is "a_1 *or* a_2". Then, the valuation function V for propositions a_o is defined as follows:

$$V((c_i^1, c_j^2), a_{1,o} \vee a_{2,o}) = \begin{cases} 1 & \text{if } \mathrm{at}_1(o) \in a_1(c_i^1) \vee \mathrm{at}_2(o) \in a_2(c_j^2) \\ 0 & \text{otherwise} \end{cases}$$

With this notation, we are now ready to define the compatible degree of any object $o \in O$ to the composed linguistic term "a_1 *or* a_2" in the model M by

$$\mu_A(o) = \mu_{A_1 \cup A_2}(o) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \omega_{ij} V((c_i^1, c_j^2), a_{1,o} \vee a_{2,o}) \tag{8}$$

where A_1, A_2 denote fuzzy sets which represent component linguistic terms a_1, a_2, respectively.

Similar for the case where a is "a_1 and a_2". The valuation function V for propositions a_o is then defined as follows:

$$V((c_i^1, c_j^2), a_{1,o} \wedge a_{2,o}) = \begin{cases} 1 & \text{if } \mathrm{at}_1(o) \in a_1(c_i^1) \wedge \mathrm{at}_2(o) \in a_2(c_j^2) \\ 0 & \text{otherwise} \end{cases}$$

and the compatible degree of any object $o \in O$ to the composed linguistic term "a_1 and a_2" in the model M is defined by

$$\mu_A(o) = \mu_{A_1 \cap A_2}(o) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \omega_{ij} V((c_i^1, c_j^2), a_{1,o} \wedge a_{2,o}) \tag{9}$$

Notice that in the case without the weighting function Ω in the model M, the membership expressions of composed fuzzy sets defined in (8) and (9) are comparable with those given in [35].

Now we examine the behaviours of operators $\cup, \cap$ in this formulation. Let us denote by

$$C'_1 = \{c_i^1 \in C_1 \mid \mathrm{at}_1(o) \in a_1(c_i^1)\}$$
$$C'_2 = \{c_j^2 \in C_2 \mid \mathrm{at}_2(o) \in a_2(c_j^2)\}$$

It is easy to see that

$$V((c_i^1, c_j^2), (a_{1,o} \vee a_{2,o})) = \begin{cases} 1 & \text{if } (c_i^1, c_j^2) \in (C'_1 \times C_2 \cup C_1 \times C'_2) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

and

$$V((c_i^1, c_j^2), (a_{1,o} \wedge a_{2,o})) = \begin{cases} 1 & \text{if } (c_i^1, c_j^2) \in (C_1' \times C_2') \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Furthermore, we have the following representation:

$$(C_1' \times C_2 \cup C_1 \times C_2') = (C_1' \times C_2 \uplus C_1 \times C_2') \setminus (C_1' \times C_2') \tag{12}$$

where $\uplus$ denotes an joint union which permits an iterative appearance of elements.

With these notation, we have the following.

Proposition 3. *For any $o \in O$, we have*

$$\mu_{A_1 \cap A_2}(o) = \mu_{A_1}(o)\mu_{A_2}(o) \tag{13}$$

$$\mu_{A_1 \cup A_2}(o) = \mu_{A_1}(o) + \mu_{A_2}(o) - \mu_{A_1}(o)\mu_{A_2}(o) \tag{14}$$

Proof. By the definition of the valuation function V and Propositions 1 and 2, it immediately implies (14) from (8) and (12). Similarly, (13) directly follows from (9). □

Expressions (13) and (14) show that fuzzy intersection and fuzzy union operators by this model are truth-functional, and, moreover, they are a well-known dual pair of *product t-norm* and *probabilistic sum t-conorm* [20]. This justifies for the situation when linguistic terms belong to different universes of discourse, for example *tall* and *high income*, there is no constraint of semantic consistency between them, and reflecting such independence, the product–sum rule is appropriate in applications. We should also note that for the purpose of finding new operators for using in the fuzzy expert system shell FLOPS, the authors in [4] have used elementary statistical calculations on binary data for the truth of two fuzzy propositions to present new *t*-norm and *t*-conorm for computing the truth of AND, and OR propositions. Interestingly, their *t*-norm and *t*-conorm are also reduced to product *t*-norm and probabilistic sum *t*-conorm in the case where the sample correlation coefficient equals to 0.

4. Fuzzy sets by context model and mass assignments

In this section we establish a mass assignment interpretation of fuzzy concepts within the context model. The mass assignment for a fuzzy concept was firstly introduced by Baldwin et al. [1,2] and can be interpreted as a probability distribution over possible definitions of the concept. These varying definitions may be provided by a population of voters where each is asked to give a crisp definition of the concept.

Let F be a fuzzy subset of a finite universe U such that the range of the membership function μ_F is $\{y_1, \ldots, y_n\}$, where $y_i > y_{i+1} > 0$, for $i = 1, \ldots, n-1$. Then the mass assignment of F, denoted by m_F, is a probability distribution on 2^U satisfying $m_F(\emptyset) = 1 - y_1, m_F(F_i) = y_i - y_{i+1}$, for $i = 1, \ldots, n-1$, and $m_F(F_n) = y_n$, where $F_i = \{u \in U | \mu_F(u) \geqslant y_i\}$, for $i = 1, \ldots, n$. $\{F_i\}_{i=1}^n$ are referred to as the focal elements of m_F. The mass assignment of a fuzzy concept is then considered as providing a probabilistic based semantics for membership function of the fuzzy concept. Furthermore, mass assignment of fuzzy sets have been applied in some fields such as induction of decision trees [3], computing with words [27,28], and fuzzy logic [29].

Given a context model $\mathscr{C} = \langle D, C, A_C(D), \Omega \rangle$. Assume $a \in A_C(D)$ and μ_A denotes the fuzzy set induced from a as defined by (1) in the preceding section. The weighting function Ω can be extended to 2^C as a probability measure by

$$\Omega(X) = \sum_{c \in X} \Omega(c), \quad \text{for any } X \in 2^C$$

Denote $\{\omega_1, \ldots, \omega_k\}$ the range of Ω defined on 2^C such that $\omega_i > \omega_{i+1} > 0$, for $i = 1, \ldots, k-1$. Clearly, $\omega_1 = 1$.

Set $C_i = \{X \in 2^C | \Omega(X) = \omega_i\}$, for $i = 1, \ldots, k$. We now define $\{A_i\}_{i=1}^k$ inductively as follows:

$$A_1 = \bigcap_{c \in C} a(c)$$

$$A_i = A_{i-1} \cup \bigcup_{X \in C_i} \bigcap_{c \in X} a(c) \quad \text{for } i > 1$$

Let s be the least number such that $A_s \neq \emptyset$.

Obviously, $A_s \subset A_{s+1} \subset \cdots \subset A_k$. If a is consistent then we have $s = 1$. In this case let us define $m : 2^D \to [0, 1]$ by

$$m(E) = \begin{cases} \omega_i - \omega_{i+1} & \text{if } E = A_i \\ 0 & \text{otherwise} \end{cases}$$

where, by convention, $\omega_{k+1} = 0$.

In the case where $s > 1$, i.e. that a is not consistent, we define $m : 2^D \to [0, 1]$ by

$$m(E) = \begin{cases} 1 - \omega_s & \text{if } E = \emptyset \\ \omega_i - \omega_{i+1} & \text{if } E = A_i \text{ and } i > s \\ 0 & \text{otherwise} \end{cases}$$

Clearly, in both cases m is a probability distribution over 2^D with $\{A_i\}_{i=s}^k$ is a nested family of focal elements of m. Consequently, the following holds.

Proposition 4. *We have $m = m_{\mu_A}$, where m_{μ_A} denotes the mass assignment of the fuzzy set μ_A in the sense of Baldwin as defined above.*

Proof. Assume $a \in A_C(D)$ and μ_A denotes the fuzzy set induced from a as defined by (1). Equivalently, for any $x \in D$, we have

$$\mu_A(x) = \sum_{c \in C} \Omega(c)\mu_{a(c)}(x)$$

where $\mu_{a(c)}$ is the characteristic function of $a(c)$. As such the range of the membership function μ_A is a subset of $\{\omega_1, \ldots, \omega_k\}$, the range of Ω defined on 2^C with $\omega_i > \omega_{i+1} > 0$, for $i = 1, \ldots, k-1$. More particularly, the range of μ_A is $\{\omega_s, \ldots, \omega_k\}$ with s being defined as above.

Now it is sufficient to prove that A_i is exactly ω_i-cut of the fuzzy set μ_A, for $i = s, \ldots, k$. Indeed, as denoted previously, for each $i = s, \ldots, k$, we have $C_i = \{X \in 2^C | \Omega(X) = \omega_i\}$. This follows by the definition of ω_i's that for each $X \in C_i$ and $x \in \cap_{c \in X} a(c)$, we obtain

$$\mu_A(x) = \sum_{c \in C} \Omega(c)\mu_{a(c)}(x) = \sum_{c \in X} \Omega(c)\mu_{a(c)}(x) = \omega_i$$

Thus we have $\mu_A(x) = \omega_i$ for only x's those belong to the set $\cup_{X \in C_i} \cap_{c \in X} a(c)$. Consequently, by the definition of A_i's as above, it follows that A_i is exactly ω_i-cut of the fuzzy set μ_A. This concludes the proof. □

On the other hand, for $a \in A_C(D)$, it naturally generates a mass distribution m_a over 2^D defined as follows:

$$m_a(E) = \Omega(\{c \in C | a(c) = E\}), \quad \text{for any } E \in 2^D$$

In this case, if the mass assignment for a fuzzy concept could be interpreted as a probability distribution over possible definitions of the concept, it would seem desirable that the natural mass distribution m_a coincides with the mass assignment of the fuzzy set μ_A induced by a up to a permutation of C. However, this is not generally the case. Actually, due to the additive property imposed on Ω, we have the following.

Proposition 5. *Given a context model $\mathscr{C} = \langle D, C, A_C(D), \Omega \rangle$, and $a \in A_C(D)$. Assume that μ_A is the fuzzy set induced from a as defined by* (1). *Then $m_a = m_{\mu_A}$ if and only if the family $\{a(c) | c \in C\}$ forms a nested family of subsets in D.*

Proof

(⇒): This part follows directly from the definitions of m_a and m_{μ_A}.

(⇐): Assume that $\{a(c) | c \in C\}$ is a nested family of subsets in D. Let

$$\{a(c) | c \in C\} = \{A_i\}_{i=1}^n$$

with $A_1 \subset A_2 \subset \cdots \subset A_n$. We now show that $m_a(A_i) = m_{\mu_A}(A_i)$, for any $i = 1, \ldots, n$. Clearly, for each $i = 1, \ldots, n$, A_i is the α_i-cut of the fuzzy set μ_A with α_i being defined as follows:

$$\alpha_i = \sum_{j=i}^{n} m_{\mu_A}(A_j) \tag{15}$$

On the other hand, we have

$$\mu_A(x) = \sum_{c \in C} \Omega(c)\mu_{a(c)}(x)$$

for any $x \in D$. Thus, it follows from the assumption $\{a(c)|c \in C\} = \{A_i\}_{i=1}^{n}$ that, for any $x \in A_i \setminus A_{i-1}$,

$$\alpha_i = \mu_A(x) = \sum_{j=i}^{n} \sum_{c \in C:a(c)=A_j} \Omega(c) = \sum_{j=i}^{n} m_a(A_j) \tag{16}$$

From (15) and (16) it easily implies that $m_a(A_i) = m_{\mu_A}(A_i)$, for any i. This completes the proof. □

5. Conclusions

Interesting relationships between context model, modal logic and fuzzy concept analysis have been explored in this paper. As is well-known, the two following important problems should be taken into account in most fuzzy set-based applications. The first problem is how to construct efficiently membership functions of fuzzy sets in a given particular application. This one has been studied by many distinguished fuzzy scholars including Turksen [36], Pedrycz [30], Klir et al. [25] among others. The second problem is how to use suitably connectives in the fuzzy setting. As observed from practical applications of fuzzy sets, if fuzzy sets of interest model linguistic terms with positive semantic consistency, for example *tall* and *very tall*, the min–max rule is more correct. In the other hand, when linguistic terms belong to different universes of discourse, for example *tall* and *high income*, there is no constraint of semantic consistency between them, and reflecting such independence, the product–sum rule is appropriate in applications. As such, if context model provides a semantic interpretation of forming fuzzy concepts, it gives a theoretical justification for appropriate use of t-norm based connectives such as min–max and product–sum rules in practical applications, as well. Furthermore, this paper also established an interpretation of mass assignments of fuzzy concepts within the context model.

Acknowledgement

The authors would like to thank the referees for their valuable comments and suggestions.

References

[1] J.F. Baldwin, The management of fuzzy and probabilistic uncertainties for knowledge based systems, in: S.A. Shapiro (Ed.), The Encyclopaedia of AI, Wiley, New York, 1992, pp. 528–537.

[2] J.F. Baldwin, J. Lawry, T.P. Martin, A mass assignment theory of the probability of fuzzy events, Fuzzy Sets and Systems 83 (1996) 353–367.

[3] J.F. Baldwin, J. Lawry, T.P. Martin, Mass assignment based induction of decision trees on words, in: Proceedings of IPMU'98, 1996, pp. 524–531.

[4] J.J. Buckley, W. Siler, A new *t*-norm, Fuzzy Sets and Systems 100 (1998) 283–290.

[5] B.F. Chellas, Modal Logic: An Introduction, Cambridge University Press, 1980.

[6] D. Dubois, H. Prade, Possibility Theory—An Approach to Computerized Processing of Uncertainty, Plenum Press, New York, 1987.

[7] D. Dubois, H. Prade, The three semantics of fuzzy sets, Fuzzy Sets and Systems 90 (1997) 141–150.

[8] B. Ganter, R. Wille, Formal Concept Analysis: Mathematical Foundations, Springer-Verlag, Berlin, Heidelberg, 1999.

[9] J. Gebhardt, R. Kruse, The context model: an integrating view of vagueness and uncertainty, International Journal of Approximate Reasoning 9 (1993) 283–314.

[10] J. Gebhardt, R. Kruse, Parallel combination of information sources, in: D.M. Gabbay, P. Smets (Eds.), Handbook of Defeasible Reasoning and Uncertainty Management Systems, vol. 3, Kluwer, Dordrecht, The Netherlands, 1998, pp. 393–439.

[11] J. Gebhardt, Learning from data—possibilistic graphical models, in: D.M. Gabbay, P. Smets (Eds.), Handbook of Defeasible Reasoning and Uncertainty Management Systems, vol. 4, Kluwer, Dordrecht, The Netherlands, 2000, pp. 314–389.

[12] I.R. Goodman, Fuzzy sets as equivalence classes of random sets, in: R. Yager (Ed.), Fuzzy Set and Possibility Theory, Pergamon Press, Oxford, 1982, pp. 327–342.

[13] I.R. Goodman, H.T. Nguyen, Uncertainty Models for Knowledge Based Systems, North Holland, 1985.

[14] D. Harmanec, R. Klir, G. Resconi, On modal logic interpretation of Demspter–Shafer theory of evidence, International Journal of Intelligent Systems 9 (1994) 941–951.

[15] D. Harmanec, G. Resconi, R. Klir, Y. Pan, On the computation of the uncertainty measure for the Demspter–Shafer theory, International Journal of General Systems 25 (1996) 153–163.

[16] D. Harmanec, R. Klir, Z. Wang, Modal logic interpretation of Demspter–Shafer theory: an infinite case, International Journal of Approximate Reasoning 14 (1996) 81–93.

[17] V.N. Huynh, G. Resconi, A modal logic interpretation of rough set theory, Advances in Natural Sciences 1 (2000) 85–93.

[18] V.N. Huynh, Y. Nakamori, Fuzzy concept formation based on context model, in: N. Baba et al. (Eds.), Knowledge-Based Intelligent Information Engineering Systems & Allied Technologies, IOS Press, Amsterdam, 2001, pp. 687–691.

[19] V.N. Huynh, Y. Nakamori, T.B. Ho, G. Resconi, A context model for constructing membership functions of fuzzy concepts based on modal logic, in: T. Eiter, K.-D. Schewe (Eds.), Foundations of Information and Knowledge Systems, LNCS 2284, Springer-Verlag, Berlin, Heidelberg, 2002, pp. 93–104.

[20] E.P. Klement, Some mathematical aspects of fuzzy sets: triangular norms fuzzy logics, and generalized measures, Fuzzy Sets and Systems 90 (1997) 133–140.

[21] R. Klir, Multi-valued logic versus modal logic: alternate framework for uncertainty modelling, in: P.P. Wang (Ed.), Advances in Fuzzy Theory and Technology, vol. II, Duke University Press, Durham, NC, 1994, pp. 3–47.

[22] R. Klir, T. Folger, Fuzzy Sets, Uncertainty and Information, Prentice-Hall, New Jersey, 1988.

[23] R. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice-Hall, PTR, Upper Saddle River, NJ, 1995.

[24] R. Klir, D. Harmanec, On modal logic interpretation of possibility theory, International Journal of Uncertainty Fuzziness and Knowledge-Based Systems 2 (1994) 237–245.

[25] R. Klir, Z. Wang, D. Harmanec, Constructing fuzzy measures in expert systems, Fuzzy Sets and Systems 92 (1997) 251–264.

[26] R. Kruse, J. Gebhardt, F. Klawonn, Numerical and logical approaches to fuzzy set theory by the context model, in: R. Lowen, M. Roubens (Eds.), Fuzzy Logic: State of the Art, Kluwer Academic Publishers, Dordrecht, 1993, pp. 365–376.

[27] J. Lawry, A methodology for computing with words, International Journal of Approximate Reasoning 28 (2001) 51–89.

[28] J. Lawry, Label semantics: a formal framework for modelling with words, in: S. Benferhat, P. Besnard (Eds.), LNAI 2143, Springer-Verlag, Berlin, Heidelberg, 2001, pp. 374–384.

[29] J. Lawry, A voting mechanism for fuzzy logic, International Journal of Approximate Reasoning 19 (1998) 315–333.

[30] W. Pedrycz, Fuzzy equalization in the construction of fuzzy sets, Fuzzy Sets and Systems 119 (2001) 329–335.

[31] H. Rasiowa, C.H. Nguyen, LT-fuzzy sets, Fuzzy Sets and Systems 47 (1992) 323–339.

[32] G. Resconi, G.J. Klir, U.St. Clair, Hierarchically uncertainty metatheory based upon modal logic, International Journal of General Systems 21 (1992) 23–50.

[33] G. Resconi, G.J. Klir, U.St. Clair, D. Harmanec, On the integration of uncertainty theories, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 1 (1) (1993) 1–18.

[34] G. Resconi, G.J. Klir, D. Harmanec, U.St. Clair, Interpretations of various uncertainty theories using models of modal logic: a summary, Fuzzy Sets and Systems 80 (1996) 7–14.

[35] G. Resconi, I.B. Turksen, Canonical forms of fuzzy truthoods by meta-theory based upon modal logic, Information Sciences 131 (2001) 157–194.

[36] I.B. Turksen, Measurement of membership functions and their acquisition, Fuzzy Sets and Systems 40 (1991) 5–38.

[37] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

[38] L.A. Zadeh, The concept of linguistic variable and its application to approximate reasoning I, Information Sciences I 8 (1975) 199–249, II: 8 (1975) 310–357, III: 9 (1975) 43–80.

[39] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1 (1978) 3–38.