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Decision making under uncertainty with fuzzy targets

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Abstract This paper discusses the issue of how to use fuzzy targets in the target-based model for decision making under uncertainty. After introducing a target-based interpretation of the expected value on which it is shown that this model implicitly assumes a neutral behavior on attitude about the target, we examine the issue of using fuzzy targets considering different attitudes about the target selection of the decision maker. We also discuss the problem for situations on which the decision maker's attitude about target may change according to different states of nature. Especially, it is shown that the target-based approach can provide an unified way for solving the problem of fuzzy decision making with uncertainty about the state of nature and imprecision about payoffs. Several numerical examples are given for illustration of the discussed issues.

Keywords Decision making · Uncertainty · Fuzzy target · Expected utility · Risk attitude

1 Introduction

Traditionally, when modelling a decision maker's rational choice between acts with uncertainty, it is assumed that the uncertainty is described by a probability distribution

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on the space of states, and the ranking of acts is based on the expected utilities of the consequences of these acts. This utility maximization principle was justified axiomatically in Savage (1954) and Von Neumann and Morgenstern (1944). As Simon argued in (Simon 1955), the traditional utility theory presumes that a rational decision maker was assumed to have "a well-organized and stable system of preferences, and a skill in computation" that was unrealistic in many decision contexts (Bordley 2003). At the same time, Simon proposed a behavioral model for rational choice, by enunciating the so-called theory of bounded rationality, that implied that, due to the cost or the practical impossibility of searching among all possible acts for the optimal, the decision maker simply looked for the first 'satisfactory' act that met some predefined target. It was also concluded that human behavior should be modelled as satisficing instead of optimizing. Intuitively, the satisficing approach has some appealing features because thinking of targets is quite natural in many situations.

Particularly, in an uncertain environment, each act a may lead to different outcomes usually resulting in a random consequence X_a. Then, given a target t, the agent can only assess the probability $P(X_a \succeq t)$ of the act a's consequence meeting the target. In this case, according to the optimizing principle, the agent should choose an act a that maximizes the probability $v(a) = P(X_a \succeq t)$ (Manski 1998). Although simple and appealing from this target-based point of view, its resulted model is still not complete because there may be uncertainty about the target itself. Therefore, Castagnoli and LiCalzi (1996) and Bordley and LiCalzi (2000) have relaxed the assumption of a known target by considering a random consequence T instead. Then the target-based decision model prescribes that the agent should choose an act a that maximizes the probability $v(a) = P(X_a \succeq T)$ of meeting an uncertain target T, provided that the target T is stochastically independent of the random consequences to be evaluated. Interestingly, despite the differences in approach and interpretation, both target-based decision procedure and utility-based decision procedure essentially lead to only one basic model for decision making. In particular, Castagnoli and LiCalzi (1996) provided a formal equivalence of von Neumann and Morgenstern's expected utility model and the target-based model with reference to preferences over lotteries and laterly, Bordley and LiCalzi (2000) showed a similar result for Savage's expected utility model with reference to preferences over acts. More details on target-based decision models as well as their potential applications and advantages could be referred to Abbas and Matheson (2005, 2004), Bordley (2002), Bordley and Kirkwood (2004), Castagnoli and LiCalzi (2006) and LiCalzi (1999).

In this paper,[1] we consider the problem of decision making in the face of uncertainty that can be most effectively described using the decision matrix shown in Table 1; see, e.g., Brachinger and Monney (2002), Chankong and Haimes (1983), Yager (1999, 2000, 2002b). In this matrix, $A_i (i = 1, \dots, n)$ represent the alternatives (or acts) available to a decision maker (shortly, DM), one of which must be selected. The elements $S_j (j = 1, \dots, m)$ correspond to the possible values/states associated with the so-called state of nature S. Each element c_{ij} of the matrix is the payoff the DM receives if alternative A_i is selected and state S_j occurs. The uncertainty associated with this

[1] This paper is a substantially expanded and revised version of the paper (Huynh et al. 2006) presented at FUZZ–IEEE 2006.

problem is generally a result of the fact that the value of S is unknown before DM must choose an alternative A_i.

Generally, as indicated in the literature, the procedure used to select the optimal alternative should depend upon the type of uncertainty assumed over the domain $\mathcal{S} = \{S_1, \ldots, S_m\}$ of variable S. Most often, it is assumed that there exists a probability distribution $P_{\mathcal{S}}$ over $\mathcal{S}$ such that $p_j = P_{\mathcal{S}}(S = S_j)$ and $\sum_{j=1}^{m} p_j = 1$. In this case we call the problem decision making under risk. The most classical method for decision making under risk is to use the expected value:

- For each alternative A_i, calculate its expected payoff as $v(A_i) \triangleq EV_i = \sum_{j=1}^{m} p_j c_{ij}$.
- Select as the best alternative the one which maximizes the expected value, i.e. that

$$A_{best} = \arg\max_i \{v(A_i)\}$$

In the case if probability information is not available, the problem is called decision making under ignorance, and various decision strategies as *maximin*, *maximax*, *average* and *Hurwicz rules* are often used depending on different attitudes of the decision maker.

Recently in Yager (1999), by arguing that the use of the expected value as our decision function may not be appropriate in many circumstance, Yager has focused on the construction of decision functions which allows for the inclusion of information about decision attitude and probabilistic information about the uncertainty. This approach has been further discussed in Liu (2004), Yager (2000, 2002b, 2004), with the help of OWA operators (Yager 1988) and/or fuzzy systems modelling (Yager and Filev 1994). Basically, the main point in these work is to define a valuation function for alternatives taking decision attitude and probabilistic information in uncertainty into account without using the notion of utility. In other words, this valuation-based approach does not consider the risk attitude factor in terms of utility functions as in the traditional utility-based paradigm, but focusing on a mechanism for combining probabilistic information about state of nature with information about DM's attitude in the formulation of a valuation function.

The main focus of this paper is put on a fuzzy target-based approach to the issue of decision making under uncertainty. Essentially, instead of trying to get the payoff-based valuation for alternatives, it tries to calculate the (expected) probability of meeting some predesigned fuzzy target for each alternative, then select the alternative which maximizes this probability according to the optimizing principle. From this target-based point of view, the DM may also have his *attitude about the target selection*, we then discuss the problem of formulating targets which simultaneously considers the DM's attitude about target selection. An interesting link between the DM's different attitudes about target and different risk attitudes in terms of utility functions is also established. More interestingly, this target-based approach allows the DM to assess his target changeable according to the state of nature, which makes it can be classified as *context dependent*. It should be worth noting that different targets for different states can be naturally understood and easily formulated. Furthermore, we also discuss the issue of how this target-based approach could be applied for the problem of fuzzy decision making with uncertainty.

The organization of this paper is as follows. In Sect. 2, a target-based interpretation of the expected value is presented. Then Sect. 3 discusses the issue of decision making under risk using fuzzy targets considering different attitudes about the target selection. In Sect. 4, we introduce context-dependent fuzzy targets and provide a practical way to implement such a context-dependent target from the decision matrix. Section 5 suggests a general target-based procedure for the problems of decision making under uncertainty where the payoff matrix may be inhomogeneous. Finally, some concluding remarks are presented in Sect. 6.

2 Target-based model of the expected value

Let us consider the decision problem as described in Table 1 with assuming a probability distribution $P_{\mathcal{S}}$ over $\mathcal{S}$. Here, we restrict ourselves to a bounded domain of the payoff variable that $D = [c_{min}, c_{max}]$, i.e. $c_{min} \leq c_{ij} \leq c_{max}$.

As mentioned above, the most commonly used method for valuating alternatives A_i is to use the expected payoff value:

$$v(A_i) \triangleq EV_i = \sum_{j=1}^{m} p_j c_{ij} \tag{1}$$

On the other hand, each alternative A_i can be formally considered as a random payoff having the probability distribution P_i defined, with an abuse of notation, as follows:

$$P_i(A_i = c) = P_{\mathcal{S}}(\{S_j : c_{ij} = c\}) \tag{2}$$

Then, similar to Bordley and LiCalzi's (2000) result, we now define a random target T which has a uniform distribution on D with the probability density function P_T defined by

$$P_T(c) = \begin{cases} \frac{1}{c_{max} - c_{min}}, & c_{min} \leq c \leq c_{max} \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Table 1 The general decision matrix

Alternatives	State of nature			
	S_1	S_2	$\dots$	S_m
A_1	c_{11}	c_{12}	$\dots$	c_{1m}
A_2	c_{21}	c_{22}	$\dots$	c_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_n	c_{n1}	c_{n2}	$\dots$	c_{nm}

Under the assumption that the random target T is stochastically independent of any random payoffs A_i (Bordley and LiCalzi 2000), we have

$$\begin{aligned} v^{\dagger}(A_i) &\triangleq P(A_i \succeq T) \\ &= \sum_c P(c \succeq T) P_i(A_i = c) \\ &= \sum_c \left[\int_{-\infty}^{c} P_T(t)\mathrm{d}t \right] P_i(A_i = c) \end{aligned} \tag{4}$$

where

$$P(c \succeq T) = \int_{-\infty}^{c} P_T(t)dt$$

is the cumulative distribution function of the target T. It should be also mentioned here that, in a different but similar context, a similar idea has been used in Huynh and Nakamori (2005) to develop the so-called satisfactory-oriented decision model for multi-expert decision making with linguistic assessments.

Due to (2)–(3) and the additive property of the probability measure, from (4) we easily obtain

$$\begin{aligned} v^{\dagger}(A_i) &= \sum_{j=1}^{m} \left[\int_{-\infty}^{c_{ij}} P_T(t)\mathrm{d}t \right] P_{\mathcal{S}}(S = S_j) \\ &= \sum_{j=1}^{m} \frac{c_{ij} - c_{min}}{c_{max} - c_{min}} p_j \end{aligned} \tag{5}$$

From the Eqs. (1) and (5), we easily see that there is no way to tell if the DM selects an alternative by maximizing the expected value or by maximizing the probability of meeting the uncertain target T. In other words, the target-based decision model with decision function $v^{\dagger}(A_i)$ above is equivalent to the expected value model.

In this target-based language, if in a situation where there is no uncertainty about targets and assuming that the decision maker could assess a certain target t_0, then he would naturally select the alternative A_i which maximizes the probability

$$v^{\dagger}(A_i) = P_{\mathcal{S}}(\{S_j : c_{ij} \geq t_0\})$$

This clearly is not a good model and inapplicable as the target does not reflect the DM's inherent objective that is to select alternatives towards maximizing profit associated with payoff value.

Interestingly, in the target-based model of the expected value above, we can think of T as an interval target represented as a membership function $T(c) = 1$ for $c_{min} \leq c \leq c_{max}$, and $T(c) = 0$ otherwise. Then the uniform distribution P_T considered as a probabilistic interpretation of this interval target can be obtained via a simple normalization proposed in (Yager et al. 2001, Yager 2002a). This observation suggests an interesting extension of target-based decision models using fuzzy targets as in following. It would be worth noting that defining fuzzy targets is much easier and intuitively natural than directly defining random targets.

3 Decision making with targets: expected probability of meeting the target

Before discussing about the problem of decision making using fuzzy targets, it is necessary to briefly recall that when expressing the value of a variable as a fuzzy set (Zadeh 1965), we are inducing a possibility distribution (Zadeh 1978) over the domain of the variable. Formally, the soft constraint imposed on a variable V in the statement "V is F", where F is a fuzzy set, can be considered as inducing a possibility distribution Π on the domain of V such that $F(x) = \Pi(x)$, for each x. From now on, we can use "membership function" and "possibility distribution" interchangeably.

Since the introduction of the idea of possibility, the relationship between possibility and probability has been received much attention from the research community. Particularly, the issue of associating probability distributions with possibility distributions has been also discussed extensively. Recently, Yager (2002a) has proposed a procedure for instantiating a possibility variable over a discrete domain by converting its possibility distribution into a probability distribution, via a simple normalization as follows:

$$P_F(x) = \frac{F(x)}{\sum_x F(x)}$$

This conversion has been extended into a continuous domain and applied to ranking fuzzy numbers (Yager et al. 2001) and to defining an "important weight" for OWA aggregation over a continuous interval argument (Yager 2004) as follows

$$P_F(x) = \frac{F(x)}{\int_x F(x)dx} \tag{6}$$

In the following we will use this conversion of a possibility distribution into a probability distribution for decision making with fuzzy targets.

3.1 State-independent targets

Let us turn back to the problem of decision making in the face of uncertainty shown in Table 1. We now discuss this problem using fuzzy targets. By a fuzzy target we mean a possibility variable T over the payoff domain D represented, with an abuse of notation, by a possibility distribution $T\colon D \rightarrow [0, 1]$. We also assume further that T is a piecewise continuous function having a bounded support and $\int_D T(x)\mathrm{d}x > 0$. In the following of this section, we suppose that $\mathrm{Supp}(T) = [c_{min}, c_{max}]$, where $\mathrm{Supp}(T)$ is the support of T.

Given a fuzzy target T, let P_T be its associated probability distribution and, without any danger of confusion, we use the same notation T for the random variable having the probability distribution P_T. Then the target-based decision model suggests the ranking of alternatives be obtained by using the value function defined by

$$v^{\dagger}(A_i) \triangleq P(A_i \succeq T) \tag{7}$$

According to (2) and the additive property of the probability measure, we have

$$\begin{aligned} v^{\dagger}(A_i) &= \sum_{j=1}^{m} P(c_{ij} \succeq T) P_{\mathcal{S}}(S = S_j) \\ &= \frac{1}{K(T)} \sum_{j=1}^{m} \left[\int_{c_{min}}^{c_{ij}} T(x)\mathrm{d}x \right] p_j \end{aligned} \tag{8}$$

where $K(T)$ is the area under the membership function T, i.e.,

$$K(T) = \int_{c_{min}}^{c_{max}} T(x)\mathrm{d}x$$

Probabilistically, we may think of the value function (8) as the expected probability of an alternative meeting a pre-defined uncertain target T.

As we have discussed in the preceding section, the uncertain target in the target-based model of the expected value can be seen as an interval target represented by the possibility distribution $T_{neutral}(c) = 1$ for $c_{min} \leq c \leq c_{max}$, and $T_{neutral}(c) = 0$ otherwise. Intuitively, this particular target clearly implies a *neutral* behavior on attitude about the target selection of the DM. This prototypical decision attitude has been used very often in the literature. In this case, as shown in preceding section, we have

$$v^{\dagger}(A_i) = \sum_{j=1}^{m} \frac{c_{ij} - c_{min}}{c_{max} - c_{min}} p_j \tag{9}$$

which is equivalent to the expected value model. Fig. 1 graphically depicts the membership function $T_{neutral}$, its associated probability distribution and the corresponding cumulative probability function.

In the following, we shall consider the cases for other two prototypical attitudes that are *pessimist* and *optimist*. Let us first consider the case of a decision maker who is optimistic in the target selection. In a target-based language, the optimistic attitude about the target selection represents that the DM assesses higher possibility about his target towards the maximal payoff, which corresponds to the attitude that he believes the best thing may happen. Formally, the optimistic fuzzy target, denoted by T_{opt}, can be defined as follows

$$T_{opt}(c) = \frac{c - c_{min}}{c_{max} - c_{min}} \tag{10}$$

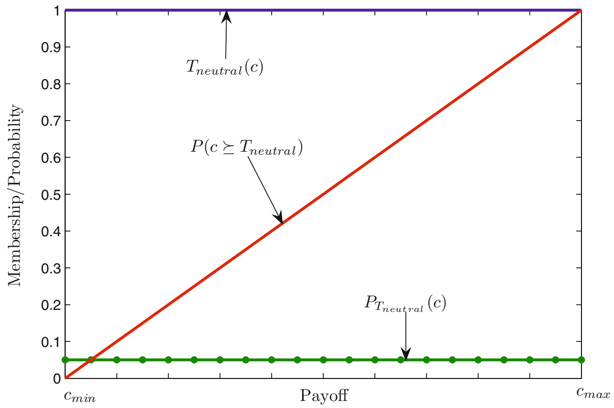

Fig. 1 Neutral attitude about target

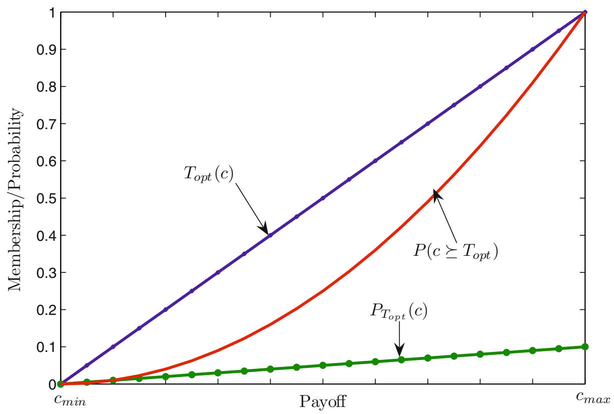

Fig. 2 Optimistic attitude about target

Then we easily get

$$P(c \succeq T_{opt}) = \frac{(c - c_{min})^2}{(c_{max} - c_{min})^2}$$

and therefore the value function in this case is

$$v^{\dagger}(A_i) = \sum_{j=1}^{m} \frac{(c_{ij} - c_{min})^2}{(c_{max} - c_{min})^2} p_j \tag{11}$$

This situation is graphically illustrated by Fig. 2.

Looking at Fig. 2 we see that the optimistic target T_{opt} leads to the convex c.p.f. $P(c \succeq T_{opt})$, which is equivalent to a convex utility function and therefore, exhibits a risk-seeking behavior. This is because of having an aspiration towards the maximal

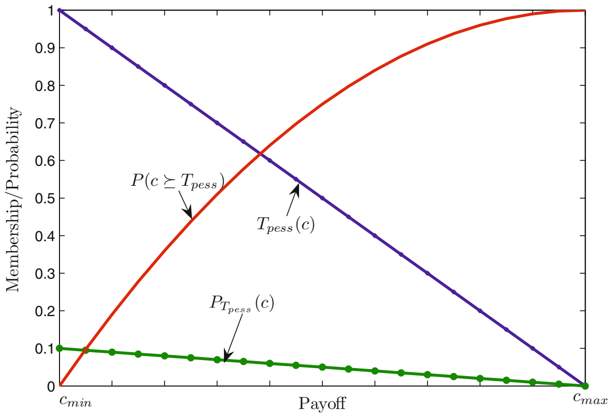

Fig. 3 Pessimistic attitude about target

payoff, the DM always feels loss over the whole domain except the maximum, which would produce a risk-seeking behavior globally.

Let us now consider the case of a decision maker who is pessimistic in the target selection. In this case, the DM assesses lower possibility about his target towards the maximal payoff, which corresponds to that he believes the worst thing may happen. Formally, the pessimistic fuzzy target, denoted by T_{pess}, is defined as follows

$$T_{pess}(c) = \frac{c_{max} - c}{c_{max} - c_{min}} \tag{12}$$

The behavior of the target in this case is graphically illustrated in Fig. 3. Then we have as the value function:

$$v^{\dagger}(A_i) = \sum_{j=1}^{m} \left[1 - \frac{(c_{max} - c_{ij})^2}{(c_{max} - c_{min})^2} \right] p_j \tag{13}$$

As we have seen, the pessimistic target T_{pess} induces a concave c.p.f. $P(c \succeq T_{pess})$, or equivalently a concave utility function, and then, associates with a risk-aversion behavior of the DM.

By observing the behavior of cumulative probability functions derived from these targets, interestingly we can see that the cumulative probability function of the neutral target has an uniform increasing behavior, and those of the other two have "opposite" behaviors. That is, while the cumulative probability function of the pessimistic target drastically increases at the beginning and becomes nearly stable at around the maximal payoff, the cumulative probability function of the optimistic target has a very little increasing at the beginning but becomes rapidly increasing at around the maximal payoff. To see how different attitudes about the target selection may lead to different results, let us consider the following example.

Assume the following decision structure over the domain $D = [-10, 10]$:

	S_1	S_2	S_3	S_4
A_1	-8	5	-6	9
A_2	-1	-4	2	2

where $p_1 = 0.3$, $p_2 = 0.1$, $p_3 = 0.2$, $p_4 = 0.4$. If the DM has a neutral behavior about the target and, as a consequent, assesses the interval $[-10, 10]$ as his target, we obtain

$$v^{\dagger}(A_1) = v^{\dagger}(A_2) = 0.525$$

which results in that the DM is indifferent between the two alternatives. Note that the same thing happen if we use the expected value as the ranking criterion, as we also have $EV_1 = EV_2 = 0.5$.

Let us consider now for the case of a pessimistic decision maker. In this case the DM assesses as the membership function for his corresponding target T_{pess}

$$T_{pess}(c) = 0.5 - \frac{1}{20}c$$

Then we obtain by (13) the value function of alternatives as follows

$$v^{\dagger}(A_1) = 0.622 \text{ and } v^{\dagger}(A_2) = 0.764$$

which implies that A_2 is the preferred choice.

Assuming now we have an optimistic decision maker. Then he assesses as the membership function for his corresponding target T_{opt}

$$T_{opt}(c) = \frac{1}{20}c + 0.5$$

Similarly we obtain by (11) the value function of alternatives as follows

$$v^{\dagger}(A_1) = 0.428 \text{ and } v^{\dagger}(A_2) = 0.286$$

which obviously implies that A_1 is the preferred choice. As we have seen in this example, while the optimistic DM pays much more attention on the alternative which dominates the other in the towards the maximal payoff, the pessimistic DM pays much more attention on the alternative which dominates the other in the backwards the minimal payoff.

We have just discussed three prototypical attitudes about the target selection, namely neutral, pessimistic and optimistic. Other assessments of the target would be also worth to be mentioned as following.

In practice, based on his feel/experience about the decision environment under consideration, the DM may also often assess his target via linguistic statements like

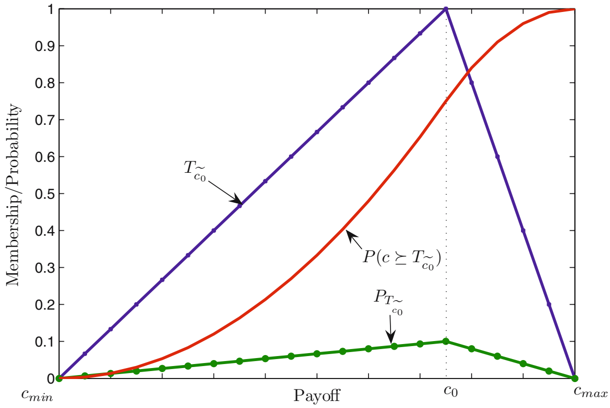

Fig. 4 The target "*About* c_0"

"*about* c_0", "*at most about* c_0" or "*at least about* c_0". For example, let us assume that the DM assesses as the membership function for his target of *about* c_0

$$T_{\widetilde{c_0}}(c) = \begin{cases} \dfrac{c - c_{min}}{c_0 - c_{min}}, & c_{min} \le c \le c_0 \\ \dfrac{c_{max} - c}{c_{max} - c_0}, & c_0 \le c \le c_{max} \end{cases}$$

Then we get (14) as the cumulative probability function induced from this target.

$$P(c \succeq T_{\widetilde{c_0}}) = \begin{cases} \dfrac{(c - c_{min})^2}{(c_{max} - c_{min})(c_0 - c_{min})}, & c_{min} \le c \le c_0 \\ \dfrac{(c_0 - c_{min})}{(c_{max} - c_{min})} + \dfrac{(c - c_0)(2c_{max} - (c + c_0))}{(c_{max} - c_0)(c_{max} - c_{min})}, & c_0 \le c \le c_{max} \end{cases} \tag{14}$$

This fuzzy target characterizes the situation at which the DM establishes a modal value c_0 as the most likely target and assesses the possibilistic uncertain target as distributed around it. We call this target the *unimodal*. Figure 4 graphically depicts the membership function of the unimodal target, its associated probability distribution $P_{T_{\widetilde{c_0}}}$ and the corresponding cumulative probability function $P(c \succeq T_{\widetilde{c_0}})$.

As illustrated, the unimodal target induces the S-shape c.p.f. $P(c \succeq T_{\widetilde{c_0}})$ that is equivalent to the S-shape utility function of Kahneman and Tversky's prospect theory (Kahneman and Tversky 1979), according to which people tend to be risk averse over gains and risk seeking over losses. In the fuzzy target-based language, as the DM assesses his uncertain target as distributed around the modal value, he feels loss (respectively, gain) over payoff values that are coded as negative (respectively, positive) changes with respect to the modal value. This would lead to the behavior consistent with that described in the prospect theory. A link of this behavior to unimodal probabilistic targets has been established by LiCalzi (1999). Further, it has been also suggested in the literature that this sort of target be the most natural one to occur.

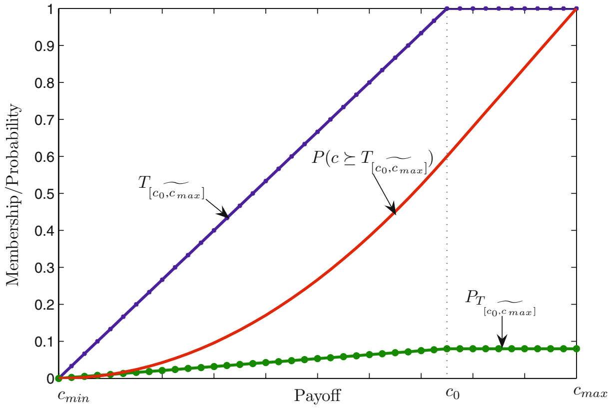

Fig. 5 The target *At least about* c_0

Similarly, we now assume that the DM assesses as the membership function for his target of *at least about* c_0 represented as the fuzzy interval $\widetilde{[c_0, c_{max}]}$

$$T_{\widetilde{[c_0,c_{max}]}}(c) = \begin{cases} \dfrac{c - c_{min}}{c_0 - c_{min}}, & c_{min} \leq c \leq c_0 \\ 1, & c_0 \leq c \leq c_{max} \end{cases}$$

Then we easily obtain the following cumulative probability function induced from this target as (15).

$$P(c \succeq T_{\widetilde{[c_0,c_{max}]}}) = \begin{cases} \dfrac{(c - c_{min})^2}{(2c_{max} - (c_0 + c_{min}))(c_0 - c_{min})}, & c_{min} \leq c \leq c_0 \\ \dfrac{(2c - (c_{min} + c_0))}{(2c_{max} - (c_0 + c_{min}))}, & c_0 \leq c \leq c_{max} \end{cases} \tag{15}$$

The membership function $T_{\widetilde{[c_0,c_{max}]}}$, its associated probability distribution $P_{T_{\widetilde{[c_0,c_{max}]}}}$ and the corresponding cumulative probability function $P(c \succeq T_{\widetilde{[c_0,c_{max}]}})$ are graphically depicted as in Fig. 5.

In the case if the DM assesses his target as *at most about* c_0 represented as the fuzzy interval $\widetilde{[c_{min}, c_0]}$ and we get as the membership function for this target

$$T_{\widetilde{[c_{min},c_0]}}(c) = \begin{cases} 1, & c_{min} \leq c \leq c_0 \\ \dfrac{c_{max} - c}{c_{max} - c_0}, & c_0 \leq c \leq c_{max} \end{cases}$$

Then we obtain (16) as the cumulative probability function induced from this target and, similarly, the related functions of this target are graphically illustrated as in Fig. 6.

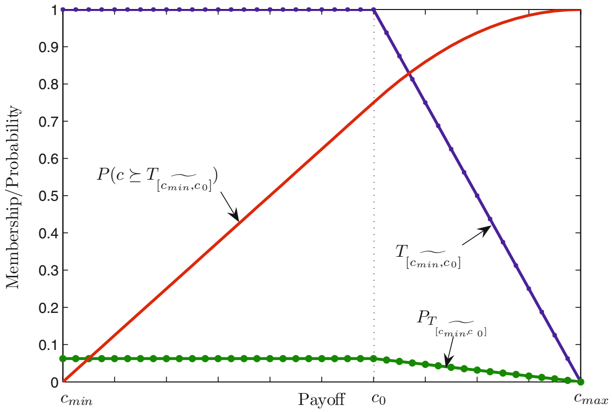

Fig. 6 The target *At most about* c_0

$$P(c \succeq T_{\widetilde{[c_{min},c_0]}}) = \begin{cases} \dfrac{2(c-c_{min})}{(c_{max}+c_0-2c_{min})}, & c_{min} \le c \le c_0 \\ \dfrac{-c^2+2c_{max}c-2c_{min}(c_{max}-c_0)-c_0^2}{(c_{max}+c_0-2c_{min})(c_{max}-c_0)}, & c_0 \le c \le c_{max} \end{cases} \tag{16}$$

3.2 A numerical example

Let us consider the following example from Samson (1988) to illustrate some points of the above discussion. Payoffs are shown in thousands of dollars for a problem with three acts and four states as described in Table 2. It is also assumed a proper prior over the four possible states of $p_1 = 0.2$, $p_2 = 0.4$, $p_3 = 0.3$, $p_4 = 0.1$.

Applying the above computational results of the cumulative probability function of different targets and (8), we obtain the value functions for acts shown in Table 3.

As we have seen in the table, while the act A_2 is the preferred choice according to a decision maker who has a neutral (equivalently, who abides by the expected value) or optimistic-oriented behavior about targets, a decision maker having pessimistic-oriented behavior about targets selects A_1 as his preferred choice. In addition, though the act A_3 is not selected in all cases, its value is much improved with respect to a pessimistic-oriented decision maker.

Table 2 The payoff matrix

Acts	States			
	1	2	3	4
A_1	400	320	540	600
A_2	250	350	700	550
A_3	600	280	150	400

Table 3 The target-based value matrix

Targets	The value function		
	$v^\dagger(A_1)$	$v^\dagger(A_2)$	$v^\dagger(A_3)$
Neutral	0.51	0.555	0.304
Optimist	0.3	0.41	0.18
Pessimist	0.72	0.7	0.43
$\widetilde{300}$	0.62	0.59	0.33
$[\widetilde{150, 300}]$	0.7	0.67	0.41
$[\widetilde{300, 700}]$	0.43	0.49	0.24

4 Context-dependent targets

In the above discussion, we have assumed that according to the decision attitude of the DM, he can assess a common target that does not depend on different states of nature for the purpose of making decisions. However, quite often in practice the DM's attitude about target may change according to different states of nature, i.e., that the DM may assess different targets for different states (Bordley and LiCalzi 2000). For example, assume that we are considering the decision problem of what stock we should buy from different stock portfolios and there are two states $S_1 = H$ and $S_2 = L$, respectively, representing high and low inflation. In such a situation we may assess a target T_H for high inflation and a target T_L (possibly different from T_H) for low inflation.

Once having determined different targets for different states of nature, by conditional on the states of nature we would then use the following as the value function

$$
\begin{aligned}
v^\dagger(A_i) &= P(A_i \succeq T) \\
&= \sum_{j=1}^{m} P(A_i \succeq T_j | S_j) P_S(S = S_j) \\
&= \sum_{j=1}^{m} P(c_{ij} \succeq T_j) P_S(S = S_j)
\end{aligned}
\tag{17}
$$

where $T = [T_1, \ldots, T_m]$ represents the state-dependent target with T_j being the target associated with the state S_j.

4.1 An illustrated example

Let us consider the following example for illustration. Assume that the management of a production company is faced with a decision problem for a new investment and there are four possible options (alternatives) for its selection. The profit/loss (in thousand of dollars) associated with each alternative depends on the future state of market which is currently not known. Further assuming that three states of nature have been considered as favorable market (S_1), normal market (S_2) and unfavorable market (S_3)

Table 4 Profit/loss matrix

Alternatives	States		
	S_1:0.25	S_2:0.5	S_3:0.25
A_1	250	10	−200
A_2	150	30	−120
A_3	100	10	−20
A_4 (do nothing)	0	0	0

with associated probabilities of 0.25, 0.5 and 0.25, respectively. The decision matrix is given in Table 4.

Applying the expected value model then yields the following ranking order among alternatives

$$A_3 \succ A_2 \succ A_1 \succ A_4$$

Therefore, the DM who obeys the expected value model will select alternative A_3 as the most preferred one. However, if the DM thought that the state S_2 would most likely occur, he may assess a state-dependent target as follows

$$T = [T_{neutral}, T_{opt}, T_{neutral}]$$

where the profit domain can be defined as $D = [-200, 250]$; i.e., the DM assesses an optimistic target for the state S_2 and the neutral target for the others. Then, we easily obtain the following ranking order among alternatives by using the value function (17):

$$A_2 \succ A_1 \succ A_3 \succ A_4$$

with which the alternative A_2 becomes the most preferred choice.

4.2 Target-based version of the expected opportunity loss

Let us now consider a special case of the context-dependent target where the DM has a neutral behavior about targets at every state of nature. In such a case, we can also define

$$T_j = [c_{min,j}, c_{max,j}]$$

where $c_{min,j} = \min_i \{c_{ij}\}$, $c_{max,j} = \max_i \{c_{ij}\}$. With this state-dependent target, we easily get the following value function

$$v^{\dagger}(A_i) = \sum_{j=1}^{m} \frac{(c_{ij} - c_{min,j})}{(c_{max,j} - c_{min,j})} p_j$$
$$= 1 - \sum_{j=1}^{m} \frac{(c_{max,j} - c_{ij})}{(c_{max,j} - c_{min,j})} p_j \tag{18}$$

Essentially, viewing the term

$$r_{ij} = \frac{(c_{max,j} - c_{ij})}{(c_{max,j} - c_{min,j})}$$

as the relative opportunity loss or regret of selecting A_i at state S_j, the maximization of the expected probability $v^{\dagger}(A_i)$ of meeting the target is equivalent to the minimization of the expected relative regret

$$ER(A_i) = \sum_{j=1}^{m} r_{ij} p_j \tag{19}$$

This can be seen as a target-based version of the expected opportunity loss or regret (Samson 1988).

Continuing with the example given in Sect. 3.2, we obtain the relative regret table derived from the payoff table as follows

With this derived table, according to (19) we select A_2 as the preferred choice as having the minimal expected relative regret of 0.225.

5 Target-based decision making with fuzzy payoffs

In the preceding section, we have discussed a target-based approach for probabilistic decision making, making use of Yager's method for converting a possibility distribution into a probability distribution and the procedure of comparing two random variables. Essentially we first begin with a random variable interpretation of alternatives and then proceed with defining the decision function as the probability of each alternative meeting a designed uncertain target. As we have seen, this target-based method of uncertain decision making is formally equivalent to a procedure which, once having designed the target, consists of the following two steps:

1. For each alternative A_i and state S_j, we define

   $$p_{ij} = P(c_{ij} \succeq T)$$

   where $T = T_j$ if the target is state-dependent, and then form a "probability of meeting the target" table described Table 6 from the payoff table (i.e., Table 1).

Table 5 The relative regret matrix

Alternatives	States			
	1	2	3	4
A_1	0.57	0.43	0.29	0
A_2	1	0	0	0.25
A_3	0	1	1	1

Table 6 The derived decision matrix

Alternatives	The state of nature S			
	S_1	S_2	$\cdots$	S_m
A_1	p_{11}	p_{12}	$\cdots$	p_{1m}
A_2	p_{21}	p_{22}	$\cdots$	p_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_n	p_{n1}	p_{n2}	$\cdots$	p_{nm}

2. Define the value function as the expected probability of meeting the target

$$v^{\dagger}(A_i) = \sum_{j=1}^{m} p_{ij} p_j \tag{20}$$

5.1 A target-based procedure for fuzzy decision making

We now consider the problem of decision making under uncertainty where payoffs may be given imprecisely. Let us turn back to the general decision matrix shown in Table 1, where c_{ij} can be a crisp number, an interval value or a fuzzy quantity. Clearly in this case we have an inhomogeneous decision matrix and traditional methods can not be applied directly. One of methods to deal with this decision problem is to use fuzzy set based techniques with help of the so-called *extension principle* and many procedures of ranking fuzzy quantities developed in the literature. Here we shall provide a target-based procedure for solving this problem.

First using the preceding mechanism, once having assessed a target T, we need to transform the payoff table into the one of probabilities of meeting the target. For each alternative A_i and state S_j, the probability of payoff value c_{ij} meeting the target is defined by

$$p_{ij} = P(c_{ij} \succeq T)$$

If c_{ij} is a crisp number, as previously discussed we have

$$p_{ij} = \frac{\int_{-\infty}^{c_{ij}} T(x)\mathrm{d}x}{\int_{-\infty}^{+\infty} T(x)\mathrm{d}x}$$

If c_{ij} is an interval value or a fuzzy quantity, the procedure for computing p_{ij} is as follows.

In the case where c_{ij} is an interval value, say $c_{ij} = [a, b]$, we consider c_{ij} as a random variable with the uniform distribution on $[a, b]$. If c_{ij} is a fuzzy quantity represented by a possibility distribution F_{ij}, we have the associated probability distribution of F_{ij} defined by

$$P_{F_{ij}}(x) = \frac{F_{ij}(x)}{\int_{-\infty}^{+\infty} F_{ij}(x)\mathrm{d}x}$$

and also denote, with an abuse of notation, c_{ij} as the random variable associated with the distribution $P_{F_{ij}}$. Recall that the associated probability distribution of the target T is

$$P_T(x) = \frac{T(x)}{\int_{-\infty}^{+\infty} T(x)\mathrm{d}x}$$

and also denote T as the random variable associated with the distribution P_T.

Having considered c_{ij} and T as two random variables in both these cases, we can define the probability of c_{ij} meeting the target T as

$$\begin{aligned} p_{ij} &= P(c_{ij} \succeq T) \\ &= \int_{-\infty}^{\infty} P_T(x) P(c_{ij} \succeq x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} P_T(x) \left[\int_{x}^{\infty} P_{F_{ij}}(y)\mathrm{d}y \right] \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \int_{x}^{\infty} P_{F_{ij}}(y) P_T(x)\mathrm{d}y\mathrm{d}x \qquad (21) \end{aligned}$$

provided accepting the independent assumption of c_{ij} and T.

As such, we have provided a method of transforming an inhomogeneous decision matrix into the derived decision matrix described by Table 6, where each element p_{ij} of the derived decision matrix is uniformly interpreted as the probability of payoff c_{ij} meeting the target T. From this derived decision matrix, we can then use the value function (20) for ranking alternatives and making decisions. It is worth to emphasize that as an important characteristic of this target-based approach, it allows for including the DM's attitude, which is expressed in assessing his target, into the formulation of decision functions. Consequently, different attitudes about target may lead to different results of the selection.

Note that in the fuzzy set method (Rommelfanger 2002), we first apply the extension principle to obtain the fuzzy expected payoff for each alternative and then utilize either a defuzzification method or a ranking procedure for fuzzy numbers for the purpose of making the decision. Therefore, we may also get different results if different methods of ranking fuzzy numbers or defuzzification are used. However, this difference of results caused by using different ranking methods does not reflect the influence of the

DM's attitude. Furthermore, a bunch of methods for ranking fuzzy numbers developed in the literature may also make it even difficult for people in choosing a most suitable method for each particular problem.

It should be mentioned here that in the research topic of ranking fuzzy numbers, the authors in Lee-Kwang and Lee (1999) proposed a ranking procedure based on the so-called satisfaction function (SF, for short) (Lee et al. 1994), which is denoted by S and defined as follows. Given two fuzzy numbers A and B,

$$S(A > B) = \frac{\int_{-\infty}^{\infty} \int_{y}^{\infty} \mu_A(x) \odot \mu_B(y) \mathrm{d}x\mathrm{d}y}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_A(x) \odot \mu_B(y) \mathrm{d}x\mathrm{d}y}$$

where $\odot$ is a T-norm and $S(A > B)$ is interpreted as "the possibility that A is greater than B" (or, the evaluation of A in the local viewpoint of B). Clearly, by a simple transformation we easily show that the probability p_{ij} of c_{ij} meeting the target T defined in (21) above is the SF $S(c_{ij} > T)$ with T-norm $\odot$ selected as being the multiplication operator. However, as having discussed, our motivation and interpretation here are different.

5.2 A numerical example

For illustrating the applicability of the target-based model in fuzzy decision problems with uncertainty, let us consider the following application example adapted from Rommelfanger (2002).

LuxElectro is a manufacturer of electro-utensils and currently the market demand for its products is higher than the output. Therefore, the management is confronted with the problem of making a decision on possible expansion of the production capacity. Possible alternatives for the selection are as following:

- A_1: Enlargement of the actual manufacturing establishment with an increase in capacity of 25%.
- A_2: Construction of a new plant with an increase in total capacity of 50%.
- A_3: Construction of a new plant with an increase in total capacity of 100%.
- A_4: Renunciation of an enlargement of the capacity, the status quo.

Table 7 Fuzzy profit matrix $\tilde{U}_{ij} = \tilde{U}(A_i, S_j)$

Alternatives	States		
	S_1:0.3	S_2:0.5	S_3:0.2
A_1	(80;90;100;110)	(75;85;90;100)	(50;60;70)
A_2	(135;145;150;165)	(120;130;140)	(−40;−30;−20)
A_3	(170;190;210;230)	(100;110;125)	(−90;−80;−70;−60)
A_4	70	70	70

The profit earned with the different alternatives depends upon the demand, which is not known with certainty. Due to the amount of information the management estimates three states of nature corresponding to 'high', 'average' and 'low' demand with associated prior probabilities of $p_1 = 0.3$, $p_2 = 0.5$, and $p_3 = 0.2$, respectively. Moreover, assume that the prior matrix of fuzzy profits $\tilde{U}_{ij}$ (measured in million Euro) is given in Table 7, where fuzzy profits are represented parametrically by triangular and trapezoidal fuzzy numbers. Then, the expected fuzzy profit of each alternative A_i ($i = 1, \ldots, 4$) can be calculated as

$$\tilde{E}(A_i) = \bigoplus_{j=1}^{3} (p_j \otimes \tilde{U}_{ij}) \tag{22}$$

where $\oplus$ and $\otimes$ stand for the extended addition and multiplication, respectively, and risk neutrality is assumed. Using Zadeh's extension principle for (22) then results in the expected fuzzy profits of alternatives as shown in Table 8 below. Now, to make a decision one can apply one of ranking methods developed in the literature on these fuzzy profits. Looking at the membership functions of the expected profits depicted in Fig. 7, we can intuitively see that the alternatives A_4 and A_1 are much worse than the alternatives A_3 and A_2. However, it is not so easy to say which alternative is dominated by the other among these two better alternatives. Here, if using for example the centroid of fuzzy numbers as the ranking criterion we get the ranking order as $A_2 \succ A_3 \succ A_1 \succ A_4$.

Table 8 Expected fuzzy profits via extension principle

Alternatives	Expected fuzzy profit	Centroid value
A_1	(71.5; 81.5; 87; 97)	84.25
A_2	(92.5; 102.5; 104; 115.5)	103.73
A_3	(83; 96; 104; 119.5)	100.76
A_4	70	70

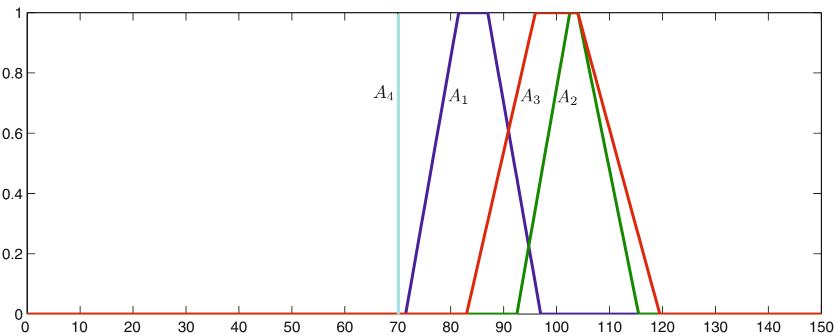

Fig. 7 Membership functions of expected profits

Table 9 Derived decision matrix $p_{ij} = P(\tilde{U}_{ij} \succeq T_{opt})$

Alternatives	States		
	S_1	S_2	S_3
A_1	0.3346	0.3079	0.2199
A_2	0.5584	0.4728	0.0353
A_3	0.8229	0.3974	0.0026
A_4	0.25	0.25	0.25

Table 10 Derived decision matrix $p_{ij} = P(\tilde{U}_{ij} \succeq T_{neut})$

Alternatives	States		
	S_1	S_2	S_3
A_1	0.5781	0.5547	0.4688
A_2	0.747	0.6875	0.1875
A_3	0.9063	0.6302	0.0469
A_4	0.5	0.5	0.5

Table 11 Derived decision matrix $p_{ij} = P(\tilde{U}_{ij} \succeq T_{pess})$

Alternatives	States		
	S_1	S_2	S_3
A_1	0.8216	0.8014	0.7176
A_2	0.9356	0.9022	0.3397
A_3	0.9896	0.8630	0.0911
A_4	0.75	0.75	0.75

Now let us apply the target-based procedure suggested above for solving this problem. According to the information given by this problem, let us define the domain of profits as $D = [-90, 230]$. Assuming for instance that a fuzzy optimistic target T_{opt} has been estimated based upon the optimistic attitude of the management, where

$$T_{opt}(c) = \frac{c + 90}{320}$$

Then with this optimistic target, using the above procedure we obtain the derived decision matrix as shown in Table 9 below.

In the same way, we also obtain the derived decision matrices corresponding to neutral and pessimistic targets, denoted, respectively, by T_{neut} and T_{pess}, as shown in Table 10 and Table 11. After assessing a target and obtaining the derived decision matrix accordingly, the value function (20) is then applied for making the decision. Table 12 shows the results of the value function for three above targets and the corresponding ranking orders of alternatives.

From Table 12 we see that the result reflects very well the behavior of the DM which is expressed in assessing the target. In particular, the ranking order of alternatives

Table 12 The ranking result using different targets

Targets	Expected probability				Ranking order
	A_1	A_2	A_3	A_4	
Optimistic	0.2983	0.411	0.4461	0.25	$A_3 \succ A_2 \succ A_1 \succ A_4$
Pessimistic	0.7907	0.7997	0.7466	0.75	$A_2 \succ A_1 \succ A_4 \succ A_3$
Neutral	0.5445	0.6053	0.5964	0.50	$A_2 \succ A_3 \succ A_1 \succ A_4$

Table 13 Result for a "more pessimistic target"

Alternatives	States			Expected values
	S_1	S_2	S_3	
A_1	0.9244	0.9113	0.8498	0.9029
A_2	0.9835	0.9693	0.4632	0.8723
A_3	0.9987	0.9491	0.133	0.8008
A_4	0.875	0.875	0.875	0.875

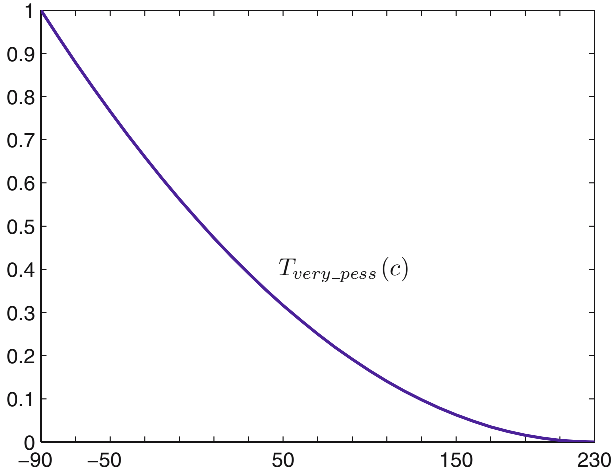

Fig. 8 Membership function of a more pessimistic target

corresponding to the neutral target is the same as that obtained by using the fuzzy expected profits with centroid-based ranking criterion, where the risk neutrality is assumed. As noted in Sect. 3, the neutral target T_{neut} induces a linear utility function $U(c) = P(c \succeq T_{neut})$ which is also equivalent to risk neutrality behavior. For the case of optimistic target T_{opt}, it induces a convex utility function $U(c) = P(c \succeq T_{opt})$ (refer to Fig. 2) which is equivalent to a risk-seeking behavior. In this case the DM wishes to have profit as big as possible accepting a risk that if the desirable state will not occur, he may get a big loss. This attitude leads to the selection of alternative A_3 which has the biggest profit in case of a high demand occurs. By the contrast, the pessimistic target T_{pess} yields a concave utility function $U(c) = P(c \succeq T_{pess})$

corresponding to a risk-aversion behavior (refer to Fig. 3). In this case, we see that though A_2 is still selected, alternatives A_1 and A_4 become more preferred over the A_3. This reflects the situation that the DM is looking for sure of getting profit. It should be noted here that we have defined membership degrees for T_{pess} linearly decrease over the profit domain, which exhibits a neutral-pessimistic attitude and consequently the DM is not enough risk averse to refuse alternative A_2. Let us now assume a more pessimistic attitude of the DM and the corresponding target, denoted by $T_{very\text{-}pess}$, is estimated for example by the following membership function:

$$T_{very\text{-}pess}(c) = \frac{(230 - c)^2}{320^2}$$

and graphically depicted as in Fig. 8. Then we obtain the result corresponding to this target as shown in Table 13, which yields the ranking order as $A_1 \succ A_4 \succ A_2 \succ A_3$. In other words, with a more pessimistic attitude the DM is enough risk averse to refuse alternative A_2 and select A_1 for surely getting best profit.

6 Concluding remarks

In this paper, we have explored a fuzzy target-based approach to decision making under uncertainty. In particularly, we have discussed the issue of how to bring fuzzy targets within the reach of the target-based decision model and also considered different attitudes of the decision maker in assessing the target. Moreover, the target-based formulation for the problem of decision making in the face of uncertainty about the state of nature and imprecision about payoffs has been also provided. This basically suggests that the target-based approach would provide an appealing and unified one for decision analysis under uncertainty. Furthermore, this target-based approach would also allow for higher generality in the formulation of decision functions taking the decision maker's attitudes about different aspects of decision problems into account (Yager 1999). However, this requires further research in the future.

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