Initial semantics in logics with constructors

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November 6, 2012

Abstract

The constructor-based logics constitute the logical foundation of the so-called OTS/CafeOBJ method, a modeling, specification and verification method of the observational transition systems. It is well known the important role played in algebraic specifications by the initial algebras semantics. Free models along presentation morphisms provide semantics for the modules with initial denotation in structured specification languages. Following Goguen and Burstall, the notion of logical system over which we build specifications is formalized as an institution. The present work is an institution-independent study of the existence of free models along sufficient complete presentation morphisms in logics with constructors in the signatures.

1 Introduction

Algebraic specification represents one of the most important classes of formal methods assisting the development of more reliable and efficient software systems. The fundamental mathematical structure underlying the theory of algebraic specifications is that of institution [20], a formal concept of logical system which arose within computing science with the ambition of covering the population of logics used in practice and working as much as possible at the abstract level, independent of any particular institution. In the present paper we study the existence of initial models for theories and free models along theory morphisms in arbitrary constructor-based institutions. The logics with constructors constitute the foundation of the OTS/CafeOBJ method, a modeling, specification and verification method of the observational transition systems, which has been previously explored in many case studies [27, 17, 26, 16].

Constructor-based institutions [3, 4, 19] are obtained from a base institution basically by enhancing the syntax with a sub-signature of constructor operators and by restricting the semantics to reachable models which consist of constructor-generated elements. The sentences and the satisfaction condition are preserved from the base institution and the signature morphisms are restricted such that the reducts of models which are reachable in the target signature are again reachable in the source signature. The result sorts of the constructors are called constrained and a sort that is not constrained it is called loose.
By introducing the constructor operators in the signatures we gain more expressivity for the specifications but some of the basic important properties are lost. For example, the constructor-based variants of Horn institutions\(^1\) are not complete. However, in [19] we obtained a \(\omega\)-completeness\(^2\) result that depends on sufficient completeness. Intuitively, a presentation \((\Sigma, \Gamma)\), where \(\Sigma\) is a signature and \(\Gamma\) a set of formulas, is sufficient complete if every term can be reduced to a term formed with constructors and operators of loose sorts using the equations in \(\Gamma\). We argued that a complete and compact system of proof rules cannot be provided due to the Gödel’s famous incompleteness result.

Firstly, we study the existence of pushouts in a concrete category of constructor-based signature morphisms (see Section 3). The method of constructing the pushouts can be used in other cases as well. As a main result it is shown that the existence can be guaranteed by imposing certain conditions on the signature morphisms. This result is important as the pushout construction is one of the most used approaches to combine specifications. A property closely related to combining specifications coherently with respect to the semantics is semi-exactness.

The institution theoretic concept of semi-exactness is a basic property of institutions that has been intensively used by various works in algebraic specification [29, 15, 8, 30]. In the present work we conduct an institution dependent study of the semi-exactness property that is used later on to prove the existence of free models.

Secondly, we investigate the existence of initial and free models in arbitrary constructor-based institutions.

1. ‘Reachable’ institutions are obtained from constructor-based institutions by restricting the category of signatures such that all operators of constrained sorts are constructors. In Section 4, we prove in arbitrary ‘reachable’ institutions the existence of initial models of any set of Horn sentences of the form \((\forall X) \bigwedge H \Rightarrow C\), where \(H\) is a set of atoms, and \(C\) is an atom. Our hypothesis are natural and easy to check in concrete examples of institutions. Our results are much general than the ones obtained within the framework of factorization systems [2, 33, 1] or inclusion systems [10]. In ‘reachable’ institutions the class of models of a set of Horn sentences do not form a quasi-variety, and therefore the initiality results derived from preservation (i.e. the model class of a set of Horn sentences forms a quasi-variety, and quasi-varieties have initial objects) such as [2, 33, 1] or [10] cannot be applied. Initiality is then easily extended to the constructor-based institutions via sufficient completeness.

2. The existence of left adjoints of the forgetful functors gives the free models along the presentation morphisms which constitute the semantic of the modules with initial denotation of the structured algebraic specification languages. This was studied in the literature under the name of liberality and it has played a central role in institution theory from its beginning [20] (see also [23, 31]). In Section 5 we apply the results in [10] to extend the existence of initial models of sufficient complete presentations to the existence of free models along sufficient complete presentation morphisms in arbitrary constructor-based institutions.

\(^1\)Horn institutions are obtained from a base institution, for example first-order logic, by restricting the sentences to the so-called Horn sentences of the form \((\forall X) \bigwedge H \Rightarrow C\), where \(H\) is a set of atoms in the base institution, and \(C\) is an atom.

\(^2\)Some proof rules contain infinite premises which can only be checked with induction schemes. As a consequence, the resulting entailment system is not compact.
We assume that the reader is familiar with the basic notions of category theory. See [25] for the standard definitions of category, functor, pushout, etc., which are omitted here.

2 Institutions

An institution $I = (\text{Sig}^I, \text{Sen}^I, \text{Mod}^I, \vdash^I)$ consists of

1. a category $\text{Sig}^I$, whose objects are called signatures,
2. a functor $\text{Sen}^I : \text{Sig}^I \rightarrow \text{Set}$, providing for each signature a set whose elements are called (\Sigma-)sentences,
3. a functor $\text{Mod}^I : \text{Sig}^I \rightarrow \text{Cat}^{\text{op}}$, providing for each signature $\Sigma$ a category whose objects are called (\Sigma-)models and whose arrows are called (\Sigma-)morphisms,
4. a relation $\models^I \subseteq |\text{Mod}^I(\Sigma)| \times |\text{Sen}^I(\Sigma)|$ for each signature $\Sigma \in |\text{Sig}^I|$, called (\Sigma-)satisfaction, such that for each morphism $\varphi : \Sigma \rightarrow \Sigma'$ in $\text{Sig}^I$, the following satisfaction condition holds:

$$M' \models^I \ell \text{Sen}^I(\varphi)(e) \text{ iff } \text{Mod}^I(\varphi)(M') \models^I e$$

for all $M' \in |\text{Mod}^I(\Sigma')|$ and $e \in |\text{Sen}^I(\Sigma)|$.

We denote the reduct functor $\text{Mod}^I(\varphi)$ by $_\varphi \models$ and the sentence translation $\text{Sen}^I(\varphi)$ by $\varphi(\_ \varphi)$. When $M = M' \models _\varphi$ we say that $M$ is the $\varphi$-reduct of $M'$ and $M'$ is a $\varphi$-expansion of $M$. When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\text{Sig}^I$ may be simply denoted by $\text{Sig}$.

Example 1 (First Order Logic (FOL) [20]) The signatures are triplets $(S,F,P)$, where $S$ is the set of sorts, $F = \{F_{w \rightarrow s}\}_{(w,s) \in S^\times S}$ is the $(S^\times S)$-indexed set of operation symbols, and $P = \{P_s\}_{w \in S^s}$ is the $(S^s)$-indexed set of relation symbols. If $w = \lambda$, an element of $F_{w \rightarrow s}$ is called a constant symbol, or a constant. By a slight notational abuse, we let $F$ and $P$ also denote $\bigcup_{(w,s) \in S^\times S} F_{w \rightarrow s}$ and $\bigcup_{w \in S^s} P_w$ respectively. A signature morphism between $(S,F,P)$ and $(S',F',P')$ is a triplet $\varphi = (\varphi^t, \varphi^{op}, \varphi^l)$, where $\varphi^t : S \rightarrow S'$, $\varphi^{op} : F \rightarrow F'$, $\varphi^l : P \rightarrow P'$ such that $\varphi^{op} F_{w \rightarrow s} \subseteq F'_{\varphi^t(w) \rightarrow \varphi^t(s)}$ and $\varphi^l(P_w) \subseteq P'_{\varphi^l(w)}$ for all $(w,s) \in S^\times S$. When there is no danger of confusion, we may let $\varphi$ denote each of $\varphi^t$, $\varphi^l$ and $\varphi^{op}$. Given a signature $\Sigma = (S,F,P)$, a $\Sigma$-model $A$ is a triplet $A = \{A_s\}_{s \in S}, \{A_{w,s}(\sigma)\}_{(w,s) \in S^\times S, \sigma \in F_{w \rightarrow s}}, \{A_w(R)\}_{w \in S^s, R \in P_w}$ interpreting each sort $s$ as a set $A_s$, each operation symbol $\sigma \in F_{w \rightarrow s}$ as a function $A_{w,s}(\sigma) : A^w \rightarrow A_s$ (where $A^w$ stands for $A_{s_1} \times \ldots \times A_{s_n}$ if $w = s_1 \ldots s_n$), and each relation symbol $R \in P_w$ as a relation $A_w(R) \subseteq A^w$. When there is no danger of confusion we may let $A_s$ and $A_w$ denote $A_{w,s}(\sigma)$ and $A_w(R)$ respectively. Morphisms between models are the usual $\Sigma$-morphisms, i.e., $S$-sorted functions that preserve the structure. The $\Sigma$-sentences are obtained from

- equality atoms $t_1 = t_2$, where $t_1, t_2 \in |(T_{(S,F)})_s|$ and $T_{(S,F)}$ is the $(S,F)$-algebra of ground terms, or
- relational atoms $R(t_1, \ldots, t_n)$, where $R \in P_{s_1 \ldots s_n}$ and $t_i \in |(T_{(S,F)})_{s_i}|$ for all $i \in \{1, \ldots, n\}$, by applying for a finite number of times:
• negation, conjunction, disjunction,
• universal or existential quantification over finite sets of constants (variables).

Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of
ground terms \( t \) as elements \( A_t \) in models \( A \). The definitions of functors \( \text{Sen} \) and \( \text{Mod} \) on
morphisms are the natural ones: for any signature morphism \( \phi : \Sigma \to \Sigma' \), \( \text{Sen}(\phi) : \text{Sen}(\Sigma) \to \text{Sen}(\Sigma') \) translates sentences symbol-wise, and \( \text{Mod}(\phi) : \text{Mod}(\Sigma') \to \text{Mod}(\Sigma) \) is the forgetful
functor.

**Example 2 (Constructor-based First Order Logic (CFOL))** The signatures are of the form
\((S, F, F^c, P)\), where \((S, F, P)\) is a first-order signature, and \( F^c \subseteq F \) (for all \((w, s) \in S' \times S \) we
have \( F^c_{w \to s} \subseteq F_{w \to s} \)) is a distinguished subfamily of sets of operation symbols called
constructors. The constructors determine the set of constrained sorts \( S' \subseteq S \): \( s \in S' \) iff there exists a
constructor \( \sigma \in F^c_{w \to s} \). We call the sorts in \( S' = S - S' \) loose. The \((S, F, F^c, P)\)-sentences are the
FOL sentences for the signature \((S, F, P)\).

The \((S, F, F^c, P)\)-models are the usual first-order structures \( M \) with the carrier sets for the
constrained sorts consisting of interpretations of terms formed with constructors and elements
of loose sorts, i.e. there exists a \( S\)-sorted set \( Y = \{ Y_s \}_{s \in S} \) of variables of loose sorts (i.e. for
all \( s \in S' \) we have \( Y_s = \emptyset \) and a \( S\)-sorted function \( f = \{ f_s : Y_s \to M_s \}_{s \in S} \) such that for every
constrained sort \( s \in S' \), the function \( f^\#_s : (T_{(S, F^c)}(Y))_s \to M_s \) is a surjection, where \( T_{(S, F^c)}(Y) \)
is the \((S, F^c)\)-algebra of terms with variables from \( Y \) and \( f^\#_s : T_{(S, F^c)}(Y) \to M \bigrfloor_{(S, F^c)} \) is the
unique extension of \( f \) to a \((S, F^c)\)-morphism.

A signature morphism \( \phi : (S, F, F^c, P) \to (S', F', F'^c, P') \) in CFOL is a first-order signature
morphisms \( \phi : (S, F, P) \to (S', F', P') \) such that the constructors are preserved along the
signature morphisms (i.e. if \( \sigma \in F^c \) then \( \phi(\sigma) \in F'^c \)) and no ‘new’ constructors are introduced
for ‘old’ constrained sorts (i.e. if \( s \in S' \) and \( \sigma' \in (F^c)'_{w' \to \phi(s)} \) then there exists \( \sigma \in F^c_{w \to s} \) such
that \( \phi(\sigma) = \sigma' \).

**Lemma 3** [18] For every CFOL signature morphism \( \phi : (S, F, F^c, P) \to (S', F', F'^c, P') \), and any \((S', F', F'^c, P')\)-model \( M' \), we have \( M' \bigrfloor_{(S, F, P)} \in \text{Mod}(S, F, F^c, P) \).

Variants of constructor-based first-order logic were studied in[4] and [3].

**Example 4 (Horn Clause Logic (HCL))** A Horn sentence for a FOL signature \((S, F, P)\) is
a sentence of the form \( (\forall X)(\bigwedge H) \Rightarrow C \), where \( X \) is a finite set of variables, \( H \) is a finite set
of (relational or equational) atoms, and \( C \) is a (relational or equational) atom. Classically,
Horn clauses are Horn sentences in first-order logic without equality. Here, we call Horn clauses all Horn sentences of FOL. Thus HCL has the same signatures and models as FOL
but only Horn clauses as sentences. One can define the constructor-based variant of HCL
(i.e. Constructor-based Horn Clause Logic (CHCL)) as the restriction of CFOL to Horn
sentences.

**Example 5 (Preorder algebra (POA) [14, 13])** The POA signatures are just the ordinary
algebraic signatures. The POA models are preordered algebras which are interpretations of
the signatures into the category of preorders \( \mathbb{P}re \) rather than the category of sets \( \mathbb{Set} \). This
means that each sort gets interpreted as a preorder, and each operation as a preorder functor,
which means a preorder-preserving (i.e. monotonic) function. A preordered algebra morphism
is just a family of preorder functors (preorder-preserving functions) which is also an
algebra morphism.
The sentences have two kinds of atoms: equations and preorder atoms. A preorder atom \( t \leq t' \) is satisfied by a preorder algebra \( M \) when the interpretations of the terms are in the preorder relation of the carrier, i.e. \( M_t \leq M_{t'} \). Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first-order quantification.

Horn preorder algebra (HPOA) and its constructor-based variant (CHPOA) are obtained as the restrictions of POA and CPOA, respectively, to Horn sentences.

Below we introduce a less refined class of constructor-based institutions that is used to import initiality and liberality to constructor-based institutions.

**Example 6 (Reachable First Order Logic (RFOL))** This institution is obtained from CFOL by restricting the signatures such that all operation symbols of constrained sorts are constructors, i.e. for each \( (S, F, F^c, P) \) we have \( F^c = F^S \) where

\[
F^S_{w \rightarrow s} = \begin{cases} 
F_{w \rightarrow s} & : s \in S^c \\
\emptyset & : s \notin S^c
\end{cases}
\]

By restricting the sentences to Horn sentences we obtain RHCL. A similar construction can also be done for preorder algebra.

### 2.1 Substitutions

In CFOL, consider \( \Sigma \overset{X_1}{\leftarrow} \Sigma(X_1) \) and \( \Sigma \overset{X_2}{\leftarrow} \Sigma(X_2) \) two inclusion signature morphisms, where \( \Sigma = (S, F, F^c, P) \) is a CFOL signature, \( X_i \) is a set of non-constructor constant symbols disjoint from the constants of \( F \), and \( \Sigma(X_i) = (S, F \cup X_i, F^c, P) \). A substitution between \( \chi_1 \) and \( \chi_2 \) in CFOL can be represented by a function \( \theta : X_1 \rightarrow T_F(X_2) \). One can easily notice that \( \theta \) can be extended to a function

\[
\operatorname{Sen}(\theta) : \operatorname{Sen}(\Sigma(X_1)) \rightarrow \operatorname{Sen}(\Sigma(X_2))
\]

that replaces all the symbols in \( X_1 \) by the corresponding \( F \cup X_2 \)-terms according to \( \theta \). On the semantics side, \( \theta \) determines a functor

\[
\operatorname{Mod}(\theta) : \operatorname{Mod}(\Sigma(X_2)) \rightarrow \operatorname{Mod}(\Sigma(X_1))
\]

such that for all \( \Sigma(X_2) \)-models \( M \) we have

- \( \operatorname{Mod}(\theta)(M)_z = M_z \), for each sort \( z \in S \), or operation symbol \( z \in F \), or relation symbol \( z \in P \), and
- \( \operatorname{Mod}(\theta)(M)_z = M_{\theta(z)} \) for each \( z \in X_1 \).

**Proposition 7** For every CFOL signature \( \Sigma \) and each substitution \( \theta : X_1 \rightarrow \Sigma(X_2) \)

\[
\operatorname{Mod}(\theta)(M) \models \rho \text{ iff } M \models \operatorname{Sen}(\theta)(\rho)
\]

for all \( \Sigma(X_2) \)-models \( M \) and all \( \Sigma(X_1) \)-sentences \( \rho \).

The proof is the same as the one for FOL, which can be found in [11].

**Assumption 8** Throughout this paper, for all institutions above, we assume that the signature morphisms allow mappings of constants to ground terms.
This makes it possible to treat first-order substitutions in the comma category \(^3\) of signature morphisms.

**Definition 9** Consider two signature morphisms \(\Sigma \xrightarrow{\theta} \Sigma_1 \) and \(\Sigma \xrightarrow{\theta} \Sigma_2 \) of an institution. A signature morphisms \(\theta : \Sigma_1 \to \Sigma_2 \) such that \(\chi_1 \theta = \chi_2 \) is called a \(\Sigma\)-substitution between \(\chi_1\) and \(\chi_2\).

### 2.2 Presentations

Let \(I = (\text{Sig}, \text{Sen}, \text{Mod}, \models)\) be an institution. A presentation \((\Sigma, E)\) consists of a signature \(\Sigma\) and a set \(E\) of \(\Sigma\)-sentences. A presentation morphism \(\phi : (\Sigma, E) \to (\Sigma', E')\) is a signature morphism \(\phi : \Sigma \to \Sigma'\) such that \(E' \models \phi(E)\). The presentation morphisms form a category denoted \(\text{Sig}^{\text{pres}}\) with the composition inherited from the category of signatures. The model functor \(\text{Mod}\) can be extended from the category of signatures \(\text{Sig}\) to the category of presentations \(\text{Sig}^{\text{pres}}\), by mapping a presentation \((\Sigma, E)\) to the full subcategory \(\text{Mod}(\Sigma, E)\) of \(\text{Mod}(\Sigma)\) consisting of models that satisfy \(E\). The correctness of the definition of the overloaded model functor \(\text{Mod} : \text{Sig}^{\text{pres}} \to \text{Cat}^{\text{op}}\) is guaranteed by the satisfaction condition of the base institution. This leads to the institution of presentations \(\text{Pres} = (\text{Sig}^{\text{pres}}, \text{Sen}, \text{Mod}, \models)\) over the base institution \(I\), where the sentence functor \(\text{Sen}\) and the satisfaction condition \(\models\) of the base institution \(I\) are overloaded such that

- \(\text{Sen}(\Sigma, E) = \text{Sen}(\Sigma)\), and
- for all \(M \in \text{Mod}(\Sigma, E)\) and sentences \(\rho \in \text{Sen}(\Sigma, E)\), \(M \models (\Sigma, E) \rho\) iff \(M \models \rho\).

### 2.3 Basic set of sentences

A set of sentences \(B \subseteq \text{Sen}(\Sigma)\) is called basic \([9]\) if there exists a \(\Sigma\)-model \(M_B\) such that, for all \(\Sigma\)-models \(M\), \(M \models B\) iff there exists a morphism \(M_B \to M\). We say that \(M_B\) is a basic model of \(B\). If in addition the morphisms \(M_B \to M\) is unique then the set \(B\) is called epi basic.

**Remark 10** Any set of epi basic sentences has an initial model.

It is well-known that any set of atoms in \(\text{FOL}\) and \(\text{POA}\) is epi basic (see for example \([9]\) or \([12]\)). In the followings we lift this result to reachable institutions.

**Lemma 11** Any set of atoms in \(\text{RFOL}\) and \(\text{RPOA}\) is epi basic.

**Proof.** Let \((S, F, F^c, P)\) be a \(\text{RFOL}\)-signature. Since all operators of constrained sorts are constructors we have \(T_{(S, F, F^c, P)} \subseteq \text{Mod}(S, F, F^c, P)\), where \(T_{(S, F, P)}\) is the term model interpreting each relation symbol as empty set. Moreover, for any congruence relation \(\equiv \subseteq T_{(S, F, F^c, P)} \times T_{(S, F, F^c, P)}\) we have that \((T_{(S, F, P)})_{\equiv} \subseteq \text{Mod}(S, F, F^c, P)\).

Let \(B\) be a set of atomic \((S, F, F^c, P)\)-sentences. The basic model \(M_B\) it is the initial model of \(B\) and it is constructed as follows: on the quotient \((T_{(S, F, P)})_{\equiv} B\) of the term model \(T_{(S, F, P)}\)

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\(^3\) Given a category \(C\) and an object \(A \in |C|\), the comma category \(A/\downarrow C\) has arrows \(A \xrightarrow{f} B \in C\) as objects, and \(h \in C(B, B')\) with \(f, h = f'\) as arrows.

\(^4\) \(C'\) is a full subcategory of \(C\) if \(C'\) is a subcategory of \(C\) such that \(C(A, B) = C'(A, B)\) for all objects \(A, B \in |C'|\).
by the congruence generated by the equational atoms of \( B \), we interpret each relation symbol \( \pi \in P \) by \( (M_B)_\pi = \{(t_1/\equiv_B, \ldots, t_n/\equiv_B) \mid \pi(t_1, \ldots, t_n) \in B\} \).

By defining a notion of congruence compatible with the preorder (see [13] or [7]) one may obtain the same result for RPOA.

A direct consequence of Lemma 11 is the following corollary.

**Corollary 12** In RFOL and RPOA, any set of atoms has an initial model.

Given a CFOL signature \((S, F, F^c, P)\), the model \( T_{(S, F, P)} \) is not reachable by the constructors in \( F^c \), i.e. \( T_{(S, F, P)} \not\in \text{Mod}(S, F, F^c, P) \) does not hold, in general. It follows that in CFOL not all sets of sentences are epi basic, and hence, not all sets of atoms have an initial model.

### 2.4 Reachable models

Below, we give an institution-independent characterization of the models with elements that consist of interpretations of terms.

**Definition 13** Let \( D \) be a broad subcategory \(^5\) of signature morphisms of an institution \( I = (\text{Sig}, \text{Sen}, \text{Mod}, \models) \). We say that a \( \Sigma \)-model \( M \) is \( D \)-reachable if for each span of signature morphisms \( \Sigma_1 \xleftarrow{\chi_1} \Sigma_0 \xrightarrow{\chi} \Sigma \) in \( D \), each \( \chi_1 \)-expansion \( M_1 \) of \( M \mid_{\chi_1} \) determines a substitution \( \theta : \chi_1 \rightarrow \chi \) such that \( M \mid_{\theta} = M_1 \).

The proof of the following proposition can be found in [18].

**Proposition 14** In FOL and POA assume that \( D \) is the class of signature extensions with a finite number of constants. A model \( M \) is \( D \)-reachable iff its elements are exactly the interpretations of ground terms.

In concrete institutions underlying the algebraic specification languages, \( D \) consists of signature extensions with a finite number of constants. Since \( D \)-reachable models have elements which consist of interpretations of ground terms, we may refer to \( D \)-reachable models as ground reachable models.

**Remark 15** In FOL and POA, the models defining a set of atoms as basic set of sentences are ground reachable.

For each RFOL \( \Sigma \)-model \( M \) there exists a signature extension \( \Sigma \hookrightarrow \Sigma' \) with constants of loose sorts, and a ground reachable \( \Sigma' \)-model \( M' \) such that the reduct of \( M' \) to the signature \( \Sigma \) is \( M \). Note that \( \Sigma' \) can be the extension of \( \Sigma \) with the elements of loose sorts of the model \( M \). In this case \( M' \) is just like \( M \) but interpreting each element of loose sort by itself. The RFOL models are called reachable. The CFOL models are reachable in the sub-signature of constructors. Actually, there is an abstract characterization of reachable models (see [19, 18]) which may be applied to a base institution in order to obtain the constructor-based variant.

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\(^5\) \( C' \) is a broad subcategory of \( C \) if \( C' \) is a subcategory of \( C \) and \( |C'| = |C| \).
2.5 Internal logic

The following institutional notions dealing with logical connectives and quantifiers were defined in [32].

Let \( \Sigma \in [\text{Sig}] \) be a signature of an institution \( I = (\text{Sig}, \text{Sen}, \text{Mod}, \models) \).

- A \( \Sigma \)-sentence \( \neg e \) is a (semantic) negation of the \( \Sigma \)-sentence \( e \) when for every \( \Sigma \)-model \( M \) we have \( M \models \neg e \iff M \not\models e \).
- A \( \Sigma \)-sentence \( e_1 \land e_2 \) is a (semantic) conjunction of the \( \Sigma \)-sentences \( e_1 \) and \( e_2 \) when for every \( \Sigma \)-model \( M \) we have \( M \models e_1 \land e_2 \iff M \models e_1 \) and \( M \models e_2 \).
- A \( \Sigma \)-sentence \( (\forall \chi)e' \), where \( \Sigma \xrightarrow{\chi} \Sigma' \in \text{Sig} \) and \( e' \in \text{Sen}(\Sigma') \), is a (semantic) universal \( \chi \)-quantification of \( e' \) when for every \( \Sigma \)-model \( M \) we have \( M \models (\forall \chi)e' \iff M' \models_{\Sigma'} e' \) for all \( \chi \)-expansions \( M' \) of \( M \).

We will use the symbol \( \land \) to denote the conjunction of a set of sentences even if it is infinite. Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in \( \text{FOL} \) considers only the signature extensions with a finite number of constants.

3 Pushouts of constructor-based signatures

There is a difference between the signature morphisms in [3] and the signature morphisms in \( \text{CFOL} \). The signature morphisms in [3] do not allow ‘new’ constructors of ‘old’ sorts, and they do not have pushouts, in general. We will study here the conditions when the pushouts do exist.

3.1 Pushouts

Let \( \phi : (S, F, F^c, P) \to (S', F', F'^c, P') \) be a signature morphism in \( \text{CFOL} \). We say that \( \phi^{op} \) is injective if for all arities \( w \in S^* \) and sorts \( s \in S \), \( \phi^{op}_{w \to s} \) is injective. The same applies to \( \phi^v \), the constructor symbols component, and \( \phi^r \), the relation symbols component. \( \phi^{op} \) is encapsulated means that no ‘new’ operation symbol, i.e., outside the image of \( \phi \), is allowed to have the sort in the image of \( \phi \). More precise, if \( \sigma' \in F'_{w' \to s'} \) then for all \( s \in S \) such that \( s' = \phi^v(s) \) there exists \( \sigma \in F_{w \to s} \) such that \( \phi^{op}(\sigma) = \sigma' \). Same applies to \( \phi^r \).

**Definition 16** ((xyz)-signature morphisms) A \( \text{CFOL} \) signature morphism

\[
\phi : (S, F, F^c, P) \to (S', F', F'^c, P')
\]

is a \((x\, y\, z)\)-morphism, with \( x, \, t \in \{i, *\} \) and \( y, \, z \in \{i, e, *\} \), where \( i \) stands for ‘injective’, \( e \) for ‘encapsulated’, and \( * \) for ‘all’, when

1. it does not map constants to terms different from constants,
2. the sort component \( \phi^u : S \to S' \) has the property \( x \),
3. the operation component \( \phi^{op} = (\phi^{op}_{w \to s} : F_{w \to s} \to F'_{\phi^v(w) \to \phi^v(s)})_{s \in S} \) has the property \( y \).
4. The constructor component \( \varphi^{ct} = (\varphi^{ct}_{w \rightarrow s} : F^{ct}_{w \rightarrow s} \rightarrow F^{ct}_{\varphi^{ct}(w) \rightarrow \varphi^{ct}(s)}_{s \in S^*}) \) has the property \( z \), and

5. The relation component \( \varphi^{rl} = (\varphi^{rl}_{w} : P^{rl}_{w} \rightarrow P^{rl}_{\varphi^{rl}(w)}) \) has the property \( t \).

This notational convention can be extended to other institutions too, such as, for example, FOL or CPOA. In case of FOL, because there are no constructor symbols, the third component is missing. In case of CPOA, because there are no relation symbols, the last component is missing.

**Proposition 17** [3] The subcategory of CFOL signature \((**e*)\)-morphisms has pushouts.

Consider the following example of parameterized specification.

**Example 18** Consider the following parameterized specification of lists of arbitrary elements:

\[
\begin{array}{c}
\text{sorts Triv, List} \\
\text{op nil : List} \\
\text{op con : Triv \rightarrow List} \\
\end{array}
\]

\[
\begin{array}{c}
\text{sort Triv} \\
\text{op \varphi : Triv \rightarrow Nat} \\
\end{array}
\]

If we want to obtain lists of natural numbers then we need to construct the pushout of the following span of signature morphisms:

\[
\begin{array}{c}
\text{sorts Triv, List} \\
\text{op nil : List} \\
\text{op con : Triv \rightarrow List} \\
\end{array}
\]

\[
\begin{array}{c}
\text{sort Triv} \\
\text{op \varphi : Triv \rightarrow Nat} \\
\end{array}
\]

\[
\begin{array}{c}
\text{sort Nat} \\
\text{op \varphi : Nat \rightarrow Nat} \\
\end{array}
\]

Note that \( \chi \) is a signature morphism in the sense of [3] but \( \varphi \) is not.

We will investigate the existence of pushouts of constructor-based signature morphisms in order to cover the example above.

**Proposition 19** The category of CFOL signature morphisms has \(((**e*), (**e*))\)-pushouts\(^6\). Moreover, if \( \{\Sigma_2 \xleftarrow{\chi} \Sigma, \Sigma \xrightarrow{\varphi} \Sigma_1, \Sigma \xrightarrow{\varphi_1} \Sigma_2\} \) is a pushout of CFOL signature morphisms such that \( \varphi \) is a \((**e*)\)-morphism and \( \chi \) is a \((**e*)\)-morphism then \( \varphi_2 \) is \((**e*)\)-morphism and \( \chi_1 \) is a \((**e*)\)-morphism.

**PROOF.** Consider the following pushout of FOL signature morphisms.

\[
\begin{array}{c}
(S_2, F_2, P_2) \xrightarrow{\varphi_2} (S', F', P') \\
\chi \downarrow \quad \chi_1 \downarrow \\
(S, F, P) \xrightarrow{\varphi} (S_1, F_1, P_1)
\end{array}
\]

**Remark 20** By the pushout construction in the category of sets, the above pushout has the following properties:

\(^6\)We say that a category \( C \) has \((L, R)\)-pushouts for two subcategories \( L, R \subseteq C \), if for each span of morphisms \( A_2 \xleftarrow{u} A_0 \xrightarrow{v} A_1 \) such that \( u \in L \) and \( v \in R \) there exists a pushout \( \{A_2 \xleftarrow{u} A_0 \xrightarrow{v} A_1 \xrightarrow{\varphi} A \xrightarrow{\psi} A_2\} \).
1. \( \chi_1 \) is injective on sorts.

2. for all \( s_1 \in S_1 \) and \( s_2 \in S_2 \) such that \( \chi_1(s_1) = \varphi_2(s_2) \) there exists \( s \in S \) such that \( \varphi(s) = s_1 \) and \( \chi(s) = s_2 \), and

3. \( \varphi_2 \) is injective on \( S_2 - \chi(S) \).

Also note that \( \chi_1 \) and \( \varphi_2 \) map constants to constants only.

Let \( F^{ic} = \chi_1(F^c_1) \cup \varphi_2(F^c_2) \), where

- \( F^{ic} = \{ F^{ic}_{w' \rightarrow s'} \}_{w' \in S'} \), and

- \( F^{ic}_{w' \rightarrow s'} = (\cup \chi_1(w_1,s_1) = (w',s') \chi_1((F^c_1)_{w_1 \rightarrow s_1})) \cup (\cup \varphi_2(w_2,s_2) = (w',s') \varphi_2((F^c_2)_{w_2 \rightarrow s_2})) \).

We prove that

\[
(S_2,F_2,F^c_2,P_2) \xrightarrow{\varphi} (S,F,F^{ic},P')
\]

\[
\xrightarrow{\chi_1}
\]

is a pushout of \textbf{CFOL} signature morphisms.

Firstly, we show that \( \chi_1 \) is a \((i * e)\)-morphism. By the definition of \( F^{ic} \) all the constructors in \( F^c_1 \) are preserved by \( \chi_1 \). Now let \( \sigma' \in F^{ic}_{w' \rightarrow \chi_1(s_1)} \), where \( s_1 \in S_1 \). There are two cases:

1. \textbf{There exists} \( \sigma_1 \in (F^c_1)_{w_1 \rightarrow s_1} \) \textbf{such that} \( \chi_1(\sigma_1) = \sigma' \). Note that \( \chi_1(s_1) = \chi_1(st_1) \) and since \( \chi_1 \) is injective on sorts, \( s_1 \) = \( st_1 \). Therefore \( \chi_1(\sigma_1 : w_1 \rightarrow s_1) = \sigma' : w' \rightarrow \chi_1(s_1) \).

2. \textbf{There exists} \( \sigma_2 \in (F^c_2)_{w_2 \rightarrow s_2} \) \textbf{such that} \( \varphi_2(\sigma_2) = \sigma' \). We have \( \varphi_2(st_2) = \chi_1(st_1) \). By Remark 20 there exists \( s \) such that \( \varphi(s) = s_1 \) and \( \chi(s) = s_2 \). Since \( \chi \) is a \((i * e)\)-morphism there exists \( \sigma \in F^{ic}_{w' \rightarrow s} \) such that \( \chi(\sigma) = \sigma_2 \). Now take \( \sigma_1 : w_1 \rightarrow s_1 = \varphi(\sigma : w \rightarrow s) \). We have \( \chi_1(st_1) = \chi_1(s_1) \) and by the injectivity of \( \chi_1 \) we get \( s_1 = s_{t_1} \). Therefore \( \chi_1(\sigma_1 : w_1 \rightarrow s_{t_1}) = \sigma' : w' \rightarrow \chi_1(s_1) \).

Secondly, we show that \( \varphi_2 \) is a \textbf{CFOL} signature morphism. Again by the definition of \( F^{ic} \) all the constructors in \( F^c_2 \) are preserved by \( \varphi_2 \). Let \( \sigma' \in F^{ic}_{w' \rightarrow \varphi_2(s_2)} \), where \( s_2 \in S_2 \). We have two cases:

1. \( s_2 \in \chi(S') \). Let \( s \in S' \) such that \( \chi(s) = s_2 \). Since \( \chi_1 \) is a \((i * e)\)-morphism and \( \varphi(s) \in S_1' \), there exists \( \sigma_1 \in (F^c_1)_{w_{1} \rightarrow \varphi_2(s_2)} \) such that \( \chi_1(\sigma_1) = \sigma' \). Because \( s \in S' \), there exists \( \sigma \in F^{ic}_{w' \rightarrow s} \) such that \( \varphi(\sigma) = \sigma_1 \). Take \( \sigma_2 = \chi(\sigma) \) and we have \( \varphi_2(\sigma_2 : \chi(w) \rightarrow s_2) = \sigma' : w' \rightarrow \varphi_2(s_2) \).

2. \( s_2 \in S_2 - \chi(S') \). Since \( \chi \) is a \((i * e)\)-morphism we have \( s_2 \in S_2 - \chi(S) \). By the definition of \( F^{ic} \) we have two subcases:

   (a) \textbf{There exists} \( \sigma_1 \in (F^c_1)_{w_1 \rightarrow st_1} \) \textbf{such that} \( \chi_1(\sigma_1) = \sigma' \). Because \( \chi_1(st_1) = \varphi_2(s_2) \) there is \( s \) such that \( \chi(s) = s_2 \) and \( \varphi(s) = st_1 \) which is a contradiction with \( s_2 \in S_2 - \chi(S) \).
(b) There exists $\sigma_2 \in (F_{w_2})_{w_2 \to s_2}^*$ such that $\varphi_2(\sigma_2) = \sigma'$. We have $s_2 \in S_2 - \chi(S)$ and $\varphi_2(s_2) = \varphi_2(st_2)$ which by Remark 20 implies $s_2 = st_2$. Therefore $\varphi_2(st_2 : w_2 \to s_2) = \sigma' : w' \to \varphi_2(s_2)$.

Now we show that for all $\chi : (S_i,F_i,F'_{\text{c}},P_i) \to (S'',F'',F''_{\text{c}},P'')$, where $i \in \{1,2\}$, such that $\varphi;v_1 = \chi;v_2$ there exists a unique $v : (S',F',F'_{\text{c}},P') \to (S'',F'',F''_{\text{c}},P'')$ such that $\chi_1;v = v_1$ and $\varphi_2 ; v = v_2$. A unique $v$ exists as a FOL signature morphism. We need to prove that $v$ is a CFOL signature morphism. Let $\sigma' \in F_{w''}^{\text{c}}(s',w')$, where $s' \in S''$. By the definition of $F'_{\text{c}}$, either there is $s_1 \in S'_1$ such that $\chi(s_1) = s'$ or there is $s_2 \in S'_2$ such that $\varphi(s_2) = s'$. Assume that there exists $s_1 \in S'_1$ such that $\chi_1(s_1) = s'$ (the other case is similar). Since $v_1$ is a CFOL signature morphism there is $\sigma_1 \in (F_1)_{w_1 \to s_1}$ such that $v_1(\sigma_1) = \sigma''$. Now take $\sigma' = \chi_1(\sigma_1)$ and we have $v(\sigma' : \chi_1(w_1) \to s') = \sigma'' : w'' \to v(s')$.

The condition $\chi$ is a $(*e*)$-morphism is necessary for Proposition 19 as one can see in the examples below.

**Example 21** Consider the following pushout in FOL:

```
\begin{tikzpicture}
  \node (s) {sort \( s \)};
  \node (a) [below of=s] {\( \varphi \colon \text{op a : s \{constr\} } \)};
  \node (b) [right of=a] {\( \text{op b : s \{constr\} } \)};
  \node (x) [above of=s] {\( \chi \colon s_1 \to s \)};
  \node (y) [below of=x] {\( \chi_1 \colon s_2 \to s \)};
  \draw[-stealth] (s) -- (a) node[midway,above] {$\subseteq$};
  \draw[-stealth] (s) -- (b) node[midway,above] {$\subseteq$};
  \draw[-stealth] (s) -- (y) node[midway,above] {};\end{tikzpicture}
```

Note that $\chi$ is not injective on sorts, and since there is no constructor $\text{(c : s1)}$ such that $\chi_1(c : s1) = (b : s)$, $\chi_1$ is not a CFOL signature morphisms.

**Example 22** Consider the following pushout in FOL:

```
\begin{tikzpicture}
  \node (s) {sort \( s \)};
  \node (a) [below of=s] {\( \text{op a : s \{constr\} } \)};
  \node (b) [right of=a] {\( \text{op b : s \{constr\} } \)};
  \node (x) [above of=s] {\( \chi \colon \text{op b : s \{constr\} } \)};
  \node (y) [below of=x] {\( \chi_1 \colon \text{op a : s \{constr\} } \)};
  \draw[-stealth] (s) -- (a) node[midway,above] {$\subseteq$};
  \draw[-stealth] (s) -- (b) node[midway,above] {$\subseteq$};
  \draw[-stealth] (x) -- (s) node[midway,above] {};\end{tikzpicture}
```

Note that the condition “no constructors of ‘old’ sorts” is not fulfilled by $\chi$ and $\varphi$, and since $\chi_1$ adds the constructor $b$ of the ‘old’ constrained sort $s$, $\chi_1$ is not a CFOL signature morphism.

Proposition 19 provides a method of constructing pushouts of signatures in other constructor-based institutions such as CPOA. Initially, the construction is done in the base institution, for example in POA, and then it is extended to the constructor-based variant, such as CPOA.
3.2 Pushouts of presentation morphisms

The pushouts of presentation morphisms have been playing a very important role in algebraic specifications \([20, 15]\) as it constitutes the basis of constructing large specifications out of smaller ones.

**Proposition 23** \([20]\) Let \(I = (\text{Sig}, \text{Sen}, \text{Mod}, \models)\) be an institution, and \(\mathcal{L}, \mathcal{R} \subseteq \text{Sig}\) two subcategories of signature morphisms. If \(\text{Sig}\) has \((\mathcal{L}, \mathcal{R})\)-pushouts then the presentation morphisms have \((\mathcal{L}^{\text{pres}}, \mathcal{R}^{\text{pres}})\)-pushouts, where

- \(\mathcal{L}^{\text{pres}}\) consists of presentation morphisms \((\Sigma, E) \xrightarrow{\phi} (\Sigma', E')\) such that \(\Sigma \subseteq \Sigma' \in \mathcal{L}\),
- \(\mathcal{R}^{\text{pres}}\) consists of presentation morphisms \((\Sigma, E) \xrightarrow{\psi} (\Sigma', E')\) such that \(\Sigma \subseteq \Sigma' \in \mathcal{R}\).

Parameterization is one of the most important technique used in structuring formal specifications. A parameterized presentation is a presentation morphism \(\chi : (P, E_P) \rightarrow (B[P], E_B[E_P])\), where \((P, E_P)\) is the parameter and \((B[P], E_B[E_P])\) the body, such that \(\chi : P \rightarrow B[P]\) is

1. an inclusive signature morphism, and
2. a \((**e***)-morphism (it does not add new constructors to the parameter).

Parameterization allows to abstract away the elements of a system that are not part of the essence, and can be obtained at a later time by instantiation. The pushout construction constitutes the basis of the instantiation mechanism. To instantiate \((P, E_P)\) with \((T, E_T)\) requires a parameter mapping \(\phi : (P, E_P) \rightarrow (T, E_T)\), and the result of the instantiation is the vertex of the pushout \(\{(B[P], E_B[E_P]) \xrightarrow{\phi} (P, E_P), (B[P], E_B[E_P]) \xrightarrow{\psi} (B[T], E_B[E_T]) \xrightarrow{\psi} \} \xrightarrow{\chi} (T, E_T)\} = \phi_2(E_B[E_P]) \cup \chi_1(E_T).

3.3 Semi-exactness

A basic institutional property that is necessary for combining specifications coherently with respect to the semantics is the \textit{semi-exactness} property.

**Definition 24** An institution \(I = (\text{Sig}, \text{Sen}, \text{Mod}, \models)\) is \((\mathcal{L}, \mathcal{R})\)-semi-exact, where \(\mathcal{L}, \mathcal{R} \subseteq \text{Sig}\) are two subcategories of signature morphisms, if for each \((\mathcal{L}, \mathcal{R})\)-pushout of signature morphisms

\[
\begin{array}{ccc}
\Sigma_2 & \xrightarrow{\phi_2} & \Sigma' \\
\chi \downarrow & & \chi_1 \downarrow \\
\Sigma & \xrightarrow{\phi} & \Sigma_1
\end{array}
\]

the following diagram

\[
\begin{array}{ccc}
\text{Mod}(\Sigma_2) & \xleftarrow{\text{Mod}(\phi_2)} & \text{Mod}(\Sigma') \\
\text{Mod}(\chi) \downarrow & & \text{Mod}(\chi_1) \downarrow \\
\text{Mod}(\Sigma) & \xrightarrow{\text{Mod}(\phi)} & \text{Mod}(\Sigma_1)
\end{array}
\]

is a pullback in \textit{Cat}. The institution \(I\) is semi-exact if \(\mathcal{L} = \mathcal{R} = \text{Sig}\).

We lift up the semi-exactness property from the base institution FOL to its constructor-based variant CFOL.

Proposition 26 CFOL is \((i\ast\ast),(i\ast e\ast)\)-semi-exact.

PROOF. Assume a pushout of CFOL signature morphisms

\[
\begin{array}{c}
(S_2, F_2, F'_2, P_2) \xrightarrow{\varphi_2} (S', F', F'^c, P') \\
\chi \\ (S, F, F^c, P) \xrightarrow{\varphi} (S_1, F_1, F'_1, P_1)
\end{array}
\]

such that \(\chi\) is a \((i\ast e\ast)\)-morphism and \(\varphi\) is a \((i\ast\ast)\)-morphism. Since FOL is semi-exact and for all CFOL signatures \((S_i, F_i, F'_i, P_i)\), \(\text{Mod}(S_i, F_i, F'_i, P_i)\) is the full subcategory of \(\text{Mod}(S_1, F_1, P_1)\), it suffices to prove that for all models \(M_1 \in \text{Mod}(S_1, F_1, F'_1, P_1)\) and \(M_2 \in \text{Mod}(S_2, F_2, F'_2, P_2)\) such that \(M_1 \mid \varphi = M_2 \mid \chi\) there exists \(M' \in \text{Mod}(S', F', F'^c, P')\) such that \(M' \mid \chi_1 = M_1\) and \(M' \mid \varphi_2 = M_2\).

By Proposition 25, there exists a model \(M' \in \text{Mod}(S', F', P')\) such that \(M' \mid \chi_1 = M_1\) and \(M' \mid \varphi_2 = M_2\). Let \(M'^d\) be the \(S'\)-sorted set such that for all \(s' \in S', M'^d_s = \begin{cases} \{M'_{s'} \mid s' \in S' \} & \text{if } s' \in S' \\
\emptyset & \text{if } s' \not\in S'.\end{cases}\)

Let \(\text{con}_{M'} : T_{(S', F'^c)}(M'^d) \rightarrow M' \mid (S', F'^c)\) be the unique extension of the inclusion \(\{M'^d_s \rightarrow M'_s\}_{s' \in S'}\) to a model \((S', F'^c)\)-morphism. If we prove that \(\text{con}_{M'}\) is surjective on the sorts in \(\chi_1(S_1)\) and \(\varphi_2(S_2)\), since \(S' = \chi_1(S_1) \cup \varphi_2(S_2)\), it follows that \(\text{con}_{M'}\) is surjective, meaning that \(M' \in \text{Mod}(S', F', F'^c, P')\).

1. We define the \(S_1\)-sorted set \(M'_1\) such that for all sorts \(s_1 \in S_1\), we have \((M'_1)_{s_1} = \begin{cases} \{M_1\}_{s_1} & \text{if } s_1 \in S' \\
\emptyset & \text{if } s_1 \not\in S'.\end{cases}\). Let \(\text{con}_{M_1} : T_{(S_1, F_1)}(M'_1) \rightarrow M_1 \mid (S_1, F_1)\) be the unique extension of the inclusion \(\{(M'_1)_{s_1} \rightarrow (M_1)_{s_1}\}_{s_1 \in S_1}\) to a \((S_1, F_1)\)-morphism. By Proposition 19, \(\chi_1\) is a \((i\ast e\ast)\)-morphism, which implies that \(\chi_1(S'_1) \subseteq S'.\) It follows that the following inclusion exists \(\{\chi_1(M'_1)_{s_1} \rightarrow (T_{(S', F'^c)}(M'^d))\}_{s_1 \in S_1}\) to a \((S_1, F'_1)\)-morphism. Note that for all \(m \in M'_1\), \(\text{con}_{M_1}(m) = (h_{M_1}; \text{con}_{M'} \mid (S_1, F'_1))(m)\), which implies that the following diagram is commutative.

\[
\begin{array}{ccc}
T_{(S_1, F_1)}(M'_1) & \xrightarrow{\text{con}_{M_1}} & M_1 \mid (S_1, F_1) \\
\downarrow h_{M_1} & & \downarrow \text{con}_{M'} \mid (S_1, F'_1) \\
T_{(S', F'^c)}(M'^d) & \xrightarrow{\text{con}_{M'}} & M' \mid (S_1, F'_1)
\end{array}
\]

Since \(\text{con}_{M_1}\) is surjective we obtain that \(\text{con}_{M'} \mid (S_1, F'_1)\) is surjective, meaning that \(\text{con}_{M'}\) is surjective on the sorts in \(\chi_1(S_1)\).
2. Let $M'_2$ be the $S_2$-sorted set such that for all $s_2 \in S_2$, $(M'_2)_{s_2} = \begin{cases} (M_2)_{s_2} & \text{if } s_2 \in S'_2 \\ \emptyset & \text{if } s_2 \notin S'_2 \end{cases}$.

Let $\text{con}_{M'_2} : T(S_2,F_2)(M'_2) \to M'_2 \sb{S_2,F_2}$ be the unique extension of following inclusion $\{(M'_2)_{s_2} \hookrightarrow (M_2)_{s_2}\}_{s_2 \in S_2}$ to a $(S_2,F_2)$-morphism. We define the $S_2$-sorted function $h_{M'_2} = \{(h_{M'_2})_{s_2} : (M'_2)_{s_2} \to (T(S,F_2))_{\varphi_2(s_2)} \}_{s_2 \in S_2}$. For all $s_2 \in S'_2$ such that $\varphi_2(s_2) \in S^t$, and $m \in (M'_2)_{s_2}$, we define $h_{M'_2}(m) = m$. Let $s_2 \in S'_2$ such that $\varphi_2(s_2) \in S^t$, and $m \in (M'_2)_{s_2}$. By the pushout construction, $S^t = \chi_1(S^t) \cup \varphi_2(S^t)$. There exists $s_1 \in S^t$ such that $\chi_1(s_1) = \varphi_2(s_2)$. Since $\text{con}_{M'}$ is surjective on the sorts in $\chi_1(S^t)$, there exists $t' \in (T(S,F_2))_{\varphi_2(s_2)}$ such that $\text{con}_{M'}(t') = m$. We define $h_{M'_2}(m) = t'$. Note that for all sorts $s_2 \in S_2$, we have $h_{M'_2} = (\text{con}_{M'})_{\varphi_2(s_2)} = (\text{con}_{M'_2})_{s_2}$.

Let $h^{g}_{M'_2} : T(S_2,F_2)(M'_2) \to T(S,F_2)(M^g) \sb{S_2,F_2}$ be the unique extension of $h_{M'_2}$ to a model $(S_2,F_2)$-morphism, and note that the following diagram is commutative.

$$
\begin{array}{ccc}
T(S_2,F_2)(M'_2) & \xrightarrow{\text{con}_{M'_2}} & M'_2 \sb{S_2,F_2} \\
\downarrow{h^{g}_{M'_2}} & & \downarrow{\text{con}_{M'}} \\
T(S,F_2)(M^g) \sb{S_2,F_2} & & \\
\end{array}
$$

Since $\text{con}_{M'_2}$ is surjective, $\text{con}_{M'} \sb{S_2,F_2}$ is surjective, which implies that $\text{con}_{M'}$ is surjective on the sorts in $\varphi_2(S_2)$.

The ideas of lifting semi-exactness from the base institution to the constructor-based variant provided by Proposition 26 can be applied in other cases such as preorder algebra.

## 4 Initial models

In this section we provide sufficient institution-independent conditions for a set of Horn sentences to have an initial model.

**Definition 27** Let $I = (\text{Sig}, \text{Sen}, \text{Mod}, \models)$ be an institution with a sub-functor $\text{Sen}_0 : \text{Sig} \to \text{Set}$ of $\text{Sen}$, and a subcategory $\mathcal{D} \subseteq \text{Sig}$ of signature morphisms. A $\mathcal{D}$-Horn sentence over $\text{Sen}_0(\Sigma)$ is a sentence semantically equivalent to $(\forall \chi) \land H \Rightarrow C$, where $\Sigma \xrightarrow{\Sigma'} \mathcal{D}, C \in \text{Sen}_0(\Sigma')$ and $H \subseteq \text{Sen}_0(\Sigma')$.

When the sub-functor $\text{Sen}_0$ and the subcategory $\mathcal{D}$ are fixed, we call $(\forall \chi) \land H \Rightarrow C$, simply, Horn sentence.

An example of institution $I = (\text{Sig}, \text{Sen}, \text{Mod}, \models)$ as in Definition 27 is $\text{CHCL}$, where the institution $I_0 = (\text{Sig}, \text{Sen}_0, \text{Mod}, \models)$ is the restriction $\text{CHCL}_0$ of $\text{CHCL}$ to atomic sentences, and $\mathcal{D}$ consists of signature extensions with a finite number of constants.

Note that not all sets of sentences in $\text{CHCL}$ have an initial model. If we restrict the signatures such that all operators of constrained sorts are constructors then we obtain initiality for Horn sentences (i.e. any set of sentences in $\text{RHCL}$ has an initial model). Abstractly we assume
1. an institution $I$ as in Definition 27, and
2. a subcategory $\text{Sig}' \subseteq \text{Sig}$ of signature morphisms.

Let $\text{Sen}' : \text{Sig}' \to \text{Set}$, $\text{Sen}_0' : \text{Sig}' \to \text{Set}$ and $\text{Mod}' : (\text{Sig}') \to \text{Cat}^{op}$ be the functors define as the restrictions of $\text{Sen} : \text{Sig} \to \text{Set}$, $\text{Sen}_0 : \text{Sig} \to \text{Set}$ and $\text{Mod} : \text{Sig} \to \text{Cat}^{op}$, respectively, to $\text{Sig}'$. We also define $|=_{\text{def}} \Sigma \equiv \{|=\}_{\Sigma \in |\text{Sig}'|}$.

**Fact 28** $I' = (\text{Sig}', \text{Sen}', \text{Mod}', |=')$ and $I_0' = (\text{Sig}', \text{Sen}_0', \text{Mod}', |=')$ are institutions.

We provide ‘easy-to-check’ institution-independent conditions for any set of sentences in $I'$ to have an initial model. Our methodology of proving initiality follows exactly the structure of the sentences:

1. we assume that all sentences of $I_0'$ are epi basic;
2. based on this assumption, which is checked in concrete examples of institutions, we prove that any set of quantifier-free Horn sentences in $I'$ has an initial model;
3. we extend the initiality result to the quantified Horn sentences of $I'$.

One may wonder what is the role played by $I$ in the abstract setting. The answer is simple: $I$ provides the subcategory $\mathcal{D}$ of signature morphisms and the satisfaction relation for quantified sentences. If $I$ is CHCL and $I'$ is RHCL then it is easy to notice that a signature extension with constants of constrained sorts is not a signature morphism in RHCL. Therefore in concrete examples we have $\mathcal{D} \not\subseteq \text{Sig}'$.

### 4.1 Quantifier-free layer

Assume an institution $I = (\text{Sig}, \text{Sen}, \text{Mod}, |=)$ and a sub-functor $\text{Sen}_0 : \text{Sig} \to \text{Set} \text{ of } \text{Sen}$. We denote by $I_0$ the institution $(\text{Sig}, \text{Sen}_0, \text{Mod}, |=)$. A concrete example of such institution is RHCL such that $I_0$ is the restriction of RHCL to atomic sentences.

**Proposition 29** If all sets of sentences in $I_0$ are epi basic then every set $\Gamma$ of $\Sigma$-sentences semantically equivalent to $\bigwedge H \Rightarrow C$, where $H \subseteq \text{Sen}_0(\Sigma)$ and $C \in \text{Sen}_0(\Sigma)$, has an initial model.

**Proof.** Let $\Sigma \in |\text{Sig}|$ and $\Gamma \subseteq \text{Sen}(\Sigma)$. We define $\Gamma_0 = \{e \in \text{Sen}_0(\Sigma) \mid \Gamma \models e\}$. Let $M_{\Gamma_0}$ be the basic model of $\Gamma_0$. We prove that $M_{\Gamma_0}$ is the initial model of $\Gamma$. If $M \models \Gamma$ then $M \models \Gamma_0$, and since $\Gamma_0$ is epi basic, there exists a unique morphism $M_{\Gamma_0} \to M$. We only need to show $M_{\Gamma_0} \models \Gamma$. Let $\bigwedge H \Rightarrow C \in \Gamma$ and assume $M_{\Gamma_0} \models H$. Since $H$ is basic, there exists a model morphism $M_H \to M_{\Gamma_0}$, which implies $\Gamma_0 \models H$, and we obtain $\Gamma \models H$. Since $\bigwedge H \Rightarrow C \in \Gamma$ and $\Gamma \models H$, we get $\Gamma \models C$. It follows that $C \in \Gamma_0$ and hence $M_{\Gamma_0} \models C$. □

**Corollary 30** In RHCL and RHPOA, any set of quantifier-free sentences has an initial model that is ground reachable.

**Proof.** We set the parameters of Proposition 29 for RHCL. The other case is similar. $I = \text{RHCL}$ and $I_0 = \text{RHCL}_0$ is the restriction RHCL to atomic sentences. By Lemma 11, the RHCL$_0$ sentences are epi basic, and by Proposition 29, any set of quantifier-free RHCL sentences has an initial model. Since the initial models are basic models, by Remark 15, they are also ground reachable. □
4.2 Quantification layer

Assume an institution $I = (\text{Sig}, \text{Sen}, \text{Mod}, \models)$ with a broad subcategory $\mathcal{D} \subseteq \text{Sig}$ of signature morphisms, and a sub-functor $\text{Sen}_1$ of $\text{Sen}$ such that all sentences of $I$ are semantically equivalent to $(\forall \chi) \rho$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and $\rho \in \text{Sen}_1(\Sigma')$. An example of such institution is $\text{CHCL}$ such that $\mathcal{D}$ is the subcategory of signature extensions with a finite number of constants, $I_1 = (\text{Sig}, \text{Sen}_1, \text{Mod}_1, \models)$ is the restriction $\text{CHCL}_1$ of $\text{CHCL}$ to quantifier-free sentences.

Any subcategory of signature morphisms $\text{Sig}' \subseteq \text{Sig}$ determines two institutions $I' = (\text{Sig}', \text{Sen}', \text{Mod}', \models')$ and $I'_1 = (\text{Sig}', \text{Sen}_1', \text{Mod}', \models')$ as in Fact 28. If $\text{Sig}'$ is the full subcategory of $\text{CHCL}$ signature morphisms such that all operators of constrained sorts are constructors then $I' = \text{RHCL}$, and $I'_1 = \text{RHCL}_1$, the restriction of $\text{RHCL}$ to quantifier-free sentences.

**Theorem 31** All sets of sentences in $I'$ have an initial model when all sets of sentences in $I'_1$ have an initial model, which is $\mathcal{D}$-reachable in $I$.

**Proof.** Let $\Sigma \in [\text{Sig}']$ and $\Gamma \subseteq \text{Sen}(\Sigma)$. We define $\Gamma_1 = \{ e \in \text{Sen}_1(\Sigma) \mid \Gamma \models e \}$. The set $\Gamma_1$ has an initial model $M_{\Gamma_1}$ which is $\mathcal{D}$-reachable. If $M \models \Gamma$ then $M \models \Gamma_1$. Since $M_{\Gamma_1}$ is the initial model of $\Gamma_1$, there exists a unique morphism $M_{\Gamma_1} \to M$. We only need to prove $M_{\Gamma_1} \models \Gamma$. Let $(\forall \chi) \rho \in \Gamma$, where $\Sigma \xrightarrow{\chi} \Sigma \in \mathcal{D}$ and $\rho \in \text{Sen}_1(\Sigma')$, and $N$ be a $\chi$-expansion of $M_{\Gamma_1}$. Since $M_{\Gamma_1}$ is $\mathcal{D}$-reachable, there exists a substitution $\theta : \chi \to 1_\Sigma$ such that $M_{\Gamma_1} \mid_\theta = N$. Since $(\forall \chi) \rho \models \theta(\rho)$, we obtain $\theta(\rho) \in \Gamma_1$. It follows $M_{\Gamma_1} \models \theta(\rho)$, and by the satisfaction condition $N \models \rho$.

**Corollary 32** In $\text{RHCL}$ and $\text{RHPOA}$, any set of sentences has an initial model.

**Proof.** We set the parameters of Theorem 31 for $\text{RHCL}$, the other case is similar. $I$ is $\text{CHCL}$, $I_1$ is the restriction of $\text{CHCL}$ to the quantifier-free sentences, $I'$ is $\text{RHCL}$ and $I'_1$ is the restriction of $\text{RHCL}$ to the quantifier-free sentences. By Corollary 30, every set of quantifier-free sentences in $\text{RHCL}$ has an initial model which is ground reachable. By Theorem 31, every set of sentences in $\text{RHCL}$ has an initial model.

4.3 Sufficient completeness

For any $\text{CHCL}$ signature $(S, F, F^c, P)$, recall that $F^S = \left\{ F_{s \rightarrow s} : s \in S \right\}$. A $\text{CHCL}$ presentation $((S, F, F^c, P), E)$ is called sufficient complete if for all $(S, F, F^c, P)$-models $M$ that satisfies $E$ we have $M \in \text{Mod}(S, F, F^c, P)$. Let $\text{Sig}_\text{CHCL}^{\text{SC}} \subseteq \text{Sig}_\text{CHCL}^{\text{pres}}$ be the full subcategory of sufficient complete presentations: $|\text{Sig}_\text{CHCL}^{\text{SC}}|$ consists of $\text{CHCL}$ presentations $((S, F, F^c, P), E)$ that are sufficient complete. We define the institution $\text{CHCL}^{\text{SC}}$ of sufficient complete presentations as the the restriction of $\text{CHCL}^{\text{pres}}$ to the sufficient complete presentations:

- $\text{Sen}_\text{CHCL}^{\text{SC}} : \text{Sig}_\text{CHCL}^{\text{SC}} \to \text{Set}$ and $\text{Mod}_\text{CHCL}^{\text{SC}} : \text{Sig}_\text{CHCL}^{\text{SC}} \to \text{Cat}^{op}$ are the restrictions of the $\text{Sen}_\text{CHCL}^{\text{pres}} : \text{Sig}_\text{CHCL}^{\text{pres}} \to \text{Set}$ and $\text{Mod}_\text{CHCL}^{\text{pres}} : \text{Sig}_\text{CHCL}^{\text{pres}} \to \text{Cat}^{op}$, respectively, to $\text{Sig}_\text{CHCL}^{\text{SC}}$;
- $\models_\text{CHCL}^{\text{SC}} = \{(\models_{\Sigma, E}) : (\Sigma, E) \in |\text{Sig}_\text{CHCL}^{\text{SC}}|\}$. 

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Similarly, one can define the institution \( \text{CHPOA}^{sc} \) of sufficient complete \( \text{CHPOA} \) presentations.

**Proposition 33** In \( \text{CHCL}^{sc} \) and \( \text{CHPOA}^{sc} \), any set of sentences has an initial model.

**Proof.** Let \( ((S,F,F^c,P),E) \) be a sufficient complete \( \text{CHCL} \) presentation, and \( \Gamma \) a set of \( (S,F,F^c,P) \)-sentences. By Corollary 32, \( \text{Mod}((S,F,F^c,P),E \cup \Gamma) \) has an initial model \( O_{E \cup \Gamma} \). Since \( ((S,F,F^c,P),E) \) is sufficient complete, \( O_{E \cup \Gamma} \in \text{Mod}(S,F,F^c,P) \). Since \( O_{E \cup \Gamma} \) is the initial model of \( \text{Mod}((S,F,F^c,P),E \cup \Gamma) \) and \( \text{Mod}((S,F,F^c,P),E \cup \Gamma) \) is the full subcategory of \( \text{Mod}((S,F,F^c,P),E \cup \Gamma) \), we obtain that \( O_{E \cup \Gamma} \) is the initial model of \( \text{Mod}((S,F,F^c,P),E \cup \Gamma) \).

The case of \( \text{CHPOA}^{sc} \) is similar. \( \square \)

5 Free models

In this section we give sufficient institution independent conditions for proving the existence of free models along sufficient complete presentation morphisms in constructor-based institutions. Our ideas are based upon [31] and especially [10]. More concrete, we notice that one of the conditions of the result in [10] is too restrictive and we replace it with a somewhat less restrictive variant, such that we can apply it in more cases. We also define the concept of weak diagrams for \( \text{CFOL} \) which is essential for proving the existence of free models and provides ideas to define it for other constructor-based institutions.

### 5.1 Institution-independent diagrams

In this paper we present a simplified version of institutional diagrams of [10], which omits the compatibility of this notion with the signature morphisms.

**Definition 34** An institution \( I = (\text{Sig},\text{Sen},\text{Mod},\vdash) \) has weak diagrams if for each signature \( \Sigma \) and \( \Sigma \)-model \( A \) there exists a signature morphism \( i_{\Sigma}(A) : \Sigma \rightarrow \Sigma_A \) called the extension of \( \Sigma \) via \( A \), and a set \( E_A \) of \( \Sigma_A \)-sentences called the diagram of the model \( A \) such that \( \text{Mod}(\Sigma_A,E_A) \) and the comma category \( A/\text{Mod}(\Sigma) \) are naturally isomorphic, i.e. the following diagram commutes by the isomorphism \( i_{\Sigma,A} \) natural in \( \Sigma \) and \( A \).

\[
\begin{array}{ccc}
\text{Mod}(\Sigma_A,E_A) & \overset{i_{\Sigma,A}}{\longrightarrow} & A/\text{Mod}(\Sigma) \\
\downarrow_{\text{Mod}(i_{\Sigma}(A))} & & \downarrow_{\text{forgetful}} \\
\text{Mod}(\Sigma) & & \\
\end{array}
\]

**Remark 35** Since the comma category \( A/\text{Mod}(\Sigma) \) has an initial model \( A \stackrel{1_A}{\rightarrow} A \) and \( i_{\Sigma,A} \) is an isomorphism of categories, the followings hold:

1. \( \text{Mod}(\Sigma_A,E_A) \) has an initial model \( A_A \) such that \( A_A \downarrow_{i_{\Sigma}(A)} = A \), and

2. for all \( M \in \text{Mod}(\Sigma_A,E_A) \), \( i_{\Sigma,A}(M) = (A_A \rightarrow M) \downarrow_{i_{\Sigma}(A)} \), where \( A_A \rightarrow M \) is the unique morphism from \( A_A \) to \( M \).

We define the diagrams for \( \text{CFOL} \). Let \( A \) be a \( \Sigma \)-model, where \( \Sigma = (S,F,F^c,P) \).
\begin{itemize}
\item \(\Sigma_A\) consists of the signature \(\Sigma\) enriched with all elements of loose sorts of \(A\), i.e. \(\Sigma_A = (S, F_A, F^c, P)\), where \((F_A)_{w \to s} = \begin{cases} F_{w \to s} \cup A_s & \text{if } w = \lambda \text{ and } s \in S^i \\ F_{w \to s} & \text{otherwise} \end{cases}\).
\item \(\tau_\Sigma(A) : (S, F, F^c, P) \mapsto (S, F_A, F^c, P)\) is the inclusion of signatures,
\item \(A_A\) is the \(\tau_\Sigma(A)\)-expansion of \(A\) interpreting each \(a \in A\) as \(a\), and
\item \(E_A\) is the set of all \(\Sigma_A\)-atoms satisfied by \(A_A\).
\end{itemize}

**Proposition 36** In \(\text{CFOL}\), for each \(\Sigma\)-model \(A\), where \(\Sigma = (S, F, F^c, P)\) there exists an isomorphism of categories \(\iota_{\Sigma, A} : \text{Mod}(\Sigma_A, E_A) \to A/\text{Mod}(\Sigma)\).

**Proof.** Let \(f_{A_A} : T_{(S,F_A,P)} \to A_A\) be the unique morphisms from \(T_{(S,F_A,P)}\) to \(A_A\). Since \(A\) is a reachable model, \(f_{A_A}\) is a surjection.

We show that \(A_A\) is the basic model of \(E_A\). Assume a \(\Sigma_A\)-model \(M\), and let \(f_M : T_{(S,F_A,P)} \to M\) be the unique morphism from \(T_{(S,F_A,P)}\) to \(M\).

1. If \(M \models E_A\) then \(\text{Ker}(f_{A_A}) \subseteq \text{Ker}(f_M)\), and there exists a unique model morphism \(h_M : A_A \to M\) such that \(f_{A_A} : h_M \to f_M\).

2. If there exists \(h_M : A_A \to M\), then by the initiality of \(T_{(S,F_A,P)}\), we have \(f_{A_A} : h_M = f_M\).

Since \(A_A\) is reachable, for all atomic \(\Sigma_A\)-sentences \(e\), \(A_A \models e\) implies \(M \models e\). It follows that \(M \models E_A\).

Since \(A_A\) is the basic model of \(E_A\), \(A_A/\text{Mod}(\Sigma) = A_A/\text{Mod}(\Sigma_A, E_A)\). Because \(A_A\) is the initial model of \(\text{Mod}(\Sigma, E_A)\), the forgetful functor \(F : A_A/\text{Mod}(\Sigma, E_A) \to \text{Mod}(\Sigma, E_A)\) is an isomorphism of categories. Also, the functor \(\iota_{\Sigma, A} : A_A/\text{Mod}(\Sigma) \to A/\text{Mod}(\Sigma)\) defined by \(\iota_{\Sigma, A}(A_A \mathcal{M} \to M) = (A_A \mathcal{M} \to M) \mid_{\Sigma(A)}\) on models, and by \(\iota_{\Sigma, A}(h_M \xrightarrow{h} h_N) = (h_M \xrightarrow{h} h_N) \mid_{\Sigma(A)}\) on morphisms, is an isomorphism of categories. Then \(\iota_{\Sigma, A} = F \circ \iota_{\Sigma, A}\) is also an isomorphism of categories as a composition of two isomorphisms.

In institutions with no constructors, such as \(\text{FOL}\), all sorts are loose, and the extension \(\tau_\Sigma(A)\) of a signature \(\Sigma\) via a \(\Sigma\)-model \(A\) is obtained by adding all the elements of the model \(A\) to the signature \(\Sigma\). This is the classical approach, which can be found, for example, in [10].

**Proposition 37** [12] If \(I\) is an institution with weak diagrams \(t\) then the institution of presentations \(I^{\text{pres}}\) has also elementary diagrams.

**Proof.** [Sketch] Let \((\Sigma, E)\) be a presentation, and \(A\) a \(\Sigma\)-model that satisfies \(E\). The extension of \((\Sigma, E)\) via \(A\) in \(I^{\text{pres}}\) is \(\tau_\Sigma(A) : (\Sigma, E) \to (\Sigma_A, \tau_\Sigma(A)(E))\) and the diagram of the \((\Sigma, E)\)-model \(A\) is \(E_A\).

**Corollary 38** \(\text{CHCL}^{sc}\) has weak diagrams.

**Proof.** Assume a sufficient complete \(\text{CHCL}\) presentation \((S, F, F^c, P), E)\), and let \(A\) be a \((S, F, F^c, P)\)-model that satisfies \(E\). Note that \((S, F_A, F^c, P), E)\) is sufficient complete, where \(\tau_{(S,F,F^c,P)}(A) : (S, F, F^c, P) \mapsto (S, F_A, F^c, P)\) is the extension of \((S, F, F^c, P)\) via \(A\). Indeed, for any \(M \in \text{Mod}((S, F_A, F^c, P), E)\), \(M \mid_{(S,F,P)} \in \text{Mod}((S, F, F^c, P), E)\); since \((S, F, F^c, P), E)\) is sufficient complete, \(M \mid_{(S,F,P)} \in \text{Mod}((S, F, F^c, P), E)\), which implies \(M \in \text{Mod}((S, F_A, F^c, P), E)\).

In \(\text{CHCL}^{sc}\), let
• \( \iota_{(S,F,F^c,P)}(A) : ((S,F,F^c,P), E) \mapsto ((S,F_A,F^c,P), E) \) be the extension of the presentation \((S,F,F^c,P), E)\) via \(A\), and

• the diagram \(E_A\) of \(A\) in \(\text{CHCL}\) be the diagram of \(A\) in \(\text{CHCL}^{sc}\).

By Proposition 37, the following diagram is commutative.

\[
\begin{array}{ccc}
\text{Mod}(((S,F,A,F^c,P), E), E_A) & \xrightarrow{\iota_{(S,F,A,F^c,P)}(A)} & A/\text{Mod}(((S,F,F^c,P), E)) \\
\text{Mod}(\iota_{(S,F,F^c,P)}(A)) & \searrow & \text{Forgetful} \\
\text{Mod}(((S,F,F^c,P), E)) & \nearrow &
\end{array}
\]

Note that the signature morphisms in \(\text{CFOL}\) do not preserve the loose sorts, in general. It follows that our notion of diagrams is not ‘functorial’ in the sense of [10], which implies that \(\text{CFOL}\) does not have elementary diagrams.

5.2 Liberality

An institution \(I = (\text{Sig}, \text{Sen}, \text{Mod}, \models)\) is \(L\)-liberal [20], where \(L \subseteq \text{Sig}\), if for all presentation morphisms \((\Sigma, E) \xrightarrow{\phi} (\Sigma', E')\) such that \(\Sigma \xrightarrow{\phi} \Sigma' \in L\) the functor

\[
\text{Mod}(\Sigma', E') \xrightarrow{\text{Mod}(\phi)} \text{Mod}(\Sigma, E)
\]

has a left adjoint.

The following theorem can be proved as in [10].

**Theorem 39** Let \(I = (\text{Sig}, \text{Sen}, \text{Mod}, \models)\) be an institution with weak diagrams. Assume two categories \(L, R \subseteq \text{Sig}\) of signature morphisms such that

1. \(\tau_L(A) \in R\), for all \(\Sigma \in |\text{Sig}|\) and \(A \in |\text{Mod}(\Sigma)|\),
2. \(\text{Sig}\) has \((L, R)\)-pushouts,
3. \(I\) is \((L, R)\)-semi-exact, and
4. each presentation has an initial model.

Then \(I\) is \(L\)-liberal.

**Proof.** [sketch] Let \(\phi : (\Sigma, E) \to (\Sigma', E')\) be a presentation morphism such that \(\Sigma \xrightarrow{\phi} \Sigma' \in L\), and let \(A\) be a \(\Sigma\)-model that satisfies \(E\). Consider the following pushout of signature morphisms

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\phi'} & \Sigma'' \\
\tau_L(A) \downarrow & & \downarrow \psi' \\
\Sigma & \xrightarrow{\phi} & \Sigma'
\end{array}
\]

We define the free model \(A^\phi\) to be \((A^\phi)_A|_{\psi'}\) where \((A^\phi)_A\) is the initial model of \(\phi'(E_A) \cup \psi'(E')\), and the the universal arrow \(\eta_A : A \to (A^\phi)\mid_{\psi}\) to be \((A_A \to (A^\phi)_A|_{\psi})\mid_{\tau_L(A)}\).
Below is a corollary of the above theorem.

**Corollary 40** CHCL$^{sc}$ is \( \ast \ast \ast \ast \ast \)-liberal.

**Proof.** We set the parameters of Theorem 39. The institution \( I \) is CHCL$^{sc}$, \( \mathcal{L} \) consists of sufficient complete presentation \( \ast \ast \ast \ast \ast \)-morphisms, and \( \mathcal{R} \) consists of sufficient complete presentation morphisms that add non-construct constants of loose sorts. By Proposition 33, each set of sentences in CHCL$^{sc}$ has an initial model. For each span of CHCL$^{sc}$ signature morphism \( ((S, F \cup C, F^c, P), E_C) \xrightarrow{\varphi} ((S, F, F^c, P), E) \xrightarrow{\varphi_2} ((S_1, F_1, F_1^c, P_1), E_1) \) such that \( \varphi \) is a \( \ast \ast \ast \ast \ast \)-morphism, and \( C \) is set of constants of loose sorts, the following pushout

\[
((S, F \cup C, F^c, P), E_C) \xrightarrow{\varphi_C} ((S_1, F_1 \cup C', F_1^c, P_1), E')
\]

is constructed as follows: \( ((S, F \cup C, F^c, P) \xrightarrow{\varphi_3} (S, F, F^c, P) \xrightarrow{\varphi_2} (S_1, F_1, F_1^c, P_1), (S, F \cup C, F^c, P) \xrightarrow{\varphi_1} (S_1, F_1 \cup C', F_1^c, P_1)) \) is a pushout of CFOL signature morphisms, and \( E' = \varphi_C(E_C) \cup \chi_1(E_1) \). Note that

- \( \chi_1 \) is the extension of \( (S_1, F_1, F_1^c, P_1) \) with constants from \( C' \), and
- \( \varphi_C \) works like \( \varphi \) on \( (S, F, F^c, P) \).

Let \( M_1 \in \text{Mod}((S_1, F_1, F_1^c, P_1), E_1) \) and \( M_2 \in \text{Mod}((S, F \cup C, F^c, P), E_C) \) such that \( M_1 \upharpoonright \varphi = M_2 \upharpoonright \chi \). By Proposition 26, there exists \( M' \in \text{Mod}(S_1, F_1 \cup C', F_1^c, P_1) \) such that \( M' \upharpoonright \chi_1 = M_1 \) and \( M' \upharpoonright \varphi_2 = M_2 \). By the satisfaction condition, \( M' \models E' \). It follows that CHCL$^{sc}$ is \( (\mathcal{L}, \mathcal{R}) \)-semi-exact.

By Theorem 39, \( \varphi \) has a left adjoint. \( \square \)

Similar results can be formulated for CHPOA too.

The example below shows that CHCL$^{sc}$ is not liberal, in general.

**Example 41** Consider the following example of CHCL signature morphism:

\[
\text{sort Triv} \xrightarrow{\varphi} \text{sort Bool}
\]

\[
\text{op true : Bool\{constr\}}
\]

\[
\text{op false : Bool\{constr\}}
\]

where \( \text{TRIV} = (\{\text{Triv}\}, \emptyset, \emptyset, \emptyset) \) and \( \text{BOOL} = (\{\text{Bool}\}, \{\text{true} : \rightarrow \text{Bool}, \text{false} : \rightarrow \text{Bool}, \{\text{true} : \rightarrow \text{Bool}, \text{false} : \rightarrow \text{Bool}\}, \emptyset) \). Let \( A \) be a TRIV-model that consists of one element \( (a : \text{Triv}) \). Assume that \( A^\varphi \) is the free model along \( \varphi \) generated by \( A \), and \( \eta_A : A \rightarrow A^\varphi \upharpoonright \varphi \) is an universal arrow. Let \( T_{BOOL} \) be the term BOOL-model, and \( f_1 : A \rightarrow T_{BOOL} \upharpoonright \varphi \) defined by \( f_1(a) = \text{true} \). By the universality of \( \eta_A \), there exists a model morphism \( h_1 : A^\varphi \rightarrow T_{BOOL} \) such that \( \eta_A ; h_1 \upharpoonright \varphi = f_1 \). If \( A^\varphi_{\text{true}} = A^\varphi_{\text{false}} \), then \( h_1(A^\varphi_{\text{true}}) = h_1(A^\varphi_{\text{false}}) \), which implies \( T_{BOOL} \upharpoonright \text{true} = (T_{BOOL}) \upharpoonright \text{false} \), a contradiction. Since \( A^\varphi \) is reachable and \( A^\varphi_{\text{true}} \neq A^\varphi_{\text{false}} \), \( h_1 \) is an isomorphism. We have \( \eta_A(a) = h_1^{-1}(f_1(a)) = h_1^{-1}((T_{BOOL}(\text{true}) = A^\varphi_{\text{true}}) \). Let \( f_2 : A \rightarrow T_{BOOL} \upharpoonright \varphi \) defined by \( f_2(a) = \text{false} \). By the universality of \( \eta_A \), there exists a model morphism \( h_2 : A^\varphi \rightarrow T_{BOOL} \) such that \( \eta_A ; h_2 \upharpoonright \varphi = f_2 \). Since \( A^\varphi \) is reachable and \( A^\varphi_{\text{true}} \neq A^\varphi_{\text{false}} \), \( h_2 \) is an isomorphism. We have \( \eta_A(a) = h_2^{-1}(f_2(a)) = h_2^{-1}((T_{BOOL}(\text{false}) = A^\varphi_{\text{false}}) \). We obtained \( A^\varphi_{\text{true}} = \eta_A(a) = A^\varphi_{\text{false}} \), a contradiction. Therefore, \( \eta_A \) is not an universal arrow.
6 Conclusions

We investigate the existence of pushouts in the concrete category of CFOL signature morphisms. This research is important as the pushout construction constitutes the basis of building large specifications from smaller ones. In connection to the semantics of the complex system development we conduct an institution dependent study of the semi-exactness property in logics with constructors.

We proved the existence of initial models of any set of sentences in arbitrary ‘reachable’ institutions. We do not use factorization systems as [2, 33, 1] or inclusion systems as [10]. We simply require that all sets of atoms are basic and the models defining the sets of atoms as basic sets of sentences are ground reachable. Note that the restriction to reachable models can be also obtained by allowing infinitary disjunctions. For example, if $\Sigma = (\{\text{Nat}\}, \{0 :\to \text{Nat}, s_- : \text{Nat} \to \text{Nat}\}, \emptyset)$ is a first-order signature then the restriction to the models that are reachable by the constructors $0 :\to \text{Nat}$ and $s_- : \text{Nat} \to \text{Nat}$ can be obtained by using the infinitary sentence $(\forall x) \bigvee_{n \in \text{Nat}} x = s^0n$, where $s^0n$ is the term obtained by applying $n$ times the function successor to $0$. It follows that the class of models of an implicational theory in logics with constructors do not form a quasi-variety, and do not fall into the framework of [2, 33, 10]. Therefore new initiality results are required for ‘reachable’ institutions. Initiality is then extended to constructor-based institutions via sufficient completeness.

Free models along presentation morphisms provide semantics for the modules with initial denotation in algebraic specifications. In order to provide an institution independent proof of liberality, we define the diagrams for logics with constructors. Given a $\Sigma$-model $A$, the extension of $\Sigma$ via $A$ is obtained by adding all elements of loose sorts of $A$ to the signature $\Sigma$. Taking into consideration that constructor-based institutions do not have all pushouts, freeness follows as in [10], and then it is instantiated to the institution of sufficient complete presentations.

Our abstract results can be applied to other frameworks such as order sorted algebra [22, 21], higher order logic [6, 24] with intensional Henkin semantics, and partial algebra [28, 5]. In the future we are planning to study interpolation in logics with constructors. Also an axiomatizability result is desired in this case.

References


