An Indexed System for Multiplicative Additive Polarized Linear Logic

Masahiro Hamano[1]* and Ryo Takemura[2]

[1] National Institute of Informatics, Tokyo 101-8430, Japan
and SFC Research Institute, Fujisawa 252-8520, Japan
hamano@jaist.ac.jp

[2] Department of Philosophy, Keio University, Tokyo 108-8345, Japan
takemura@abelard.flet.keio.ac.jp

Abstract. We present an indexed logical system $\mathsf{MALLP}(I)$ for Laurent's multiplicative additive polarized linear logic (MALLP) [14]. The system is a polarized variant of Bucciarelli-Ehrhard's indexed system for multiplicative additive linear logic [4]. Our system is derived from a web-based instance of Hamano-Scott's denotational semantics [12] for MALLP. The instance is given by an adjoint pair of right and left multi-pointed relations. In the polarized indexed system, subsets of indexes for I work as syntactical counterparts of families of points in webs. The rules of $\mathsf{MALLP}(I)$ describe (in a proof-theoretical manner) the denotational construction of the corresponding rules of MALLP. We show that $\mathsf{MALLP}(I)$ faithfully describes a denotational model of MALLP by establishing a correspondence between the provability of indexed formulas and relations that can be extended to (non-indexed) proof-denotations.

1 Introduction

In their study of logical relations and the denotational completeness of linear logic (LL), Bucciarelli and Ehrhard [4] introduced an indexed system $\mathsf{MALL}(I)$ for multiplicative additive linear logic (MALL). In their sequel [5], this system was extended into full fragment $\mathsf{LL}(I)$. The status of this indexed syntactical system is noteworthy as it stems from relational semantics **Rel**, which is one of the simplest denotational semantics for LL. Bucciarelli-Ehrhard's indexed system is designed so that each formula corresponds to a relation and each logical rule corresponds to a denotational interpretation of the corresponding rule in LL. The crucial ingredient for this correspondence is the *domains* of formulas: Each formula A of the indexed system is equipped with a domain $d(A)$ which enumerates the *locations* of points in the corresponding relation on $|A|$. Their indexed system enjoys *basic property*, which establishes a relationship between the provability of indexed formulas and the sub-definability of the corresponding relations in the denotational semantics of LL. Later A. Bruasse-Bac [3] extended the indexed

* The first author is supported by Grant-in-Aid for Scientific Reserach (C) (19540145) of JSPS.

system to the second order by adapting relational semantics to Girard's objects of variable type.

Another logical framework in which locations play a key role is that of Girard's ludics [11]. In ludics one abstracts locations, where syntactical formulas give its occurrences through construction of proofs. Several similarities (though not rigorous) have been noticed with the indexed system discussed above. Among them, one observes a common idea in the underlying *hypersequentialized calculus* [10] on which ludics is founded. In the hypersequenatialized calculus, which is a variant of MALL, each formula is equipped with a coherent space (rather than the more primitive notion of relation), and each inference rule is defined in terms of the construction of cliques for the equipped spaces. While sharing such similar syntactical construction with reflecting semantics, the key ingredient, peculiar to ludics, is polarity (see [11]). Polarity was introduced by Girard [8]. Through Laurent's formalization of polarized linear logic LLP [14], polarity turns out to be an important parameter controlling linear proof-theory. Most fundamentally, polarity enables categorization of Andreoli's [1] dual properties of *focalization* and *reversibility* of connectives for proof-search in LL. In polarized linear logic, reversible and focusing connectives are characterized simply as *negative* and *positive*. As is the case with ludics, polarity is also a crucial tool for handling locations game-theoretically. Laurent [15] establishes how polarity dominates game-theoretical computational models arising from LL. Polarity and locations are becoming a crucial tandem for understanding the computational meaning of LL.

Given the above, a natural question arises: Is there any polarized variant of Bucciarelli-Ehrhard's indexed system that naturally accommodates polarity in its indexes? The existence of such a system would guarantee that polarity is a stable core controlling both syntax and semantics uniformly, whose combination is at the heart of indexed systems. In this paper, we answer this question affirmatively by presenting an indexed system $\mathsf{MALLP}(I)$ for the multiplicative additive fragment MALLP [14] of Laurent's LLP. Our indexed system is designed by means of multi-pointed relational semantics, a web-based instance of Hamano-Scott's denotational semantics [12] for MALLP. The cornerstone of our multi-pointed relational semantics is a pair of contravariant categories $\mathbf{PRel}_l$ and $\mathbf{PRel}_r$. Left (resp. right) multi-pointed relational semantics $\mathbf{PRel}_l$ (resp. $\mathbf{PRel}_r$) consist of multi-pointed sets (i.e., sets with distinguished multi-points) and of relations preserving the distinguished elements from left (resp. from right). Polarity shifting operators are then interpreted as a pair of adjoint functors between the contravariant pair. In addition to the adjunction, the usual relations provide *bimodule* $\widehat{\mathbf{Rel}}$ so that it is closed under left (resp. right) compositions from $\mathbf{PRel}_r$ (resp. $\mathbf{PRel}_l$). Being a polarized variant of $\mathbf{Rel}$, our framework $(\langle \mathbf{PRel}_l, \mathbf{PRel}_r\rangle, \widehat{\mathbf{Rel}})$ provides one of the simplest denotational semantics for MALLP.

Our $\mathsf{MALLP}(I)$, designed from multi-pointed relational semantics, is a polarized variant of Bucciarelli-Ehrhard's $\mathsf{MALL}(I)$: the usual multiplicative additive rules for the former coincide with those for the latter under the polarity constraint. It is remarkable that in our $\mathsf{MALLP}(I)$ there arise, corresponding to $\downarrow$, parameterized $\downarrow_K$-rules with subsets K's of I. Each $\downarrow_K$-rule comes equipped with a side condition on domains by reflecting the corresponding categorical adjunction. In $\mathsf{MALLP}(I)$ polarity behaves compatibly with indexes since focusing/reversible properties are captured by indexed positive/negative connectives. $\mathsf{MALLP}(I)$ formulas correspond bijectively to relations arising in our relational denotational semantics. The main goal of this paper is to establish a basic property (Theorem 1), that is a polarized version of Bucciarelli-Ehrhard's property established in [4]. This basic property states that a family of points is contained in a (denotation of) MALLP proof if and only if the corresponding $\mathsf{MALLP}(I)$ formula is provable in the indexed system.

2 Multi-pointed relational semantics for MALLP

In this section, we introduce *multi-pointed relational* semantics, which is a variant of relational semantics. Multi-pointed relations are shown to provide a simple denotational semantics for polarized multiplicative additive linear logic (MALLP). (See [14] for the syntax of MALLP.) This is a polarized analogy of the category **Rel** of relations, which, as is well-known, provides one of the simplest denotational semantics for usual multiplicative additive linear logic (MALL) (see [5, 2] for **Rel**). Let us begin by defining a pair of categories $\mathbf{PRel}_r$ and $\mathbf{PRel}_l$ of right and left multi-pointed relations. The right/left pair corresponds to negative/positive polarity of MALLP.

Notation: When X and Y are sets, we denote by $X \times Y$ the cartesian product of them; and by $X + Y$ the disjoint union of them, i.e., $\{1\} \times X \cup \{2\} \times Y$.

Definition 1 ($\mathbf{PRel}_r$ and $\mathbf{PRel}_l$). The categories $\mathbf{PRel}_r$ of right-multi-pointed relations and $\mathbf{PRel}_l$ of left-multi-pointed relations are defined as follows:

An object A is a pair $(|A|, \mathsf{mp}(A))$, where $|A|$ is a set called the web of A, and $\mathsf{mp}(A)$ is a finite subset of $|A|$. Moreover if $|A| \neq \emptyset$, then $\mathsf{mp}(A) \neq \emptyset$. Each element of $\mathsf{mp}(A)$ is called a distinguished element of A.

A morphism from A to B is a relation $R \subseteq |A| \times |B|$ which satisfies

$$\mathsf{mp}(A) = R[\mathsf{mp}(B)] \text{ for } \mathbf{PRel}_r$$
$$[\mathsf{mp}(A)]R = \mathsf{mp}(B) \text{ for } \mathbf{PRel}_l$$

where, for sets $X \subseteq |A|$ and $Y \subseteq |B|$, and for a relation $R \subseteq |A| \times |B|$,

$$R[Y] = \{a \mid \exists b \in Y, (a, b) \in R\} \quad \text{and} \quad [X]R = \{b \mid \exists a \in X, (a, b) \in R\}.$$

Compositions for each category are *relational* so that given $R : A \to B$ and $S : B \to C$, $S \circ R = \{(a, c) \mid \exists b \in B,\ (a, b) \in R \text{ and } (b, c) \in S\} : A \to C$. This composes in each categories because it holds that $R[S[\mathsf{mp}(C)]] = (S \circ R)[\mathsf{mp}(C)]$ and $[[\mathsf{mp}(A)]R]S = [\mathsf{mp}(A)](S \circ R)$.

There are obviously forgetful functors $|\ \ |$ both from $\mathbf{PRel}_r$ and $\mathbf{PRel}_l$ to the category **Rel** of relations.

The two categories $\mathbf{PRel}_l$ and $\mathbf{PRel}_r$ are contravariantly equivalent. This is given by $(\ )^{\perp}$, which leaves the objects invariant but reverses the relations and compositions:

$$(\ )^{\perp} : (\mathbf{PRel}_r)^{op} \simeq \mathbf{PRel}_l$$

Starting from the contravariantly equivalent pair, we give a denotational semantics for polarized MALL by the following Definition 2. See Section 3 and Definition A.1 in Hamano-Scott [12] for the definition of categorical semantics for MALLP. Note that, in the following, their general framework based on Definition A.1 is adapted so that *bimodules* play a role between the two categorical pair. See also Cockett-Seely [6] for bimodules in polarized category. Our bimodule may be seen as a concrete instance of Example 3.0.2 of [6].

Definition 2 (Polarity-changing functors and bimodule).

– *The functors* $\uparrow$ *and* $\downarrow$ are defined as follows:

$$\downarrow : \mathbf{PRel}_l \longrightarrow \mathbf{PRel}_r \tag{1}$$

$$\uparrow : \mathbf{PRel}_r \longrightarrow \mathbf{PRel}_l \tag{2}$$

(On objects) $|{\downarrow}A| = |{\uparrow}A| = \{*\} + |A|$ with $\mathsf{mp}({\downarrow}A) = \mathsf{mp}({\uparrow}A) = \{*\}$,
(On morphisms) Given a morphism $R \subseteq |A| \times |B|$,

$${\downarrow}R := {\uparrow}R := R + \{(*, *)\},$$

which are morphisms of ${\downarrow}A \to {\downarrow}B$ and of ${\uparrow}A \to {\uparrow}B$.

The unique element of $\mathsf{mp}({\downarrow}A)$ (resp. $\mathsf{mp}({\uparrow}A)$) is often denoted by $*_{\downarrow}$ (resp. by $*_{\uparrow}$) to stress that the distinguished point arises to interpret the $\downarrow$ (resp. the $\uparrow$). The above definition yields the strict form $({\downarrow}N)^{\perp} = {\uparrow}N^{\perp}$ and $({\uparrow}P)^{\perp} = {\downarrow}P^{\perp}$ of De Morgan duality between $\downarrow$ and $\uparrow$.

Note that the functors (1) and (2) factor through $|\ \ |$ to $\mathbf{Rel}$ by inducing the functors from $\mathbf{Rel}$ respectively to $\mathbf{PRel}_r$ and to $\mathbf{PRel}_l$. By abuse of notation, the induced functors are also denoted by $\downarrow$ and $\uparrow$, respectively. See the diagram depicting Lemma 1 below, where the clockwise and the anticlockwise triangles show the factorizations.

– A *bimodule* $\widehat{\mathbf{Rel}}(P, N)$ consists of maps of the form $P \to N$ for object $P \in \mathbf{PRel}_r$ and $N \in \mathbf{PRel}_l$ so that they are closed under left (respectively, right) composition of morphisms from $\mathbf{PRel}_l$ (respectively from $\mathbf{PRel}_r$). A bimodule is thus characterized by a profunctor:

$$\widehat{\mathbf{Rel}}(-,-) : (\mathbf{PRel}_r)^{op} \times \mathbf{PRel}_l \to \mathbf{Set}$$

so that each instantiation determines a set of these maps. We define

$$\widehat{\mathbf{Rel}}(P, N) = \mathbf{Rel}(|P|, |N|) \tag{3}$$

That is, the maps $P \to N$ consist of *usual relations* of $P \to N$ (i.e., of morphism of **Rel**).

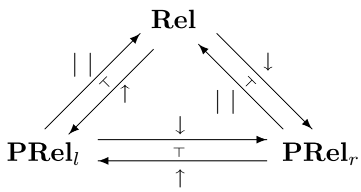
Then the bimodule obviously satisfies:

$$\widehat{\mathbf{Rel}}(\mathbf{1}, P^{\perp} \times N) \cong \widehat{\mathbf{Rel}}(P, N)$$

where $\mathbf{1}$ is an object of $\mathbf{PRel}_r$ such that $|\mathbf{1}| = \{*\}$ and $\times$ is the cartesian product of **Rel** for objects of $\mathbf{PRel}_l$.

Finally, the following series of adjunctions are crucial in order to obtain a polarized category:

Lemma 1 (Adjunctions). *The following adjunctions hold:*



That is, for every object $P \in \mathbf{PRel}_r$ and $N \in \mathbf{PRel}_l$, there are natural isomorphisms

$$\mathbf{PRel}_l(\uparrow P, N) \quad \cong \quad \widehat{\mathbf{Rel}}(P, N) \quad \cong \quad \mathbf{PRel}_r(P, \downarrow N) \tag{4}$$

Proof. We show the right-isomorphism (dually for the left). Let $R \in \mathbf{PRel}_r(P, \downarrow N)$. Since R is right-multi-pointed, we have $\mathsf{mp}(P) = R[\mathsf{mp}(\downarrow N)]$, which implies that if $(a, *_{\downarrow}) \in R$ then $a \in \mathsf{mp}(P)$ and that $\forall p \in \mathsf{mp}(P)$, $(p, *_{\downarrow}) \in R$. Hence, R is written as $\{(p, *_{\downarrow}) \mid p \in \mathsf{mp}(P)\} \cup R'$ for a unique $R' \in \widehat{\mathbf{Rel}}(P, N)$. The correspondence from R to R' is bijective and gives the natural isomorphism. ■

Proposition 1 (Multi-pointed relational model for MALLP). *The pair $\langle \mathbf{PRel}_l, \mathbf{PRel}_r \rangle$ together with the module $\widehat{\mathbf{Rel}}$ forms a polarized category, and hence a denotational semantics for* MALLP. *This polarized category is denoted by* $(\langle \mathbf{PRel}_l, \mathbf{PRel}_r \rangle, \widehat{\mathbf{Rel}})$.

The categorical framework presented so far yields the following interpretation of formulas and of proofs of MALLP.

Definition 3 (Interpretation of MALLP formulas). Positive (resp. negative) formulas A of MALLP are interpreted by objects $(|A|, \mathsf{mp}(A))$ in $\mathbf{PRel}_r$ (resp. in $\mathbf{PRel}_l$):

- $|\mathbf{1}| = \{*\}$, $|\bot| = \{*\}$ and $\mathsf{mp}(\mathbf{1}) = \mathsf{mp}(\bot) = \{*\}$.
- $|\top| = |\mathbf{0}| = \emptyset$ and $\mathsf{mp}(\top) = \mathsf{mp}(\mathbf{0}) = \emptyset$.
- $|P \otimes Q| = |P| \times |Q|$, $|M \parr N| = |M| \times |N|$ and $\mathsf{mp}(X \otimes Y) = \mathsf{mp}(X \parr Y) = \mathsf{mp}(X) \times \mathsf{mp}(Y)$.
- $|P \oplus Q| = |P| + |Q|$, $|M \,\&\, N| = |M| + |N|$ and $\mathsf{mp}(X \oplus Y) = \mathsf{mp}(X \,\&\, Y) = \mathsf{mp}(X) + \mathsf{mp}(Y)$.
- $|\uparrow P| = \{*\} + |P|$, $|\downarrow N| = \{*\} + |N|$ and $\mathsf{mp}(\downarrow N) = \mathsf{mp}(\uparrow P) = \{*\}$.

Definition 4 (Interpretation of proofs). Every MALLP-proof π is interpreted in $(\langle \mathbf{PRel}_l, \mathbf{PRel}_r \rangle, \widehat{\mathbf{Rel}})$ by π^*, which is a map either in $\mathbf{PRel}_r$ or in $\widehat{\mathbf{Rel}}$ depending on whether the end sequent contains a positive formula or not, respectively.

Rules for the usual linear logic are the same as [4]. In the following, for a sequence $\mathcal{M}$ of negative formulas $N_1, \ldots, N_n$, the sequence $\mathcal{M}$ (resp. $\mathcal{M}^\perp$) is identified with the object $N_1 \parr \cdots \parr N_n$ of $\mathbf{PRel}_l$ (resp. $N_1^\perp \otimes \cdots \otimes N_n^\perp$ of $\mathbf{PRel}_r$).

- When π is $\vdash P^\perp, P$, we define $\pi^* = \{(a, a) \mid a \in |P|\} \quad \in \mathbf{PRel}_r(P, P)$
- When π is $\dfrac{\overset{\pi_1}{\vdash \Delta, N} \quad \overset{\pi_2}{\vdash \Lambda, N^\perp}}{\vdash \Delta, \Lambda}\,cut$, we define

  $$\pi^* = \{(\delta, \lambda) \mid \exists a,\ (\delta, a) \in \pi_1^* \text{ and } (\lambda, a) \in \pi_2^*\}$$

  Since π_2^* is always a map in $\mathbf{PRel}_r$, π^* is a map either in $\mathbf{PRel}_r$ or in $\widehat{\mathbf{Rel}}$, depending on whether Δ contains a positive formula or not, respectively.
- When π is $\dfrac{\overset{\pi_1}{\vdash \mathcal{M}, N}}{\vdash \mathcal{M}, \downarrow N}\downarrow$, we define

  $$\pi^* = \pi_1^* \cup \{(p, *_\downarrow) \mid p \in \mathsf{mp}(\mathcal{M})\} \quad \in \mathbf{PRel}_r(\mathcal{M}^\perp, \downarrow N)$$

  π^* is obtained from $\pi_1^* \in \widehat{\mathbf{Rel}}(\mathcal{M}^\perp, N)$ by the right adjunction of (4).
- When π is $\dfrac{\overset{\pi_1}{\vdash \mathcal{M}, P}}{\vdash \mathcal{M}, \uparrow P}\uparrow$, we define $\pi^* = \pi_1^* \quad \in \widehat{\mathbf{Rel}}(\mathcal{M}^\perp, \uparrow P)$.

  π^* is obtained from $\pi_1^* \in \mathbf{PRel}_r(\mathcal{M}^\perp, P)$ by composing $\eta : P \to \uparrow P$, which is the unit of the left adjunction of (4); i.e., $\eta = \{(p, p) \mid p \in |P|\}$. Note that the composition of η acts identically on morphisms.

Our simple interpretation above provides a nice framework for discriminating the two proofs of Example 1 below.

Example 1 (Denotations of proofs in $(\langle\ \mathbf{PRel}_l,\ \mathbf{PRel}_r \rangle, \widehat{\mathbf{Rel}})$). Let us consider a MALLP-sequent $\vdash \uparrow\downarrow\uparrow\mathbf{1}, \uparrow\downarrow\bot$ and two different proofs π_1 and π_2 for this:

$$\pi_1 = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathbf{1}, \bot}{\vdash \uparrow\mathbf{1}, \bot}\uparrow}{\vdash \downarrow\uparrow\mathbf{1}, \bot}\downarrow}{\vdash \uparrow\downarrow\uparrow\mathbf{1}, \bot}\uparrow}{\vdash \uparrow\downarrow\uparrow\mathbf{1}, \downarrow\bot}\downarrow}{\vdash \uparrow\downarrow\uparrow\mathbf{1}, \uparrow\downarrow\bot}\uparrow \qquad \begin{array}{c} \{(1, B)\} \\ \{(1, B)\} \\ \{(\downarrow_a, B), (1, B)\} \\ \{(\downarrow_a, B), (1, B)\} \\ \{(\uparrow_a, \downarrow_c), (\downarrow_a, B), (1, B)\} \\ \{(\uparrow_a, \downarrow_c), (\downarrow_a, B), (1, B)\} \end{array}$$

$$\pi_2 = \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \mathbf{1}, \bot}{\vdash \uparrow\mathbf{1}, \bot}\uparrow}{\vdash \uparrow\mathbf{1}, \downarrow\bot}\downarrow}{\vdash \uparrow\mathbf{1}, \uparrow\downarrow\bot}\uparrow}{\vdash \downarrow\uparrow\mathbf{1}, \uparrow\downarrow\bot}\downarrow}{\vdash \uparrow\downarrow\uparrow\mathbf{1}, \uparrow\downarrow\bot}\uparrow \qquad \begin{array}{c} \{(1, B)\} \\ \{(1, B)\} \\ \{(\uparrow_b, \downarrow_c), (1, B)\} \\ \{(\uparrow_b, \downarrow_c), (1, B)\} \\ \{(\downarrow_a, \uparrow_c), (\uparrow_b, \downarrow_c), (1, B)\} \\ \{(\downarrow_a, \uparrow_c), (\uparrow_b, \downarrow_c), (1, B)\} \end{array}$$

The right-hand side of each subproof designates its interpretation, where we take $|\uparrow\downarrow\uparrow\mathbf{1}| = \{\uparrow_a, \downarrow_a, \uparrow_b, 1\}$ and $|\uparrow\downarrow\bot| = \{\uparrow_c, \downarrow_c, B\}$ so that $\downarrow_a = \uparrow_c$, $\uparrow_b = \downarrow_c$ and $1 = B$. As is seen above, the two proofs π_1 and π_2 are interpreted by different relations.

3 Indexed multiplicative additive polarized linear logic MALLP($\boldsymbol{I}$)

In this section, we present an indexed logical system MALLP(I), which is a conservative extension of MALLP. The syntactical system MALLP(I) arises from our multi-pointed relational semantics for MALLP, presented in Section 2. Each rule of MALLP(I) is designed so that it describes the denotational construction of the corresponding rule of MALLP in $(\langle\, \mathbf{PRel}_l,\ \mathbf{PRel}_r\rangle, \widehat{\mathbf{Rel}})$. Our design is inspired by Bucciarelli-Ehrhard's system [4] of MALL(I), just as their system stems from the denotational semantics **Rel** for MALL. By reflecting the adjunctions of the polarized category of Section 2, our polarity shifting rule $\downarrow_K$ for each $K \subseteq I$ is accompanied by a certain side condition. Let us begin by defining formulas of MALLP(I).

Let I be an index set which is fixed, once and for all. Each formula A of MALLP(I) is associated with a set $d(A) \subseteq I$, called the *domain* of A.

Definition 5 (Formulas and domains). *Positive* and *negative* formulas of domain J (denoted simply as P_J and N_J, respectively) are defined by the following grammar: For any sets $J, K, L \subseteq I$ such that $K \cap L = \emptyset$,

$$P_J ::= \mathbf{1}_J \mid \mathbf{0}_\emptyset \mid P_J \otimes P_J \mid P_K \oplus P_L \mid {\downarrow_K} N_L \ \text{(positive formula)}$$
$$N_J ::= \bot_J \mid \top_\emptyset \mid N_J \parr N_J \mid N_K \,\&\, N_L \mid {\uparrow_K} P_L \ \text{(negative formula)}$$

Note that, in contrast to the MALL connectives $\otimes, \parr, \oplus, \&$, the polarity shifting connectives $\downarrow_K$ and $\uparrow_K$ are provided for each K to have their own domains. To be precise, ${\downarrow_K} N$ and ${\uparrow_K} P$ are defined as follows (the other connectives are the same as those in [4]):

- For each $K \subseteq I$, we introduce two new connectives $\downarrow_K$ and $\uparrow_K$, both of which have K as their domains, i.e., $d(\downarrow_K) = d(\uparrow_K) = K$.
- For $L \subseteq I$ disjoint with K, if N is a negative formula with $d(N) = L$, then ${\downarrow_K} N$ is a positive formula with $d({\downarrow_K} N) = K + L$.
- For $L \subseteq I$ disjoint with K, if P is a positive formula with $d(P) = L$, then ${\uparrow_K} P$ is a negative formula with $d({\uparrow_K} P) = K + L$.

For any MALLP(I)-formula A with $d(A) = J$, we define its negation $A^\perp$ with $d(A^\perp) = J$ in the usual way, using the De Morgan duality for MALLP-formulas. A J-sequent is an expression of the shape $\vdash_J \Delta$ where Δ is a (possibly empty) sequence of MALLP(I)-formulas of domains J (denoted as $d(\Delta) = J$).

Definition 6 (Restriction). For a MALLP(I)-formula A with $d(A) = J$, and for $K \subseteq I$, we define the *restriction* of A by K, denoted by $A{\upharpoonright_K}$, which is a MALLP(I)-formula of domain $J \cap K$ as follows:

- $\top_\emptyset{\upharpoonright_K} = \top_\emptyset$ and $\mathbf{0}_\emptyset{\upharpoonright_K} = \mathbf{0}_\emptyset$;
- $\bot_J{\upharpoonright_K} = \bot_{J\cap K}$ and $\mathbf{1}_J{\upharpoonright_K} = \mathbf{1}_{J\cap K}$;
- $(P \otimes Q){\upharpoonright_K} = P{\upharpoonright_K} \otimes Q{\upharpoonright_K}$, $(N \parr M){\upharpoonright_K} = N{\upharpoonright_K} \parr M{\upharpoonright_K}$, $(P \oplus Q){\upharpoonright_K} = P{\upharpoonright_K} \oplus Q{\upharpoonright_K}$, $(N \,\&\, M){\upharpoonright_K} = N{\upharpoonright_K} \,\&\, M{\upharpoonright_K}$, $({\uparrow_J} P){\upharpoonright_K} = {\uparrow_{J\cap K}} P{\upharpoonright_K}$, and $({\downarrow_J} N){\upharpoonright_K} = {\downarrow_{J\cap K}} N{\upharpoonright_K}$.

Trivially $A^{\perp}\restriction_K = (A\restriction_K)^{\perp}$. If Δ is a sequence of $\mathsf{MALLP}(I)$-formulas $A_1, \ldots, A_n$ of domains J, we define $\Delta\restriction_K = A_1\restriction_K, \ldots, A_n\restriction_K$ so that $d(\Delta\restriction_K) = d(\Delta) \cap K$.

In order to introduce polarity shifting rule $\downarrow_K$ for $\mathsf{MALLP}(I)$, we give the following definition:

Definition 7 ($d(\partial\mathcal{M})$). For a sequence $\mathcal{M}$ of negative formulas, the domain $d(\partial\mathcal{M})$, which is a subset of $d(\mathcal{M})$, is defined as follows:
First let $\dagger_1, \ldots, \dagger_n$ denote all the outermost $\uparrow$'s of $\mathcal{M}$ and all the $\perp$'s outside any scope of polarity shifting operators. Then $\mathcal{M}$ is written as $\underline{\mathcal{M}}[\dagger_1 P_1, \ldots, \dagger_n P_n]$, where P_m is a positive formula (empty if $\dagger_m$ is $\perp$) and $\underline{\mathcal{M}}[*_1, \ldots, *_n]$ is an expression made from $*_1, \ldots, *_n$ by applying only negative connectives $⅋$ and &. Note that each comma of $\mathcal{M}$ is identified with $⅋$. We define

$$d(\partial\mathcal{M}) := |\underline{\mathcal{M}}|[d(\dagger_1), \ldots, d(\dagger_n)],$$

where $|\underline{\mathcal{M}}|$ is the (set-theoretical) expression resulting from the expression $\underline{\mathcal{M}}[*_1, \ldots, *_n]$ by replacing $⅋$ and & respectively with $\cap$ and $\cup$. Obviously $d(\partial\mathcal{M}) \subseteq d(\mathcal{M})$ since $d(\mathcal{M}) = |\underline{\mathcal{M}}|[d(\dagger_1 P_1), \ldots, d(\dagger_n P_n)]$.

Remark 1 ($d(\partial\mathcal{M})$ for distributed $\mathcal{M}$). By distributing $⅋$ over &, every $\mathcal{M}$ is rewritten as $M_1 \,\&\, \cdots \,\&\, M_n$ so that each M_i is $⅋(\dagger_{ij} P_{ij})$. Then we can more directly define $d(\partial\mathcal{M}) = \bigcap_j d(\dagger_{1j}) + \cdots + \bigcap_j d(\dagger_{nj})$.

We introduce the inference rules of $\mathsf{MALLP}(I)$, which consist of the polarity shifting rules on top of Bucciarelli-Ehrhard's rules [4] of $\mathsf{MALL}(I)$ with the polarity constraint.

Definition 8 (Inference rules of $\mathsf{MALLP}(I)$). Inference rules of $\mathsf{MALLP}(I)$ are defined as follows (the exchange rule is left implicit):
Axioms[3] and cut:

$$\vdash_J \mathbf{1}_J \qquad\qquad \vdash_\emptyset \Delta, \top_\emptyset \qquad\qquad \frac{\vdash_J \Delta, N \quad \vdash_J \Lambda, N^{\perp}}{\vdash_J \Delta, \Lambda}\ cut$$

For $\top$-axiom $\vdash_\emptyset \Delta, \top_\emptyset$, its context Δ contains at most one positive formula.
Multiplicative rules:

$$\frac{\vdash_J \Delta}{\vdash_J \Delta, \perp_J}\ \perp_J \qquad\qquad \frac{\vdash_J \Delta, P \quad \vdash_J \Lambda, Q}{\vdash_J \Delta, \Lambda, P \otimes Q}\ \otimes \qquad\qquad \frac{\vdash_J \Delta, N, M}{\vdash_J \Delta, N ⅋ M}\ ⅋$$

Additive rules:

$$\frac{\vdash_J \Delta, P}{\vdash_J \Delta, P \oplus Q}\ \oplus_1 \qquad\qquad \frac{\vdash_J \Delta, P}{\vdash_J \Delta, Q \oplus P}\ \oplus_2 \qquad\qquad \frac{\vdash_J \Delta\restriction_J, N \quad \vdash_K \Delta\restriction_K, M}{\vdash_{J+K} \Delta, N \,\&\, M}\ \&$$

For $\oplus_1$-,$\oplus_2$-rules, observe that Q has to have the empty domain.
For &-rule, it is assumed that $d(N) = J$ and $d(M) = K$ with $J \cap K = \emptyset$, and that $d(\Delta) = J + K$.

[3] $\mathsf{MALLP}(I)$ has no propositional variables, and every formula consists of constants. Hence, the usual identity axiom is derivable in Lemma 2.

Polarity shifting rules:

(↑-rule)

$$\frac{\vdash_J \Delta, P}{\vdash_J \Delta, \uparrow_\emptyset P}\ \uparrow$$

($\downarrow_K$-rule) For every $K \subseteq I$ such that $J \cap K = \emptyset$,

$$\frac{\vdash_J \mathcal{M}\!\upharpoonright_J, N}{\vdash_{K+J} \mathcal{M}, \downarrow_K N}\ \downarrow_K \text{ with } \ K \subseteq d(\partial\mathcal{M}) \tag{5}$$

$\downarrow_K$-rule is applicable only when the side condition is satisfied. This condition is a syntactical description of the adjunctions (4) of Section 2. (See Proposition 2 bellow.)

*Remark 2 (*MALLP $\prec$ MALLP(I)*).* For each inference rule of MALLP(I), if the conclusion sequent is of the domain $\emptyset$, then so is the premise sequent(s). Thus, the rules for sequents of the empty domain are identified with the standard rules of MALLP. Moreover, every MALLP(I)-proof σ for $\vdash_\emptyset \Delta$ contains only sequents of the empty domain. Hence σ is considered as a MALLP-proof for $\vdash \Delta$. Thus MALLP(I) is a conservative extension of MALLP.

The following lemmas hold in the same way as in [4].

Lemma 2 (Identity). $\vdash_J A, A^\perp$ *is provable for any* MALLP(I)*-formula A of domain J.*

Lemma 3 (Restriction). *If $\vdash_J \Delta$ is provable, then so is $\vdash_{J\cap K} \Delta\!\upharpoonright_K$ for any $K \subseteq I$.*

In Laurent's original polarized linear logic MALLP [14], positive/negative polarities classify the dual proof-theoretical properties for connectives: *reversible* connectives $⅋, \&, \downarrow$ for negative formulas and *focusing* connectives $\otimes, \oplus, \uparrow$ for positive formulas. Our MALLP(I) retains the dual properties, as expected.

Lemma 4 (Focalized sequent property). *If $\vdash_J \Delta$ is provable in* MALLP(I)*, then Δ contains at most one positive formula.*

Lemma 5 (Reversibility). $\{⅋, \&, \downarrow_K\}$*-rule is reversible. That is, if the conclusion sequent of $\{⅋, \&, \downarrow_K\}$-rule is provable, then so is the premise sequent.*

Proof. We prove $\downarrow_K$-rule. (The other rules are immediate.) Suppose $\vdash_{K+J} \mathcal{M}, \downarrow_K N$ is provable. Then Lemma 3, by restricting the domain of the sequent to $(K + J) \cap J$, implies that $\vdash_J \mathcal{M}\!\upharpoonright_J, \downarrow_\emptyset N$ is provable. On the other hand, $\vdash_J N^\perp, N$ is provable by Lemma 2. Thus we have the following proof of $\vdash_J \mathcal{M}\!\upharpoonright_J, N$:

$$\frac{\vdash_J \mathcal{M}\!\upharpoonright_J, \downarrow_\emptyset N \qquad \dfrac{\vdash_J N^\perp, N}{\vdash_J \uparrow_\emptyset N^\perp, N}\ \uparrow}{\vdash_J \mathcal{M}\!\upharpoonright_J, N}\ cut$$

■

Let us see, by the following example, how the side condition of $\downarrow_K$-rule is applied. For its denotational characterization, see Proposition 2.

Example 2 ($\downarrow_K$-rule and the side condition). Let us consider the following provable sequent of domain $\{1,2,3,4\}$:

$$\vdash \uparrow_{\{1\}}\mathbf{1}_{\{2\}} \,\&\, \uparrow_{\{3\}}\mathbf{1}_{\{4\}} \ ,\ \downarrow_{\{1,3\}}\bot_{\{2,4\}}$$

Let us search for a proof of the sequent. It is possible to apply $\downarrow_{\{1,3\}}$-rule because the side condition is satisfied: $d(\downarrow_{\{1,3\}}) \subseteq d(\partial(\uparrow_{\{1\}}\mathbf{1}_{\{2\}} \,\&\, \uparrow_{\{3\}}\mathbf{1}_{\{4\}})) = \{1\} \cup \{3\}$. Then we obtain $\vdash\ (\uparrow_{\{1\}}\mathbf{1}_{\{2\}} \,\&\, \uparrow_{\{3\}}\mathbf{1}_{\{4\}})\lceil_{\{2,4\}}\ ,\ \bot_{\{2,4\}}$, which coincides with $\vdash \uparrow_{\emptyset}\mathbf{1}_{\{2\}} \,\&\, \uparrow_{\emptyset}\mathbf{1}_{\{4\}}\ ,\ \bot_{\{2,4\}}$. Then by applying a &-rule and then $\uparrow$-rules, we have the following proof σ (the braces $\{,\}$ of domains are omitted for simplicity):

$$\sigma = \cfrac{\cfrac{\cfrac{\vdash \mathbf{1}_2\ ,\ \bot_2}{\vdash \uparrow_\emptyset \mathbf{1}_2\ ,\ \bot_2}\uparrow \quad \cfrac{\vdash \mathbf{1}_4\ ,\ \bot_4}{\vdash \uparrow_\emptyset \mathbf{1}_4\ ,\ \bot_4}\uparrow}{\vdash \uparrow_\emptyset \mathbf{1}_2 \,\&\, \uparrow_\emptyset \mathbf{1}_4\ ,\ \bot_{2,4}}\&}{\vdash \uparrow_1 \mathbf{1}_2 \,\&\, \uparrow_3 \mathbf{1}_4\ ,\ \downarrow_{1,3}\bot_{2,4}}\downarrow_{1,3}$$

Let us consider another example by modifying the domains of the above example: $\vdash \uparrow_{\{1\}}\mathbf{1}_{\emptyset} \,\&\, \uparrow_{\emptyset}\mathbf{1}_{\{2\}}\ ,\ \downarrow_{\{1,2\}}\bot_{\emptyset}$. This sequent is shown to be unprovable by the cut-elimination (Corollary 1): first, $\downarrow_{\{1,2\}}$-rule is not applicable since the sequent fails to satisfy the side condition: $d(\downarrow_{\{1,2\}}) \not\subseteq d(\partial(\uparrow_{\{1\}}\mathbf{1}_{\emptyset} \,\&\, \uparrow_{\emptyset}\mathbf{1}_{\{2\}})) = \{1\} \cup \emptyset$. Therefore, the last rule must be:

$$\cfrac{\vdash \uparrow_1\mathbf{1}_\emptyset\ ,\ \downarrow_1\bot_\emptyset \quad \vdash \uparrow_\emptyset\mathbf{1}_2\ ,\ \downarrow_2\bot_\emptyset}{\vdash \uparrow_1\mathbf{1}_\emptyset \,\&\, \uparrow_\emptyset\mathbf{1}_2\ ,\ \downarrow_{1,2}\bot_\emptyset}\&$$

Although the left premise sequent is provable, the right premise sequent is not: By Lemma 4 the last rule must be $\downarrow_{\{2\}}$-rule, which is not applicable because of the violation of the side condition: $d(\downarrow_{\{2\}}) \not\subseteq d(\partial(\uparrow_{\emptyset}\mathbf{1}_{\{2\}})) = \emptyset$. ■

The formulas of indexed system $\mathsf{MALLP}(I)$ are designed so that the domain of each formula indicates (syntactically) a family of points, thus a relation, in the multi-pointed relational semantics of Section 2. Hence, there is a bijective correspondence between $\mathsf{MALLP}(I)$-formulas and families of points in the webs for the corresponding MALLP-formulas. Let us describe this correspondence precisely.

Notation: For $a \in X^J$ and $j \in J$, a_j denotes the j-th element of a. For $a \in X^J$ and $b \in Y^J$, we denote by $a \times b$ the element of $(X \times Y)^J$ given by $(a \times b)_j = (a_j, b_j)$ for each $j \in J$. If $K, L \subseteq I$ are disjoint and if $a \in X^K$ and $b \in Y^L$, we denote by $a + b$ the element of $(X+Y)^{K+L}$ defined by case; $(a+b)_k = a_k$ if $k \in K$; and $(a+b)_l = b_l$ if $l \in L$. For $J \subseteq I$, we denote by $(c)_J$ the J-indexed family of the constant c, i.e., the unique element of $\{c\}^J$.

Definition 9 (Translation of MALLP-formulas). To any MALLP-formula A, and any family $a \in |A|^J$, we associate a $\mathsf{MALLP}(I)$-formula $A\langle a\rangle$ of domain J inductively as follows:

We here treat only $\downarrow N$ and $\uparrow P$. (The other connectives are the same as in [4] .)
– If $A \equiv \downarrow N$, then $a = (*_\downarrow)_K + b$ with the family $(*_\downarrow)_K \in \{*_\downarrow\}^K$ and $b \in |N|^L$ such that $K+L = J$. Then we set $A\langle a\rangle = \downarrow_K N\langle b\rangle$, which is a $\mathsf{MALLP}(I)$-formula of domain J.
– If $A \equiv \uparrow P$, then $a = (*_\uparrow)_K + b$ with the family $(*_\uparrow)_K \in \{*_\uparrow\}^K$ and $b \in |P|^L$ such that $K+L = J$. Then we set $A\langle a\rangle = \uparrow_K P\langle b\rangle$, which is a $\mathsf{MALLP}(I)$-formula of domain J.

If $\Delta = A_1, \ldots, A_n$ is a sequence of MALLP-formulas, we define $|\Delta| = |A_1| \times \cdots \times |A_n|$. If $\gamma \in |\Delta|^J$, then, using our usual notational conventions, we can write $\gamma = \gamma^1 \times \cdots \times \gamma^n$ with $\gamma^m \in |A_m|^J$, and we set $\Delta\langle\gamma\rangle = A_1\langle\gamma^1\rangle, \ldots, A_n\langle\gamma^n\rangle$.

We have the following lemma, which will be used to prove Proposition 3.

Lemma 6. *Let* $\vdash \Delta$ *be a sequent in* MALLP *and let* $\gamma \in |\Delta|^J$. *Let* $K \subset I$. *Let* $\gamma\restriction_{J\cap K}$ *be the restriction of* γ *to* $J \cap K$. *Then* $\Delta\langle\gamma\restriction_{J\cap K}\rangle = \Delta\langle\gamma\rangle\restriction_K$.

The following Lemma 7 ensures that the correspondence given in Definition 9 is bijective to the $\mathsf{MALLP}(I)$-formulas:

Lemma 7. *If* A *is a* $\mathsf{MALLP}(I)$*-formula of domain* J *and* $A\restriction_\emptyset$ *is the corresponding* MALLP*-formula, there is a unique family* $a \in |A\restriction_\emptyset|^J$ *such that* $A = A\restriction_\emptyset \langle a\rangle$.

For a typographical convenience, a MALLP-formula $A\restriction_\emptyset$ and a sequent $\mathcal{M}\restriction_\emptyset$ are sometimes denoted by A and M, respectively.

We see the above bijective correspondence by the following example.

Example 3 ($\mathsf{MALLP}(I)$*-formula as a relation).* Let us consider the $\mathsf{MALLP}(I)$-sequent $\vdash \uparrow_{\{1\}}\mathbf{1}_{\{2\}} \mathbin{\&} \uparrow_{\{3\}}\mathbf{1}_{\{4\}}\ ,\ \downarrow_{\{1,3\}}\bot_{\{2,4\}}$ of Example 2. We determine the corresponding family $\gamma \in |\uparrow\mathbf{1} \mathbin{\&} \uparrow\mathbf{1}, \downarrow\bot|^{\{1,2,3,4\}}$ as follows: Let us represent the webs $|\uparrow\mathbf{1} \mathbin{\&} \uparrow\mathbf{1}| = \{\uparrow_a, 1_a, \uparrow_b, 1_b\}$ and $|\downarrow\bot| = \{\downarrow_c, B_c\}$. The representation designates the correspondence between components of formulas and points. We determine γ_1 as follows: Since it is the domains of $\uparrow_{\{1\}}$ and $\downarrow_{\{1,3\}}$ that contain the index 1, γ_1 is a pair $(\uparrow_a, \downarrow_c)$ of the corresponding points to $\uparrow_{\{1\}}$ and $\downarrow_{\{1,3\}}$. Similar calculations for $\gamma_2, \gamma_3, \gamma_4$ yield $\gamma_1 = (\uparrow_a, \downarrow_c),\ \gamma_2 = (1_a, B_c), \gamma_3 = (\uparrow_b, \downarrow_c),\ \gamma_4 = (1_b, B_c)$. In fact, γ happens to be the denotation of the MALLP-proof, which is obtained from $\mathsf{MALLP}(I)$-proof σ of Example 2 by forgetting all the domain symbols. ■

By means of the above bijective correspondence, the side condition of $\downarrow_K$-rule turns out to be a syntactic counterpart of multi-pointedness of relations:

Proposition 2 (Semantical characterization of the side condition of $\downarrow_K$**-rule).** *Let* J *and* K *be disjoint subsets of* I. *Let* $\mathcal{M}$ *be a sequence of negative formulas of domain* $K + J$, *and* N *be a formula of domain* J *in* $\mathsf{MALLP}(I)$. *Let* $\gamma \times a \in |(\mathsf{M}, \downarrow\mathsf{N})|^{K+J}$ *be the family of points associated with* $(\mathcal{M}, \downarrow_K N)$. *Then the following two conditions are equivalent:*

1. $K \subseteq d(\partial\mathcal{M})$
2. $\gamma \times a$ *is right-multi-pointed, that is, for any index* $i \in K$, $\gamma_i \in \mathsf{mp}(\mathsf{M})$.

Proof. By induction on the number of & in $\mathcal{M}[*_1, \ldots, *_n]$. ■

Example 4 (A non-$\uparrow$-soft sequent of $\mathsf{MALLP}(I)$). Let us consider the sequent $\vdash \uparrow_{\{1\}}\mathbf{1}_{\{2\}}\ ,\ \uparrow_{\{1\}}\downarrow_{\emptyset}\bot_{\{2\}}$, which is not provable in $\mathsf{MALLP}(I)$. By the cut-elimination theorem of $\mathsf{MALLP}(I)$ (Corollary 1 below), the last rule should be a $\uparrow$-rule. However, it is impossible to apply the rule because both the outermost $\uparrow$'s have the non-empty domain $\{1\}$. The unprovability corresponds, by virtue of Theorem 1 bellow, to the non-$\uparrow$-softness of the corresponding relation $\gamma \subseteq |\uparrow\mathbf{1}, \uparrow\downarrow\bot|$ such that $\gamma_1 = (*_\uparrow, *_\uparrow)$. Note that if a relation is not $\uparrow$-soft (i.e., does not factor through any outermost $\uparrow$), it cannot be contained in any denotations of MALLP-proofs since MALLP syntax is $\uparrow$-soft. See Section 7.1.1 of [12] for the $\uparrow$-softness.

4 A correspondence between $\mathsf{MALLP}(I)$-provability and denotations of MALLP-proofs

This section is devoted to proving the *basic property* of $\mathsf{MALLP}(I)$, which is the main theorem of this paper. The property, named after Bucciarelli-Ehrhard [4], characterizes a relationship between provability of formulas of $\mathsf{MALLP}(I)$ and denotations of proofs of MALLP. The characterization is a polarized version of Bucciarelli-Ehrhard's Proposition 20 of [4]. As a corollary of the basic property, the cut-elimination theorem of $\mathsf{MALLP}(I)$ is obtained.

Theorem 1 (Basic property of $\mathsf{MALLP}(I)$). *Let Δ be a sequence of formulas of MALLP, and let $\gamma \in |\Delta|^J$. The following two statements are equivalent.*

i) There exists a proof π of Δ in MALLP such that $\gamma \in (\pi^*)^J$.
ii) The sequent $\vdash_J \Delta\langle\gamma\rangle$ is provable in $\mathsf{MALLP}(I)$.

The theorem is proved by the following Proposition 3 and Proposition 4, which are converse to each other: Proposition 3 shows an implication from (*i*) to (*ii*) and conversely for Proposition 4. In the following proofs, we sometimes denote by A_J a $\mathsf{MALLP}(I)$-formula A with domain J. We first show the following:

Proposition 3. *Let Δ be a sequent in MALLP and let π be a proof (resp. cut-free proof) of $\vdash \Delta$ in MALLP. Let $\gamma \in (\pi^*)^J$ (for some $J \subseteq I$). Then the sequent $\vdash_J \Delta\langle\gamma\rangle$ has a proof (resp. cut-free proof) σ in $\mathsf{MALLP}(I)$ such that $\sigma{\upharpoonright_\emptyset} = \pi$.*

Proof. By induction on the MALLP-proof π. We consider only the polarity shifting rules (the other rules are the same as Lemma 18 of Bucciarelli-Ehrhard [4]).

– When π is $\dfrac{\overset{\pi_1}{\vdash \mathcal{M}, P}}{\vdash \mathcal{M}, \uparrow P}\ \uparrow$

since $\gamma \in (\pi^*)^J = (\pi_1^*)^J \subseteq |\mathcal{M}, P|^J$ by the interpretation of $\uparrow$-rule, γ is of the form $\delta \times a$ with $\delta \in |\mathcal{M}|^J$ and $a \in |P|^J$. Thus by the induction hypothesis, the $\mathsf{MALLP}(I)$-sequent $\vdash_J \mathcal{M}\langle\delta\rangle, P\langle a\rangle$ has a proof σ_1 such that $\sigma_1{\upharpoonright_\emptyset} = \pi_1$. By applying $\uparrow$-rule to σ_1, we obtain the following proof σ of $\vdash_J (\mathcal{M}, \uparrow P)\langle\gamma\rangle$ so that $\sigma{\upharpoonright_\emptyset} = \pi$:

$$\dfrac{\overset{\sigma_1}{\vdash_J \mathcal{M}\langle\delta\rangle, P\langle a\rangle}}{\vdash_J \mathcal{M}\langle\delta\rangle, \uparrow_\emptyset P\langle a\rangle}\ \uparrow$$

– When π is $\dfrac{\begin{array}{c}\pi_1\\ \vdash \mathcal{N}, M\end{array}}{\vdash \mathcal{N}, \downarrow M}\downarrow$

since $\gamma \in (\pi^*)^J = \big(\{(p, *_\downarrow) \mid p \in \mathsf{mp}(\mathcal{N})\} \cup \pi_1^*\big)^J \subseteq |\mathcal{N}, \downarrow M|^J$ by the interpretation of $\downarrow$-rule, γ is of the form $\delta \times a$ with $\delta \in |\mathcal{N}|^J$ and $a \in |\downarrow M|^J$. Since $|\downarrow M|^J = (\{*_\downarrow\} + |M|)^J$, there are two uniquely defined disjoint sets K and L such that $K + L = J$, and a is of the form $(*_\downarrow)_K + b$ with $(*_\downarrow)_K \in \{*_\downarrow\}^K$ and $b \in |M|^L$. According to this decomposition of J, δ is also written as $\delta\restriction_K + \delta\restriction_L$ with $\delta\restriction_K \in |\mathcal{N}|^K$ and $\delta\restriction_L \in |\mathcal{N}|^L$. Note that we have $\delta\restriction_K \in \mathsf{mp}(\mathcal{N})$ by the interpretation of $\downarrow$-rule in the multi-pointed relational semantics. Thus γ is written as the disjoint union $\gamma = (\delta\restriction_K \times (*_\downarrow)_K) + (\delta\restriction_L \times b)$, where $\delta\restriction_K \times (*_\downarrow)_K \in \{(p, *_\downarrow) \mid p \in \mathsf{mp}(\mathcal{N})\}^K$ and $\delta\restriction_L \times b \in (\pi_1^*)^L$. By the induction hypothesis for π_1, the $\mathsf{MALLP}(I)$-sequent $\vdash_L \mathcal{N}\langle\delta\restriction_L\rangle, M\langle b\rangle$ has a proof σ_1 such that $\sigma_1\restriction_\emptyset = \pi_1$. Since $\mathcal{N}\langle\delta\restriction_L\rangle = \mathcal{N}\langle\delta\rangle\restriction_L$ by Lemma 6, we apply $\downarrow_K$-rule to σ_1 in order to obtain the following proof σ of $\vdash_J (\mathcal{N}, \downarrow M)\langle\gamma\rangle$ such that $\sigma\restriction_\emptyset = \pi$:

$$\dfrac{\begin{array}{c}\sigma_1\\ \vdash_L \mathcal{N}\langle\delta\rangle\restriction_L \ , \ M\langle b\rangle\end{array}}{\vdash_{K+L} \mathcal{N}\langle\delta\rangle \ , \ (\downarrow M)\langle(*_\downarrow)_K + b\rangle}\downarrow_K$$

The side condition for $\downarrow_K$-rule is satisfied since γ is of the form $(\delta\restriction_K \times (*_\downarrow)_K) + (\delta\restriction_L \times b)$ with $\delta\restriction_K \in \mathsf{mp}(\mathcal{N})^K$ (see Proposition 2). ■

Next, we show the converse of Proposition 3:

Proposition 4. *Let Δ be a sequent in* MALLP. *Let $\gamma \in |\Delta|^J$ (for some $J \subseteq I$), and let σ be a proof of $\vdash_J \Delta\langle\gamma\rangle$ in* $\mathsf{MALLP}(I)$. *Then $\gamma \in (\sigma\restriction_\emptyset{}^*)^J$.*

Proof. By induction on the $\mathsf{MALLP}(I)$-proof σ. We consider the polarity shifting rules since the other rules are the same as those in Lemma 19 of [4]. In the following proof, a MALLP-formula $A\restriction_\emptyset$ and a sequent $\Delta\restriction_\emptyset$ are denoted by A and $\mathsf{\Delta}$, respectively.

– When σ is $\dfrac{\begin{array}{c}\sigma_1\\ \vdash_J \Delta, P\end{array}}{\vdash_J \Delta, \uparrow_\emptyset P}\uparrow$

there is, by Lemma 7, $\gamma = \delta \times a \in |\mathsf{\Delta}, \uparrow\mathsf{P}|^J$ such that $\Delta_J, \uparrow_\emptyset P_J = (\mathsf{\Delta}, \uparrow\mathsf{P})\langle\gamma\rangle$. Note first that $a \in |\mathsf{P}|^J$ holds since the domain of the outermost $\uparrow$ of $\uparrow_\emptyset P_J = (\uparrow\mathsf{P})\langle a\rangle$ is empty. Thus $\vdash_J \Delta, P$ coincides with $\vdash_J \mathsf{\Delta}\langle\delta\rangle, \mathsf{P}\langle a\rangle$, and hence we have $\delta \times a \in (\sigma_1\restriction_\emptyset{}^*)^J$ by the induction hypothesis. Then, by the denotational interpretation of $\uparrow$-rule of MALLP, we conclude:

$$\gamma = \delta \times a \in (\sigma_1\restriction_\emptyset{}^*)^J = (\sigma\restriction_\emptyset{}^*)^J.$$

– When σ is $\dfrac{\begin{array}{c}\sigma_1\\ \vdash_L \mathcal{M}\restriction_L, N\end{array}}{\vdash_{K+L} \mathcal{M}, \downarrow_K N}\downarrow_K$ with $K \subseteq d(\partial\mathcal{M})$,

there is, by Lemma 7, $\gamma \in |\mathsf{M}, \downarrow\mathsf{N}|^{K+L}$ such that $\mathcal{M}_{K+L}, \downarrow_K N_L = (\mathsf{M}, \downarrow\mathsf{N})\langle\gamma\rangle$. Because K and L are disjoint, γ is written as the following disjoint union:

$$\gamma = (\delta\restriction_K \times (*_\downarrow)_K) + (\delta\restriction_L \times b) \in |\mathsf{M}, \downarrow\mathsf{N}|^{K+L},$$

where $\delta{\restriction_K} \times (*_\downarrow)_K \in |\mathsf{M}|^K \times \{*_\downarrow\}^K$ and $\delta{\restriction_L} \times b \in |\mathsf{M}|^L \times |\mathsf{N}|^L$. Since $\vdash_L \mathcal{M}{\restriction_L}, N$ coincides with $\vdash_L \mathsf{M}\langle\delta{\restriction_L}\rangle, \mathsf{N}\langle b\rangle$, we have $\delta{\restriction_L} \times b \in (\sigma_1{\restriction_\emptyset}\ {}^*)^L$ by the induction hypothesis. Since the side condition of $\downarrow_K$-rule implies $\delta{\restriction_K} \in \mathsf{mp}(\mathsf{M})^K$ (see Proposition 2), we obtain:

$$(\delta{\restriction_K} \times (*_\downarrow)_K) + (\delta{\restriction_L} \times b) \in \left(\{(p, *_\downarrow) \mid p \in \mathsf{mp}(\mathsf{M})\} \cup \sigma_1{\restriction_\emptyset}\ {}^*\right)^{K+L}.$$

Thus, by the interpretation of $\downarrow$-rule of MALLP, we conclude $\gamma \in (\sigma{\restriction_\emptyset}\ {}^*)^{K+L}$. ■

As a corollary, we have the following semantical cut-elimination à la Bucciarelli-Ehrhard [5].

Corollary 1 (Cut-elimination of MALLP(I)). *The sequent calculus system* MALLP(I) *enjoys cut-elimination. That is, if a sequent is provable, then it is provable without using cut-rule.*

Proof. Assume $\vdash_J \varDelta$ is provable with a MALLP(I)-proof σ. By Lemma 7, $\varDelta$ is of the form $\Delta\langle\gamma\rangle$ for a sequence Δ of MALLP-formulas and $\gamma \in |\Delta|^J$. Then by Proposition 4, $\gamma \in (\sigma{\restriction_\emptyset}\ {}^*)^J$. Since $\sigma{\restriction_\emptyset}$ is a MALLP-proof of $\vdash \Delta$, there exists a cut-free MALLP-proof π for the sequent by the cut-elimination of MALLP. Since $\sigma{\restriction_\emptyset}\ {}^* = \pi^*$ by Proposition 1, we have $\gamma \in (\pi^*)^J$. Then, by Proposition 3, there exists a cut-free MALLP(I)-proof ρ of $\vdash_J \Delta\langle\gamma\rangle$, which sequent is $\vdash_J \varDelta$. ■

5 Discussions and Future work

Let us discuss several comparisons of our polarity shifting operators with Bucciarelli-Ehrhard's exponentials of [5]. First, our multi-pointed relational interpretation of $\downarrow A$ and $\uparrow A$ is seen as a restriction of their interpretation of $!A$ and $?A$ (pg.212 of [5]) to the multisets of cardinality *at most one* (i.e., $|\uparrow A| = |\downarrow A| = \{[a] \mid a \in |A|\} \cup \{[\ ]\}$). Note that the empty multiset $[\ ]$ corresponds to our distinguished element $*$. Due to this restriction, the contraction rule is absent in our interpretation. On the other hand, the interpretation of the promotion rule of LL (pg.240 of [5]) simulates ours of the $\downarrow$-rule for MLLP (without additives) by restricting the cardinality n for the index of the family to either 0 or 1. Second, our indexed $\downarrow_K N$ and $\uparrow_K P$ of Definition 5 coincide with Bucciarelli-Ehrhard's $!_u N$ and $?_u P$ when u is the injection from L to $L + K$. Then our translation of Definition 9 corresponds to theirs (pg.213 of [5]).

As another comparison, it is straightforward to generalize our construction of this paper into a polarized variant of LL(I) of [5] with exponentials. By weakening the bijective correspondence of Lemma 7 into surjective one, the construction yields an indexed system for Laurent's $\mathsf{LL}_{\mathrm{pol}}$ augmented with polarity shifting operators $\uparrow$ and $\downarrow$.

Regarding future works, a phase semantics for MALLP(I) should be examined. Phase semantics is a standard truth-value semantics for linear logic. Such a semantics for MALLP(I) is obtained by a generalization of our polarized phase semantics [13] for MALLP. In [13] a topological structure was given to a phase space by interpreting $\downarrow$ and $\uparrow$ as interior and closure operators, respectively.

For the generalization, an I-product phase spaces become crucial analogously to Bucciarelli-Ehrhard [4, 5] and Ehrhard [7]. In our polarized setting the product topology on this phase space is important to understand parameterized $\downarrow_K$ connectives for $\mathsf{MALLP}(I)$. A phase semantics for $\mathsf{MALLP}(I)$ yields, by virtue of Theorem 1, a new denotational semantics for MALLP. Moreover a truth valued completeness of such a phase semantics leads naturally to a weak denotational completeness in the sense of Girard [9] and Bucciarelli-Ehrhard. In particular, such a denotational completeness explicates an I-indexed topological logical relations for polarized linear logic.

Acknowledgements
The authors would like to express their sincere thanks especially to one of the anonymous referees for careful reading and for valuable comments and suggestions.

References

1. Jean-Marc Andreoli, Logic Programming with Focusing Proofs in Linear Logic, *Journal of Logic and Computation*, 2(3), 297-347, 1992.
2. Richard Blute and Philip J. Scott, Category theory for linear logicians, Proceedings Linear Logic Summer School, editors: P. Ruet, T. Ehrhard, J.-Y. Girard, P. Scott, Cambridge University Press, 1-52, 2004.
3. Alexandra Bruasse-Bac, Logique linéaire indexée du second ordre, Thèse de doctorat, Institut de Mathématiques de Luminy, Université Aix-Marseille II, 2001.
4. Antonio Bucciarelli and Thomas Ehrhard, On phase semantics and denotational semantics in multiplicative-additive linear logic, *Annals of Pure and Applied Logic*, 102(3), 247-282, 2000.
5. Antonio Bucciarelli and Thomas Ehrhard, On phase semantics and denotational semantics: the exponentials, *Annals of Pure and Applied Logic*, 109(3), 205-241, 2001.
6. Robin Cockett and Robert Seely, Polarized category theory, modules, and game semantics, *Theory and Applications of Categories*, Vol. 18, No. 2, pp 4-101, 2007.
7. Thomas Ehrhard, A completeness theorem for symmetric product phase spaces, *Journal of Symbolic Logic*, Volume 69, Issue 2, 2004, 340-370.
8. Jean-Yves Girard, A New Constructive Logic: Classical Logic, *Mathematical Structures in Computer Science*, 1 (3), 255-296, 1991.
9. Jean-Yves Girard, On denotational completeness, *Theoretical Computer Science*, Volume 227 , Special issue on linear logic 1, 249-273, 1999.
10. Jean-Yves Girard, On the meaning of logical rules II: multiplicative/additive case, in Foundation of Secure Computation, NATO Series F, Vol. 175, IOS Press, Amsterdam, 183-212, 2000.
11. Jean-Yves Girard, Locus Solum: From the rules of logic to the logic of rules, *Mathematical Structures in Computer Science*, 11(3), 301-506, 2001.
12. Masahiro Hamano and Philip Scott, A Categorical Semantics for Polarized MALL, *Annals of Pure and Applied logic*, 145, 2007, 276-313.
13. Masahiro Hamano and Ryo Takemura, A Phase Semantics for Polarized Linear Logic and Second Order Conservativity, submitted to a journal, 2007.
14. Olivier Laurent, Étude de la polarisation en logique, Thèse de Doctorat, Institut de Mathématiques de Luminy, Université Aix-Marseille II, 2002.
15. Olivier Laurent, Polarized games, *Annals of Pure and Applied Logic*, 130 (1-3), 79-123, 2004.

A Multiplicative additive polarized linear logic MALLP

Formulas of MALLP are given by the following grammar:

$$P ::= X \mid P \otimes P \mid P \oplus P \mid \mathbf{1} \mid \mathbf{0} \mid {\downarrow} N$$
$$N ::= X^{\perp} \mid N \parr N \mid N \,\&\, N \mid \perp \mid \top \mid {\uparrow} P$$

Notation: P, Q (with or without subscript) (resp. N, M) denote positive (resp. negative) formulas. Δ, Λ denotes multisets of formulas; $\mathcal{N}, \mathcal{M}$ denotes sequents consist only of negative formulas, called *negative sequents*.

Inference rules of MALLP are defined as follows:

$$\vdash N, N^{\perp} \qquad \vdash \mathbf{1} \qquad \vdash \Delta, \top \ \text{(Here } \Delta \text{ contains at most one positive formula.)} \qquad \frac{\vdash \Delta, N \quad \vdash N^{\perp}, \Lambda}{\vdash \Delta, \Lambda}\ cut$$

$$\frac{\vdash \Delta}{\vdash \Delta, \perp}\ \perp \qquad \frac{\vdash \Delta, N, M}{\vdash \Delta, N \parr M}\ \parr \qquad \frac{\vdash \Delta, P \quad \vdash \Lambda, Q}{\vdash \Delta, \Lambda, P \otimes Q}\ \otimes$$

$$\frac{\vdash \Delta, N \quad \vdash \Delta, M}{\vdash \Delta, N \,\&\, M}\ \& \qquad \frac{\vdash \Delta, P}{\vdash \Delta, P \oplus Q}\ \oplus_1 \qquad \frac{\vdash \Delta, Q}{\vdash \Delta, P \oplus Q}\ \oplus_2$$

$$\frac{\vdash \mathcal{M}, N}{\vdash \mathcal{M}, {\downarrow} N}\ \downarrow \qquad \frac{\vdash \Delta, P}{\vdash \Delta, {\uparrow} P}\ \uparrow$$

B Multi-pointed relational interpretation of MALLP-proofs

Definition 4 (Interpretation of proofs in $(\langle\ \mathbf{PRel}_l,\ \mathbf{PRel}_r\rangle, \widehat{\mathbf{Rel}})$)

When π is $\vdash \mathbf{1}$, we define

$$\pi^* = \{*_{\mathbf{1}}\} \in \ \mathbf{PRel}_r$$

When π is $\dfrac{\overset{\pi_1}{\vdash \Delta}}{\vdash \Delta, \perp}\ \perp$, we define

$$\pi^* = \{(\delta, *_{\perp}) \mid \delta \in \pi_1^*\}$$

When π is $\dfrac{\overset{\pi_1}{\vdash \Delta, N, M}}{\vdash \Delta, N \parr M}\ \parr$, we define

$$\pi^* = \{(\delta, (a, b)) \mid (\delta, a, b) \in \pi_1^*\}$$

When π is $\dfrac{\overset{\pi_1}{\vdash \Delta, P} \quad \overset{\pi_2}{\vdash \Lambda, Q}}{\vdash \Delta, \Lambda, P \otimes Q}\ \otimes$, we define

$$\pi^* = \{(\delta, \lambda, (a, b)) \mid (\delta, a) \in \pi_1^* \text{ and } (\lambda, b) \in \pi_2^*\} \in \ \mathbf{PRel}_r(\Delta^{\perp} \otimes \Lambda^{\perp}, P \otimes Q)$$

When π is $\dfrac{\overset{\pi_1}{\vdash \Delta, N} \quad \overset{\pi_2}{\vdash \Delta, M}}{\vdash \Delta, N \,\&\, M}\ \&$, we define

$$\pi^* = \{(\delta, (1, a)) \mid (\delta, a) \in \pi_1^*\} \cup \{(\delta, (2, b)) \mid (\delta, b) \in \pi_2^*\}$$

When π is $\dfrac{\overset{\pi_1}{\vdash \Delta, P}}{\vdash \Delta, P \oplus Q}\ \oplus_1$, we define

$$\pi^* = \{(\delta, (1, a)) \mid (\delta, a) \in \pi_1^*\} \in \ \mathbf{PRel}_r(\Delta^\perp, P \oplus Q)$$

Similar for $\oplus_2$.
When π is $\vdash \Delta, \top$ we define

$$\pi^* = \emptyset$$

When π is $\vdash P^\perp, P$, we define

$$\pi^* = \{(a, a) \mid a \in |P|\} \in \ \mathbf{PRel}_r(P, P)$$

When π is $\dfrac{\overset{\pi_1}{\vdash \Delta, N} \quad \overset{\pi_2}{\vdash \Lambda, N^\perp}}{\vdash \Delta, \Lambda}\ cut$, we define

$$\pi^* = \{(\delta, \lambda) \mid \exists a,\ (\delta, a) \in \pi_1^* \text{ and } (\lambda, a) \in \pi_2^*\}$$

Note that π_2^* is always a map in $\mathbf{PRel}_r$. Hence π^* is a map either in $\mathbf{PRel}_r$ or in $\widehat{\mathbf{Rel}}$ depending whether Δ contains a positive formula or not, respectively.
When π is $\dfrac{\overset{\pi_1}{\vdash \mathcal{M}, N}}{\vdash \mathcal{M}, {\downarrow}N}\ \downarrow$, we define

$$\pi^* = \pi_1^* \cup \{(p, *_\downarrow) \mid p \in \mathsf{mp}(\mathcal{M})\} \qquad \in \ \mathbf{PRel}_r(\mathcal{M}^\perp, {\downarrow}N)$$

π^* is obtained from $\pi_1^* \in \widehat{\mathbf{Rel}}(\mathcal{M}^\perp, N)$ by the right adjunction of (4).
When π is $\dfrac{\overset{\pi_1}{\vdash \mathcal{M}, P}}{\vdash \mathcal{M}, {\uparrow}P}\ \uparrow$, we define

$$\pi^* = \pi_1^* \qquad \in \widehat{\mathbf{Rel}}(\mathcal{M}^\perp, {\uparrow}P).$$

π^* is obtained from $\pi_1^* \in \ \mathbf{PRel}_r(\mathcal{M}^\perp, P)$ by composing $\eta : P \to {\uparrow}P$ which is the unit of the left adjunction of (4); i.e., $\eta = \{(p, p) \mid p \in |P|\}$. Note that the composition of η acts identically on morphisms.

In the above rules for negative formulas, π^* is a map either in $\mathbf{PRel}_r$ or in $\widehat{\mathbf{Rel}}$ depending whether Δ contains a positive formula or not, respectively.

C Complete definitions

Definition 5 (Domains of formulas) Formulas and their domains of $\mathsf{MALLP}(I)$ are defined as follows:

- $\mathbf{0}$ and $\top$ are two constants, both having *empty* domain.
- For each $J \subseteq I$, we introduce two new constants $\bot_J$ and $\mathbf{1}_J$, both of which have J as their domains; i.e., $d(\mathbf{1}_J) = d(\bot_J) = J$.
- For each $J \subseteq I$,
  if P and Q are positive formulas such that $d(P) = d(Q) = J$, then $P \otimes Q$ is a positive formula with $d(P \otimes Q) = J$.
  if N and M are negative formulas such that $d(N) = d(M) = J$, then $N \parr M$ is a negative formula with $d(N \parr M) = J$.
- For $J, K \subseteq I$, with $J \cap K = \emptyset$,
  if P and Q are positive formulas with $d(P) = J$ and $d(Q) = K$, then $P \oplus Q$ is a positive formula with $d(P \oplus Q) = J + K$.
  if N and M are negative formulas with $d(N) = J$ and $d(M) = K$, then $N \mathbin{\&} M$ is a negative formula with $d(N \mathbin{\&} M) = J + K$.

Definition 9 (Translation of MALLP-formulas) To any formula A of MALLP, and any family $a \in |A|^J$, we associate a formula $A\langle a\rangle$ of $\mathsf{MALLP}(I)$ of domain J as follows:

- For $A \equiv \mathbf{0}$ or $A \equiv \top$, if $J \neq \emptyset$ then, since $|A|^J = \emptyset$ in that case, $A\langle a\rangle$ is undefined. If $J = \emptyset$, then $|A|^J$ has exactly one element, namely the empty family $\emptyset$, and we set $\mathbf{0}\langle\emptyset\rangle = \mathbf{0}$ and $\top\langle\emptyset\rangle = \top$.
- If $A \equiv \mathbf{1}$ or $A \equiv \bot$, then a is the constant family $(*)_J$, and we set $\mathbf{1}\langle(*)_J\rangle = \mathbf{1}_J$ and $\bot\langle(*)_J\rangle = \bot_J$.
- If $A \equiv P \otimes Q$, then $a = b \times c$ with $b \in |P|^J$ and $c \in |Q|^J$, and we set $A\langle a\rangle = P\langle b\rangle \otimes Q\langle c\rangle$ which is a well-formed formula of $\mathsf{MALLP}(I)$ of domain J.
  Similarly for $A \equiv N \parr M$, we set $A\langle a\rangle = N\langle b\rangle \parr M\langle c\rangle$.
- If $A \equiv P \oplus Q$, then $a = b + c$ with $b \in |P|^K$ and $c \in |Q|^L$ and $K + L = J$. Then we set $A\langle a\rangle = P\langle b\rangle \oplus Q\langle c\rangle$ which is a well-formed formula of $\mathsf{MALLP}(I)$ of domain J.
  Similarly for $A \equiv N \mathbin{\&} M$, we set $A\langle a\rangle = N\langle b\rangle \mathbin{\&} M\langle c\rangle$.

D Omitted proofs

Proof of Lemma 2 . By induction on A. We show the particular case where $A \equiv \downarrow_K N$ with $d(N) = L$ such that $K + L = J$. Other cases are shown by the same way as [4]. Since $\vdash_L N, N^\perp$ is provable by the induction hypothesis, we have the following proof:

$$\dfrac{\dfrac{\vdash_L N, N^\perp}{\vdash_L N, \uparrow_\emptyset N^\perp}\uparrow}{\vdash_{K+L} \downarrow_K N, \uparrow_K N^\perp}\downarrow_K$$

■

Proof of Lemma 3. By induction on the proof of $\vdash_J \Delta$.
When the last rule of the proof is $\dfrac{\vdash_J \mathcal{M}\restriction_J, N}{\vdash_{L+J} \mathcal{M}, \downarrow_L N} \downarrow_L$, by the induction hypothesis, $\vdash_{J\cap K} \mathcal{M}\restriction_{J\cap K}, N\restriction_K$ is provable. Then we have the following proof:

$$\dfrac{\vdash_{J\cap K} \mathcal{M}\restriction_{J\cap K}, N\restriction_K}{\vdash_{(L\cap K)+(J\cap K)} \mathcal{M}, \downarrow_{L\cap K}(N\restriction_K)} \downarrow_{L\cap K}$$

where $(L \cap K) + (J \cap K) = (L + J) \cap K$ and $\downarrow_{L\cap K}(N\restriction_K) = (\downarrow_L N)\restriction_K$. ■

Proof of Lemma 4. Straightforward by induction on the construction of proof of $\vdash_J \Delta$ as in [14]. ■

Proof of Lemma 5. $⅋$-rule is immediate. For &-rule, assume that $\vdash_{K+L} \mathcal{M}, N \,\&\, M$ is provable, where $d(N) = K$ and $d(M) = L$ with $K \cap L = \emptyset$. We show that the left premise sequent of &-rule is provable. The same applies to the right premise sequent. Lemma 3, by restricting the domain of the sequent to $(K + L) \cap K$, implies that $\vdash_K \mathcal{M}\restriction_K, (N \,\&\, M)\restriction_K$ is provable. Note that $d(M)$ is $\emptyset$ in $(N \,\&\, M)\restriction_K$ because K and L are disjoint. On the other hand, $\vdash_K N^\perp, N$ is provable by Lemma 2. Thus we have the following proof of $\vdash_K \mathcal{M}\restriction_K, N$:

$$\dfrac{\vdash_K \mathcal{M}\restriction_K, (N \,\&\, M)\restriction_K \quad \dfrac{\vdash_K N^\perp, N}{\vdash_K N^\perp \oplus M^\perp, N} \oplus_1}{\vdash_K \mathcal{M}\restriction_K, N} \mathit{cut}$$

■

Proof of Lemma 7. By induction on A. In particular, when $A \equiv \downarrow_K N_L$, by the induction hypothesis, there is unique $a \in |N_L\restriction_\emptyset|^L$ such that $N_L = N_L\restriction_\emptyset \langle a\rangle$. By taking $(*_\downarrow)_K \in \{*_\downarrow\}^K$, we have $(\downarrow_K N_L) = (\downarrow_K N_L)\restriction_\emptyset \langle (*_\downarrow)_K + a\rangle$. ■

Proof of Proposition 2. By induction on the number of & in $\mathcal{M}[*_1, \ldots, *_n]$.
(Base case) By identifying $⅋$'s with commas, $\mathcal{M}$ is of the form $\dagger_1 P_1, \ldots, \dagger_n P_n$. Then the side condition is written as $K \subseteq \bigcap_{\dagger_m \in \partial\mathcal{M}} d(\dagger_m)$. The given γ is of the form $\delta \times b$ such that $\delta \in (|\dagger_1 P_1| \times \cdots \times |\dagger_n P_n|)^{K+J}$ and $b \in |{\downarrow} M|^{K+J}$. Note that $b\restriction_K$, the restriction of J-indexed family b to K, is a constant family $(*_\downarrow)_K$. Then the side condition equivalently says that, $\delta\restriction_K = ((*_{\dagger_1}, \ldots, *_{\dagger_n}))_K$ where $(*_{\dagger_1}, \ldots, *_{\dagger_n}) \in \mathsf{mp}(\mathsf{M})$. This is equivalent that γ is right-multi-pointed: When $(x, *_\downarrow) \in \gamma$, we have $(x, *_\downarrow) \in \gamma\restriction_K = \delta\restriction_K \times b\restriction_K$, which means $x \in \mathsf{mp}(\mathsf{M})$.
(Induction case) We assume, without loss of generality, that $\mathcal{M}$ is of the form $\mathcal{L}, B_1 \,\&\, B_2$. Let $d(B_1) = K_1 + J_1$ and $d(B_2) = K_2 + J_2$ such that $K = K_1 + K_2$ and $J = J_1 + J_2$.
Since & is reversible, from the premise sequent $\vdash_J (\mathcal{L}, B_1 \,\&\, B_2)\restriction_J, N$ of given $\downarrow_K$-rule, we obtain

$$\dfrac{\vdash_{J_1} \mathcal{L}\restriction_{J_1}, B_1\restriction_{J_1}, N\restriction_{J_1} \quad \vdash_{J_2} \mathcal{L}\restriction_{J_2}, B_2\restriction_{J_2}, N\restriction_{J_2}}{\vdash_J \mathcal{L}\restriction_J, B_1\restriction_{J_1} \& B_2\restriction_{J_2}, N} \&$$

Then by applying $\downarrow_{K_i}$-rule $(i = 1, 2)$ for each premise sequent, we have

$$\frac{\vdash_{J_i} \mathcal{L}\restriction_{J_i}, B_i\restriction_{J_i}, N\restriction_{J_i}}{\vdash_{K_i+J_i} \mathcal{L}\restriction_{K_i+J_i}, B_i, \downarrow_{K_i}(N\restriction_{J_i})} \downarrow_{K_i}$$

where $\downarrow_{K_i}(N\restriction_{J_i}) = (\downarrow_K N)\restriction_{K_i+J_i}$. Note that the side conditions $K_i \subseteq d(\partial(\mathcal{L}\restriction_{K_i+J_i}, B_i))$ for the $\downarrow_{K_i}$-rules $(i = 1, 2)$ are equivalent to the condition $K \subseteq d(\partial\mathcal{M})$ for the $\downarrow_K$-rule because

$$d(\partial\mathcal{M}) = d(\partial(\mathcal{L}, B_1 \,\&\, B_2)) = d(\partial(\mathcal{L}\restriction_{K_1+J_1}, B_1)) + d(\partial(\mathcal{L}\restriction_{K_2+J_2}, B_2))$$

On the other hand, $\gamma \in |\mathsf{L}, \mathsf{B}_1 \,\&\, \mathsf{B}_2, \downarrow\mathsf{N}|^{K+J}$ is written as

$$\gamma = \gamma_1 + \gamma_2 \in |\mathsf{L}, \mathsf{B}_1, \downarrow\mathsf{N}|^{K_1+J_1} + |\mathsf{L}, \mathsf{B}_2, \downarrow\mathsf{N}|^{K_2+J_2}.$$

Thus, by the induction hypothesis, for each $i = 1, 2$, the side condition for the above $\downarrow_{K_i}$-rule is, in effects, states that γ_i is right-multi-pointed. Since the right-multi-pointedness of γ is equivalent to the right-multi-pointedness of both γ_i, we have obtained what we wanted to prove. ■

Proof of Proposition 4.

– When σ is $\dfrac{\overset{\sigma_1}{\vdash_J \Delta, N} \quad \overset{\sigma_2}{\vdash_J \Lambda, N^{\perp}}}{\vdash_J \Delta, \Lambda} \; cut$

there is, by Lemma 7, $\gamma \in |\Delta, \Lambda|^J$ such that $\Delta_J, \Lambda_J = (\Delta, \Lambda)\langle\gamma\rangle$. So γ is of the form $\delta \times \lambda$ such that $\Delta_J = \Delta\langle\delta\rangle$ and $\Lambda_J = \Lambda\langle\lambda\rangle$. For the left-upper sequent $\vdash_J \Delta, N$, there is a pair of MALLP-formula N and J-indexed family $a \in |\mathsf{N}|^J$ such that $\Delta_J, N_J = \Delta\langle\delta\rangle, \mathsf{N}\langle a\rangle$. Hence $\delta \times a \in (\sigma_1\restriction_{\emptyset}\ ^*)^J$ holds by the induction hypothesis. For the right-upper sequent $\vdash_J \Lambda, N^{\perp}$, note first that $N_J = \mathsf{N}\langle a\rangle$ yields $N_J^{\perp} = (\mathsf{N}\langle a\rangle)^{\perp} = \mathsf{N}^{\perp}\langle a\rangle$. Thus the sequence $\Lambda_J, N_J^{\perp}$ is of the form $\Lambda\langle\lambda\rangle, \mathsf{N}^{\perp}\langle a\rangle$, hence $\lambda \times a \in (\sigma_2\restriction_{\emptyset}\ ^*)^J$ holds by the induction hypothesis. To sum up, we have $\gamma = \delta \times \lambda \in |\Delta, \Lambda|^J$ with

$$\delta \times a \in (\sigma_1\restriction_{\emptyset}\ ^*)^J \quad \text{and} \quad \lambda \times a \in (\sigma_2\restriction_{\emptyset}\ ^*)^J.$$

By the relational composition to interpret cut-rule, we conclude $\gamma \in (\sigma\restriction_{\emptyset}\ ^*)^J$. ■