A Direct Independence Proof of Buchholz's Hydra Game on Finite Labeled Trees

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Abstract

We shall give a direct proof of the independence result of a Buchholz style-Hydra Game on labeled finite trees. We shall show that Takeuti-Arai's cut-elimination procedure of $(\Pi^1_1 - CA) + BI$ and of the iterated inductive definition systems can be directly expressed by the reduction rules of Buchholz's Hydra Game. As a direct corollary the independence result of the Hydra Game follows.

1 Introduction

Kirby-Paris [10] introduced the "Hercules and Hydra" game on finite *unlabeled* trees, and showed that the the statement "*Hercules eventually chops all heads of Hydra*" is true but cannot be proved in Peano Arithmetic. Martin Gardner in his Scientific American article [9] remarked that a hydra of Kirby-Paris' Hydra Game grew up only in width and did not grow up in height, and he suggested the possibility to make a much stronger version of Hydra Game which grew both in width and in height. Buchholz [4] introduced a version of Hydra Game in which each node of a hydra had a label and which grew up both in width and in height, and showed that the corresponding statement is true but cannot be proved even in a certain subsystem of analysis, called $(\Pi^1_1 - CA) + BI$. In that paper, Buchholz extended the ordinal analysis of the Hardy hierarchy by Cichon [5], Ketonen-Solovay [11] and Schmidt [15] to that of the generalized Hardy hierarchy, and connected it to an infinitary proof system which can characterize the provability (for Π^0_2-sentences) of $(\Pi^1_1 - CA) + BI$, via the generalized Hardy hierarchy. On the other hand, the aim of our paper is to directly connect Buchholz's Hydra Game to the Gentzen-style proof-reduction for traditional finitary

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proof systems, without any bypass of the Hardy hierarchy nor infinitary derivations. Our basic idea is to interpret the Gentzen-style finitary proof reduction (formulated by Takeuti [16] and by Arai [2]) in terms of the reduction rules of Buchholz's Hydra Game directly. In other words, the Gentzen-Takeuti style traditional proof reduction procedure is identified as a specific (winning) strategy of the Hydra Game. As an immediate corollary of this interpretation, with the help of Gödel's incompleteness theorem, it follows that the termination of the Hydra Game is not provable in the corresponding subsystem of analysis. The fact that the Hydra Game is actually terminating can be also immediately shown by a direct interpretation of the finite labeled trees into the ordinal diagrams from $O(\omega+1,1)$, as shown in Okada [14]. Moreover, since accessibility of any fixed ordinal of $O(\omega+1,1)$ is provable in $(\Pi^1_1-CA)+BI$ [1], with the help of the ordinal interpretation of hydras in [14], we can see;

Proposition 1 (Buchholz Theorem 2 in [4]) $(\Pi^1_1-CA)+BI \vdash$ *"Every recursive strategy starting with a fixed hydra is a winning strategy."*

Our proof technique for $(\Pi^1_1-CA)+BI$ can be extended to the system of Iterated Inductive Definitions (ID_ξ), by directly connecting Arai's cut-elimination procedure [3] with a naturally extended version of Buchholz's Hydra Game, as sketched in the note at the end of Section 2.

On the other hand, a naturally restricted version of Buchholz's Hydra Game (obtained by restricting the hydras to the one-dimensional hydras) can essentially be identified with the original Kirby-Paris' Hydra Game [10] (on the two-dimensional but unlabeled hydras), as shown in [8].

2 Buchholz's Hydra Game and Takeui-Arai's Proof Reduction for $\Pi^1_1-CA+BI$

2.1 Buchholz's Hydra Game on Finite Labeled Trees

In this section we shall introduce Buchholz's Hydra Game [4], which is an extension of the original version of Kirby-Paris's Hydra Game [10]. We shall identify the hydras with a certain subset $\mathcal{H}_0$ of the domain of Takeuti's system $O(\omega+1,1)$ of ordinal diagrams. Buchholz's Hydra Game shall be considered as a term rewriting game defined over $\mathcal{H}_0$. See Okada [14] for the identification between two-dimensional trees labeled from $\omega+1=\{0,1,\ldots\omega\}$ and Takeuti's system $O(\omega+1,1)$ of ordinal diagrams. The reader who is not familiar with the ordinal diagrams may refer to [16, 13].

Definition 1 (Set $\mathcal{H}$ of ordinal diagrams)

1. *If* $i\leq\omega$ *and* $\alpha\in\mathcal{H}\cup\{0\}$ *then* $(i,\alpha)\in\mathcal{H}$.
2. *If* $\alpha_1,\ldots,\alpha_n\in\mathcal{H}$ *then* $\alpha_1\#\cdots\#\alpha_n\in\mathcal{H}$.

$\alpha \in \mathcal{H}$ is connected if it has the form (i, β) for $\beta \in \mathcal{H}$.

Fact 1 *For each $\alpha \in \mathcal{H}$ there are uniquely determined connected $\alpha_0, \ldots, \alpha_n$ such that $\alpha = \alpha_0 \# \cdots \# \alpha_n$.*

$\mathcal{H}$ is a subset of Takeuti's system $O(\omega + 1, 1)$ of ordinal diagrams. The elements of $\mathcal{H}$ are interpreted one to one as the two-dimensional forests labeled from $\omega+1 = \{0, 1, \ldots \omega\}$ (Cf. Okada [14]). For example, $(0,(6,(4,0)\#(5,0))\#(3,0)\#(2,(1,0))) \in \mathcal{H}$ is interpreted as the Tree 1 below.

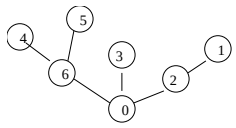

Figure 1: Tree 1

Notation 1 $\underline{1} = (0, 0)$

Definition 2 *$\mathcal{H}_0 = \{(0, \beta) \mid \beta \in \mathcal{H}\}$. An element of $\mathcal{H}_0$ is called a hydra.*

Definition 3 *$\mathcal{H}[*]$ denotes the set of all expressions which results from some $\alpha \in \mathcal{H}$ by replacing exactly one occurrence of $\underline{1}$ by $*$. An element of $\mathcal{H}[*]$ is called a context. A hydra context is a context which results from $\alpha \in \mathcal{H}_0$. If $c \in \mathcal{H}[*]$ and $u \in \mathcal{H} \cup \mathcal{H}[*]$, then $c[u]$ denotes the result of substituting u for $*$ in c. $c \in \mathcal{H}[*]$ is said to be connected if it is of the form (i, d) where $d \in \mathcal{H}[*]$.*

Fact 2

- *If $a, c \in \mathcal{H}[*]$, then $c[a] \in \mathcal{H}[*]$ and $(c[a])[u] = c[a[u]]$ for $u \in \mathcal{H} \cup \mathcal{H}[*]$.*
- *If $\alpha \in \mathcal{H}$ and $c \in \mathcal{H}[*]$, then $c[\alpha] \in \mathcal{H}$.*

r any $c \in \mathcal{H}[*]$, $c[0] \in \mathcal{H} \cup \{0\}$

Definition 4 `Succ` *is the set of all contexts of the form $\alpha\#*, \alpha\# * \#\beta, *\#\beta$, or $*$, where $\alpha, \beta \in \mathcal{H}$.*

Fact 3 *If $a, c \in$* `Succ` *then $c[a] \in$* `Succ`*.*

In the above definition of $c[u]$ we assumed $u \neq 0$ (since $0 \notin \mathcal{H} \cup \mathcal{H}[*]$). Now we extend the notation $c[u]$ for the case $u = 0$.

Definition 5 *For any $c \in \mathcal{H}[*]$, $c[0] \in \mathcal{H} \cup \{0\}$ is defined as follows.*

$$c[0] \equiv \begin{cases} \alpha & \text{if } c \in \texttt{Succ} \text{ is of the form } \alpha\#* \text{ or } *\#\alpha. \\ \alpha\#\beta & \text{if } c \in \texttt{Succ} \text{ is of the form } \alpha\# * \#\beta. \\ c[0/*] & \text{otherwise, where } c[0/*] \text{ is a term obtained from } c \text{ by substituting } 0 \text{ for } *. \end{cases}$$

Definition 6 (cf. Def 26.5 in [16]) *For $i \leq \omega$ the subset $\mathcal{C}_i$ of $\mathcal{H}[*]$ is defined inductively as follows.*

1. $* \in \mathcal{C}_i$
2. *If $\alpha \in \mathcal{H}$ and $c \in \mathcal{C}_i$, then $\alpha\#c, c\#\alpha \in \mathcal{C}_i$.*
3. *If $i < k$ and $c \in \mathcal{C}_i$ then $(k,c) \in \mathcal{C}_i$.*

We shall consider a term rewriting system $(\mathcal{H}, R)$ as follows.

Definition 7 (cf. Buchholz [4]) *$(\mathcal{H} \cup \{0\}, R)$ denotes the term rewriting system over $\mathcal{H} \cup \{0\}$ with a set R of the following three kinds of schemata of reduction rules.*

Rule 1 $(i, a[\underline{1}]) \hookrightarrow (i, a[0]) \cdot (n+1)$ *for an arbitrary $n \in N$ if $a \in \mathtt{Succ}$, where $\alpha \cdot n$ means $\underbrace{\alpha\# \cdots \#\alpha}_{n}$.*

Rule 1' $c[\underline{1}] \hookrightarrow c[0]$, *if $c \in \mathtt{Succ}$.*

Rule 2 $(j, a[(i+1, 0)]) \hookrightarrow (j, a[(i, a[\underline{1}])])$ *if $a \in \mathcal{C}_i$ and $j \leq i$.*

Rule 3 $(\omega, 0) \hookrightarrow (n+1, 0)$ *for an arbitrary $n \in N$.*

We write the above reduction schemata as $\overset{Rule1}{\hookrightarrow}$, $\overset{Rule1'}{\hookrightarrow}$, $\overset{Rule2}{\hookrightarrow}$ and $\overset{Rule3}{\hookrightarrow}$.

For the basic notions and notations of term rewriting systems the reader may refer to Klop [12]. Our term rewriting system $(\mathcal{H} \cup \{0\}, R)$ is, in particular, a ground term rewriting system composed of infinitely many ground reduction rules expressed by the four schemata. Here, a term rewriting system is a *ground* system if all the terms in the reduction rules are ground (i.e., closed).

We use notation $\hookrightarrow$ for the one step reduction relation. A *strategy* (starting with α_0) is a reduction sequence of the form $\alpha_0 \hookrightarrow \alpha_1 \hookrightarrow \cdots$. A *winning* strategy (starting with α_0) is a strategy $\alpha_0 \hookrightarrow \alpha_1 \hookrightarrow \cdots$ which eventually terminates (in a finite number of steps).

Buchholz's Hydra Game (BHG) [4] is a term rewriting game defined over terms of $\mathcal{H}_0 \cup \{0\}$ in $(\mathcal{H} \cup \{0\}, R)$: *Steps* of BHG consist of Rule 1, Rule 1', Rule 2 and Rule 3 of $(\mathcal{H} \cup \{0\}, R)$ described as above: In each step of BHG a redex in a hydra is chosen and rewritten into its contractum for an arbitrarily chosen n. A *strategy* of BHG is a strategy starting with a hydra.

It is easily observed that in $(\mathcal{H} \cup \{0\}, R)$ if a hydra has a normal form (i.e., irreducible form) it must be 0. Hence a winning strategy starting with a hydra must eventually terminate at 0. Here, for any strategy $a_0 \hookrightarrow a_1 \cdots$, if $a_0 \in \mathcal{H}_0$ then for any a_j, $a_j \in \mathcal{H}_0$ or a_j is of the normal form 0. The property that every strategy of BHG is a winning strategy was shown in [4] (cf. also in [14]). We

call this property *termination property of BHG.*

Remark 1: In Buchholz [4] the hydras are considered as *structured trees* in the sense that the direct children of a node is *ordered*. In this paper, for technical simplicity we consider the hydras as *unstructured trees* in the sense that the order of direct children of a node is not concerned, which corresponds to the associative-commutative property of natural sum # of the ordinal diagrams. The identification of the unstructured trees is represented by the equality $\sim$ of ordinal diagrams (Def 26.4 in [16]). This modification of the definition of hydras does not affect the independence result (cf. Remark 3 after Theorem 2).

Fact 4 *If* $c \in \mathtt{Succ}$ *and* $\alpha \in \mathcal{H} \cup \{0\}$, *then* $c[\alpha] \sim c[0]\#\alpha \sim \alpha\#c[0]$.

The main purpose of this Section is to show that the reduction relation $\hookrightarrow$ is strong enough for proving the consistency of $(\Pi^1_1 - CA) + BI$ by a finitary method (Theorem 1 in Subsection 2.2). Until the end of this Subsection we shall investigate various properties of the reductions of BHG, which will be necessary in Subsection 2.2. We shall use the termination property of BHG for this investigation (cf. Remark 2 at the end of this Section).

Notation 2 *Let* $\hookrightarrow^+$ *denote the transitive closure of* $\hookrightarrow$, *and* $\hookrightarrow^*$ *denote the transitive and reflexive closure of* $\hookrightarrow$ *modulo* $\sim$. *Especially* $\overset{Rule\cdots}{\hookrightarrow}{}^+$ *and* $\overset{Rule\cdots}{\hookrightarrow}{}^*$ *denote the transitive closure of* $\overset{Rule\cdots}{\hookrightarrow}$ *and transitive and reflexive closure of* $\overset{Rule\cdots}{\hookrightarrow}$ *modulo* $\sim$, *respectively.*

Lemma 1 *If* $c[\underline{1}] \in \mathcal{H}_0$, *then* $c[\underline{1}] \hookrightarrow c[0]$.

Lemma 2 *If* $c[(k,\alpha)] \in \mathcal{H}_0$ *with* $\underline{1} \neq \alpha \in \mathcal{H}$ *and* $k \leq \omega$, *then* $c[(k,\alpha)] \hookrightarrow c[(k,\alpha')]$ *for some* $\alpha' \in \mathcal{H}$.

Proof.
Case 1: $\alpha = a[\underline{1}]$. Then the assertion follows from Lemma 1.
Case 2: $\alpha = a[(i+1,0)]$. Then $c[(k,\alpha)] = c'[(i+1,0)] \hookrightarrow c'[(i,d[\underline{1}])]$ with $c' = c[(k,a)]$ and $d \in \mathcal{H}[*]$. Note that $c'[(i,d[\underline{1}])] = c[(k,a[(i,d[\underline{1}])])]$. Hence $c[(k,\alpha)] \hookrightarrow c[(k,\alpha')]$ with $\alpha' = a[(i,d[\underline{1}])]$.
Case 3: $\alpha = a[(\omega,0)]$. Then $c[(k,\alpha)] \hookrightarrow c[(k,\alpha')]$ with $\alpha' = a[(n+1,0)]$.
$\Box$.

Lemma 3 *1. If* $c[(k,\alpha)] \in \mathcal{H}_0$ *with* $\alpha \in \mathcal{H}$ *and* $k \leq \omega$, *then* $c[(k,\alpha)] \hookrightarrow^+ c[(k,0)]$.

2. If $c[(k,0)] \in \mathcal{H}_0$ *and* $j < k \leq \omega$, *then* $c[(k,0)] \hookrightarrow^+ c[(j,0)]$

Proof.
1. Consider the following reduction sequence, each step of which denotes the

reduction guaranteed by Lemma 2:

$$c[(k,\alpha)] \hookrightarrow c[(k,\alpha_1)] \hookrightarrow c[(k,\alpha_2)] \hookrightarrow \cdots$$

Since $c[(k,\alpha)] \in \mathcal{H}_0$, the termination property of BHG implies that the above reduction sequence terminates in a finite number of steps at $c[(k,\alpha_n)]$, where $\alpha_n \in \mathcal{H}$ or $\alpha_n = 0$ since $(k,\alpha_n) \in \mathcal{H}$. If $\alpha_n \in \mathcal{H}$, then there exists $\alpha_{n+1} \in \mathcal{H} \cup \{0\}$ such that $c[(k,\alpha_n)] \hookrightarrow c[(k,\alpha_{n+1})]$ by Lemma 1 and Lemma 2, which is a contradiction. Hence $\alpha_n = 0$.
2. is proved by induction on k.

- The case where $k = i+1$. $c[(i+1,0)] \overset{Rule2}{\hookrightarrow} c[(i,\alpha)]$ for some $\alpha \in \mathcal{H}$. ¿From (1), $c[(i,\alpha)] \hookrightarrow^+ c[(i,0)]$. Then either $i = j$ or $c[(i,0)] \overset{IH}{\hookrightarrow^+} c[(j,0)]$.
- The case where $k = \omega$. $c[(\omega,0)] \overset{Rule3}{\hookrightarrow} c[(j+1,0)] \overset{IH}{\hookrightarrow^+} c[(j,0)]$.

□

Lemma 4 *The following reduction relation is derivable from R, which will be called Rule 4.*
Rule 4 $c[(k,\alpha)] \hookrightarrow^+ c[(j,0)]$ *with an arbitrary* $j \leq k \leq \omega$, *if* $c[(k,\alpha)] \in \mathcal{H}_0$ *and* $\alpha \neq 0$.

Proof. The assertion follows from Lemma 3.
□

Lemma 5 *The following reduction relation is derivable from R, which will be called Rule 5.*
Rule 5 $c[b[\underline{1}]] \hookrightarrow^* c[b[0]\#\underline{1}]$, *if* $c[b[\underline{1}]] \in \mathcal{H}_0$.

Proof. We prove the assertion by induction on the length of b.

- The case where $b \in \mathtt{Succ}$. $c[b[\underline{1}]] \sim c[b[0]\#\underline{1}]$.
- The case where $b = d[(i,b')]$ with $d \in \mathtt{Succ}$. $c[b[\underline{1}]] = c[d[(i,b'[\underline{1}])]] \overset{IH^*}{\hookrightarrow} c[d[(i,b'[0]\#\underline{1})]] \overset{Rule1}{\hookrightarrow} c[d[(i,b'[0])\#(i,b'[0])]] \overset{Lemma3}{\hookrightarrow} c[d[(i,b'[0])\#\underline{1}]] \overset{Fact\,3,4}{\sim} c[d[(i,b'[0])]\#\underline{1}] = c[b[0]\#\underline{1}]$

□

Lemma 6 *The following reduction relations are derivable from R, which will be called Rule 4' and Rule 4", respectively.*

Rule 4' $c[\alpha] \hookrightarrow^+ c[\underline{1}]$ *if* $c[\alpha] \in \mathcal{H}_0$ *and* $\underline{1} \neq \alpha \in \mathcal{H}$.

Rule 4" $c[\alpha] \hookrightarrow^+ c[0]$ *if* $c[\alpha] \in \mathcal{H}_0$ *and* $\alpha \in \mathcal{H}$.

Proof.
(Derivability of Rule 4'): ¿From Fact 1, $\alpha = \alpha_0 \# \cdots \# \alpha_n$ with connected $\alpha_0, \cdots, \alpha_n \in \mathcal{H}$. Hence Rule 4' follows from Rule 4 and Lemma 1.
(Derivability of Rule 4") follows from the above Rule 4' and Lemma 1.
□

Under the tree interpretation, Rule 4 and Rule 5 correspond to the tree reductions of hydras sketched as follows.

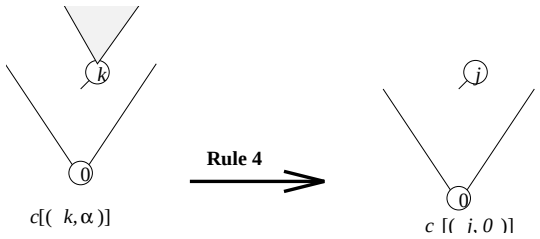

Figure 2: Decreasing the label of the root of a subtree and deleting that subtree except for the root.

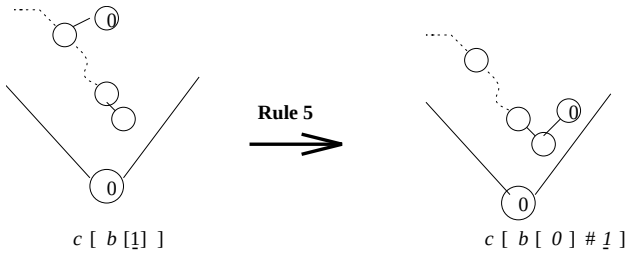

Figure 3: Shift-down of a 0-branch.

Remark 2: Each reduction in Lemma 3, Lemma 4, Lemma 5 and Lemma 6 has been shown to be derivable from Rule 1, Rule 2 and Rule 3 with the help of the termination property of BHG (i.e. strong normalizability of $(\mathcal{H}, R)$). In fact, the proofs for the derived rules show that the weak normalization property, rather than the strong normalization property, of BHG is enough. By a suitable arithmetization, we can easily verify the following:
$WA \vdash$ ((Each hydra $\in \mathcal{H}_0$ is weakly normalizable) $\Rightarrow$ (Rule 4, Rule 4', Rule 4" and Rule 5 are derivable from Rule 1, Rule 2 and Rule 3)).

Here, WA denotes a weak second order arithmetic which is a conservative extension of Primitive Recursive Arithmetic (PRA).

2.2 Direct Interpretation of Takeuti-Arai's Proof Reductions by Means of Buchholz's Hydra Game

Here we deal with subsystem ISN (Takeuti [16]) of Analysis, which is equivalent to $\Pi^1_1 - CA + BI$.

Definition 8 (Def.1.1 in Arai [2], Def.27.16 in Takeuti [16]) *By Isolated Number Theory ISN we mean the subsystem of second order arithmetic where the second order $\forall$-left rule is as follows;*

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \phi F(\phi), \Gamma \rightarrow \Delta}$$

(1) the principal formula $\forall \phi F(\phi)$ is isolated, or
(2) the abstract V in $F(V)$ is a second order free variable.

For a proof P in ISN, a proof with degree $< P, d >$ is defined as in Takeuti [16] Def 27.18. Now we introduce the notion of *proof with degree and height* by modifying Def 27.19 of Takeuti [16].

Definition 9 *Let $< P, d >$ be a proof with degree, and h a function assigning to each sequent S in P a natural number $h(S)$. Then $< P, d, h >$ is called a proof with degree and height and $h(S)$ is called a height of S in P if the following conditions are satisfied. Here $g_0(D)$, the grade of a formula D, denotes the number of occurrences of logical symbols in D.*

1. *$h(S) = 0$ if S is the end sequent of P, or S is the upper sequent of a substitution in P.*
2. *$h(S) \geq h(S')$ if S is an upper sequent of an inference except substitution, cut, ind, and isolated second order $\forall$;left, where S' is the lower sequent of the inference.*
3. *$h(S) \geq max\{h(S'), g_0(D)\}$ if S is an upper sequent of cut, or isolated second order $\forall$;left, where D is the cut formula or auxiliary formula of the inference respectively, and S' is the lower sequent of the inference.*
4. *$h(S) \geq max\{h(S'), g_0(D)\} + 1$ if S is an upper sequent of an ind, where D is the induction formula of the inference.*
5. *$h(S') = h(S'')$ if S' and S'' are the left upper sequent and the right upper sequent, respectively, of an inference.*

(The original definition(Takeuti [16]) of height is described as follows: In (2) and (3), $\geq$ is changed by $=$, and (4) is contained in (3); no distinctive assignment of height between cut, isolated $\forall$*;left, and ind is made.)*

In Takeuti-Arai's reduction method, an ordinal diagram from $O(\omega{+}1{,}1)$ is assigned to a proof with degree and height. Here in order to adjust Buchholz's Hydra Game Rules, we modify $\omega(n+1,\mu)$ of Arai [2] in the following manner.

Definition 10

- $\omega(0,\mu) = \mu$
- $\omega(n+1,\mu) = (\omega, \underline{1}\#\omega(n,\mu))$

The original definition of $\omega(n+1,\mu)$ of [2] was $(\omega, \omega(n,\mu))$.

The ordinal assignment to each sequent of a proof with degree and height is modified based on that in Arai [2], as follows.

Definition 11 (A modification of Def 1.8 in Arai [2], or of Def 27.20 in Takeuti [16]) [1]

1 $O(S) = \underline{1}$ *if* S *is an initial sequent of* P*.*

2 *Suppose that o.d.'s of S' (and of S") have been assigned for J, where*

$$\frac{S' \qquad (S'')}{S}\ J$$

Here S" does not exist when J has a unique upper sequent.
Then O(S) is determined as follows.

2.1 *If J is a weak structural inference, then* $O(S) = \omega(h(S')-h(S), O(S'))$

2.2 *If J is* $\neg$, $\wedge; left$*, first order* $\forall$*, second order* $\forall; right$*, or explicit second order* $\forall; left$*, then* $O(S) = \omega(h(S') - h(S), \underline{1}\#O(S'))$

2.3 *If J is* $\wedge; right$*, then* $O(S) = \omega(h(S') - h(S), O(S')\#O(S''))$

2.4 *If J is an implicit second order* $\forall; leftX$*, then* $O(S) = \omega(h(S') - h(S), (\omega, 0)\#O(S'))$

[1] In connection with Buchholz's Hydra Game the ordinal assignment defined in Arai [2] is more convenient than that defined in Takeuti [16]: In the case (4) below, we followed the assignment of [2]. On the other hand [16] assigns $O(S) = \omega(h(S') - h(S) + 1, O(S'))$ in the case (4). When we make a connection of *Rapid Hydra Game* described in Remark 4 in the end of Section 2, this minor difference does not matter.

2.5 *If J is an induction, then* $O(S) = \omega(h(S') - h(S), \underline{1}\#O(S'))$

2.6 *If J is a cut, then* $O(S) = \omega(h(S') - h(S), O(S')\#O(S''))$

2.7 *If J is a substitution, then* $O(S) = (d(J), \omega(1, O(S')))$

For each proof $< P, d, h >$ with degree and height in ISN we define , $O(P, d, h) := (0, \omega(1, O(S)))$, *where S is the endsequent of P and $O(S)$ is the ordinal diagram assigned to S through the above definition due to Arai. Actually $O(P, d, h)$ is a hydra.*

Through the above ordinal assignment, we can interpret a proof with degree and height as a hydra.

Example 1 (Tree interpretation of a proof in ISN via the ordinal assignment) *Let $< P, d, h >$ be a proof with degree and height of the following form;*

$$\dfrac{\dfrac{\dfrac{X(x) \overset{\underline{1}}{\rightarrow} X(x)}{\forall\phi\phi(x) \overset{\omega(1,\underline{1}\#\underline{1})}{\rightarrow} X(x)}\ 1}{\forall\phi\phi(x) \overset{(1,\omega(1,\omega(1,\underline{1}\#\underline{1})))}{\rightarrow} A(x)}\ Deg1\ 0 \qquad \dfrac{\dfrac{A(x) \overset{\underline{1}}{\rightarrow} A(x)}{\neg A(x), A(x) \overset{\underline{1}\#\underline{1}}{\rightarrow}}\ n+1}{\forall\phi\phi(x), A(x) \overset{\omega(1,\underline{1}\#\underline{1}\#\underline{1})}{\rightarrow}}\ n+1}{A(x), \forall\phi\phi(x) \overset{\omega(1,\underline{1}\#\underline{1}\#\underline{1})}{\rightarrow}}\ n \qquad n$$

$$\dfrac{\forall\phi\phi(x), \forall\phi\phi(x) \overset{\omega(n,\alpha_1\#\alpha_2)}{\rightarrow}}{\forall\phi\phi(x) \overset{\omega(n,\alpha_1\#\alpha_2)}{\rightarrow}}\ 0 \qquad 0$$

Here $n = g_0(A(x))$, $\alpha_1 = (1, \omega(1, \omega(1, \underline{1}\#\underline{1})))$, and $\alpha_2 = \omega(1, \underline{1}\#\underline{1}\#\underline{1})$. We have assigned an ordinal diagram to each sequent according to Definition 11 as the above. This yields $O(P, d, h) = (0, \omega(1, \omega(n, \alpha_1\#\alpha_2)))$. Then $O(P, d, h)$ is interpreted as Hydra 1 below.

Now we shall show that the cut elimination procedure from $< P, d >$ to $< P', d' >$ in Takeuti [16] (p335 $\sim$ p341) is interpretable by a finite number of steps of BHG with Rule 4 and Rule 5, each of which is derivable from BHG using the normalization of BHG as we showed in Remark 2.

Theorem 1 *If $< P, d, h >$ is a proof with degree and height and $< P', d' >$ is the proof with degree resulting from $< P, d >$ by Takeuti-Arai's one step proof reduction in their consistency proof for ISN [2, 16], then one can define a height h' for $< P', d' >$ such that $O(P, d, h) \hookrightarrow^* O(P', d', h')$.*

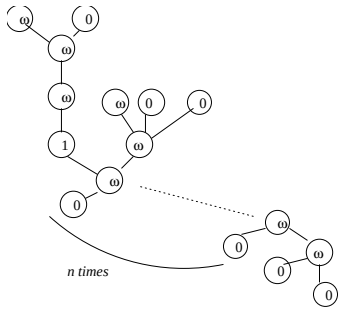

Figure 4: Hydra 1

Proof.
The following redundant inferences, called *term-replacement*, are added.

$$\frac{\Gamma_1, F(s), \Gamma_2 \to \Delta}{\Gamma_1, F(t), \Gamma_2 \to \Delta} \quad , \quad \frac{\Gamma \to \Delta_1, F(s), \Delta_2}{\Gamma \to \Delta_1, F(t), \Delta_2} \quad ,$$

where s and t are closed terms expressing the same numeral. Our definition of the assignment of height to the upper sequent of term-replacement follows Definition 9 (2), and that of the ordinal assignment of the lower sequent of term-replacement follows Definition 11 (2.1).

We shall show 1 and 2 below.

1. In each of (Preparation 0) $\sim$ (Preparation 2) below, either of the following conditions holds.

   (a) There exists such a height h' for $< P', d' >$ that $O(P', d', h') \equiv O(P, d, h)$.

   (b) There exists such a height h' for $< P', d' >$ that $O(P', d', h') \hookrightarrow^* O(P, d, h)$

2. In each of (Case 1) $\sim$ (Case 4), (b) holds but (a) does not hold.

(Preparation 0) Suppose that $< P, d, h >$ contains a free variable which is not used as an eigenvariable. $< P', d', h' >$ is obtained from $< P, d, h >$ by replacing the free variable by the constant 0. Obviously (a) holds.
(Case 1) Suppose that P contains an application of induction in its end-piece. Let I be a lower most induction in the end-piece of P.
$< P, d, h >$ is of the following form. Here l and $m + 1$ denote the height of

$A(0),\Gamma \to \Delta, A(t)$ and of $A(a),\Gamma \to \Delta, A(a')$, respectively, with $l \leq m$.

$$\dfrac{\dfrac{\dfrac{\vdots\ Q(a)}{A(a),\Gamma \overset{\omega(k-m,\alpha)}{\to} \Delta, A(a')}\ J}{A(0),\Gamma \overset{\omega(m-l+1,\underline{1}\#\omega(k-m,\alpha))}{\to} \Delta, A(t)}\ I}{\vdots \to} \quad \begin{matrix} k+1 \\ m+1 \\ l \end{matrix}$$

Hence $O(P,d,h) = c[\omega(m-l+1,\underline{1}\#\omega(k-m,\alpha))]$ with a suitable hydra context c.

(Case 1.1) The case where $t = 0$. $< P', d', h' >$ is obtained from $< P, d, h >$ by replacing the part above $A(0),\Gamma \to \Delta, A(t)$ of $< P, d, h >$ by the following.

$$\dfrac{\dfrac{\dfrac{A(0) \overset{\underline{1}}{\to} A(0)}{A(0),\Gamma \overset{\underline{1}}{\to} \Delta, A(0)}}{A(0),\Gamma \overset{\underline{1}}{\to} \Delta, A(t)}\ term-replacement}{}\ .$$

In this case the reduction sequence from $O(P,d,h)$ to $O(P',d',h')$ corresponds exactly to our Rule 4; $c[\omega(m-l+1,\underline{1}\#\omega(k-m,\alpha))] \overset{Rule4}{\hookrightarrow} c[\underline{1}]$.

(Case 1.2) The case where $t = n+1$ for some numeral n. $< P', d', h' >$ is obtained by replacing J by the following.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ Q(0)}{A(0),\Gamma \overset{\omega(k-m+1,\alpha)}{\to} \Delta, A(0')}\ J_0 \quad \dfrac{\vdots\ Q(0')}{A(0'),\Gamma \overset{\omega(k-m+1,\alpha)}{\to} \Delta, A(0'')}\ J_1}{A(0),\Gamma,\Gamma \to \Delta,\Delta, A(0'')}\ m}{A(0),\Gamma \to \Delta, A(0'')}\ m \qquad k+1$$

$$\ddots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{A(0),\Gamma \overset{\omega(k-m+1,\alpha)\cdot n}{\longrightarrow} \Delta, A(n)} \quad \dfrac{\vdots\ Q(n)}{A(n),\Gamma \overset{\omega(k-m+1,\alpha)}{\to} \Delta, A(n+1)}\ J_n}{A(0),\Gamma,\Gamma \overset{\omega(m-l,\omega(k-m+1,\alpha)\cdot(n+1))}{\longrightarrow} \Delta,\Delta, A(n+1)}\ m}{A(0),\Gamma \to \Delta, A(n+1)}\ l$$

$$\dfrac{A(0),\Gamma \to \Delta, A(t)}{\vdots \to}\ term-replacement$$

Hence $O(P', d', h') = c[\omega(m - l, \omega(k - m + 1, \alpha) \cdot (n + 1))]$.
Our reduction sequence from $O(P, d, h)$ to $O(P', d', h')$ is divided into the following two cases:
(Case 1.2.1) The case where $A(a), \Gamma \rightarrow \Delta, A(a')$ is an initial sequent of $< P, d, h >$. In this case J in $< P, d, h >$ does not exist and $J_0, \ldots, J_{n-1}$ do not exist in $< P', d', h' >$, either. Since there is no risk of confusion we may regard $k = m$ and $\alpha = \underline{1}$ in $< P, d, h >$ and $< P', d', h' >$ described the above. Then the reduction sequence is as follows. (Here the context $c[\cdots]$ is implicit.)[2]

$$
\begin{array}{rll}
 & \omega(m + 1 - l, \underline{1} \# \underline{1}) & \cdots (1) \\
\overset{Rule4}{\hookrightarrow^+} & \omega(m + 1 - l, 0) & \\
= & \omega(m - l, (\omega, \underline{1})) & \cdots (2) \\
\overset{Rule1}{\hookrightarrow} & \omega(m - l, (\omega, 0) \cdot (n + 1)) & \cdots (3) \\
\overset{Rule4}{\hookrightarrow^+} & \omega(m - l, (0, 0) \cdot (n + 1)) & \cdots (4)
\end{array}
$$

Under the tree interpretation the above reduction sequence is sketched as follows.

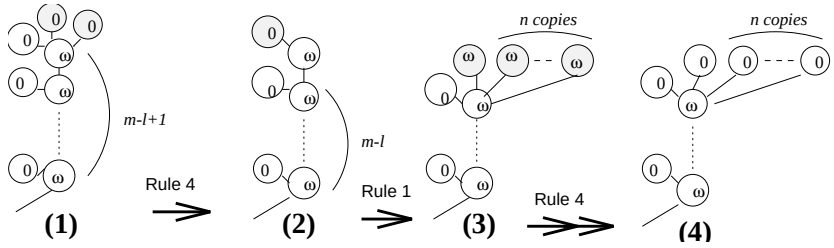

Figure 5: Tree Reduction of Case 1.2.1

(Case 1.2.2) The case where $A(a), \Gamma \rightarrow \Delta, A(a')$ is not an initial sequent of $< P, d, h >$. Let J be the inference in P whose lower sequent is $A(a), \Gamma \rightarrow \Delta, A(a')$. From the definition of *proof with degree* in ISN (cf. [16] Def 27.18 (1)), J must be an inference except a substitution. Hence h(an upper sequent of J)$= k$ with $m \leq k$. In $< P', d' >$, let J_i be a copy of J in $Q(i)$, and S_i be an upper sequent of J_i where $0 \leq i < n$. Therefore we shall define height h' for $< P', d' >$ as

$$
\begin{cases}
h'(A(i), \Gamma \rightarrow \Delta, A(i')) = m \\
h'(S_i) = k + 1
\end{cases}
$$

[2]For the simplicity of the representation, from now on, in describing a reduction sequence of term rewriting the context $c[\cdots]$ is not always explicitly written , where $c[\cdots]$ is a hydra context. We would like to remind the reader that Rule 4 (4', 4") and Rule 5 are derived rules of BHG only when the context $c[\cdots]$ is a hydra context. We shall not mention the underlying hydra context in every time.

(cf. Definition 9). By virtue of this height h' of $< P', d' >$, the reduction sequence from $O(P, d, h)$ to $O(P', d', h')$ is as follows.

$$\begin{array}{rl} & \omega(m-l+1, \underline{1}\#\omega(k-m, \alpha)) \\ = & \omega(m-l, \omega(1, \underline{1}\#\omega(k-m, \alpha))) \\ \overset{Rule1}{\hookrightarrow} & \omega(m-l, \omega(1, \omega(k-m, \alpha)) \cdot (n+1)) \\ = & \omega(m-l, \omega(k-m+1, \alpha) \cdot (n+1)) \end{array}$$

(Preparation 1) Suppose that the end-piece of P contains an equality axiom of the form $s = t, A(s) \to A(t)$ as an initial sequent with the height h. Let m and n denote numerals equal to s and t respectively.

If $\quad m = n \to \quad$ is an axiom, then $< P', d', h' >$ is obtained from $< P, d, h >$ by replacing the initial sequent $\quad s = t, A(s) \to A(t) \quad$ by the following.

$$\begin{array}{ll} \dfrac{\dfrac{\dfrac{}{m = n \overset{\underline{1}}{\to}}}{m = n, A(m) \to A(n)}}{term - replacements} & \begin{array}{l} h \\ \vdots \\ \end{array} \\ s = t, A(s) \overset{\underline{1}}{\to} A(t) & h \end{array}$$

By virtue of the assignment of ordinal diagrams concerned with a term-replacement defined in (Preparation 0), the ordinal diagram assigned to $s = t, A(s) \to A(t)$ remains the same through this replacement; it remains $\underline{1}$. Hence (a) holds. It is treated in the same way when $\quad \to m = n \quad$ is an axiom.

(Case 2) The case where the end-piece of P contains a logical initial sequent. $< P, d, h >$ is of the following form.

$$\begin{array}{ll} \dfrac{\begin{array}{cc} \vdots & D \to D \\ & \vdots \\ \Gamma \overset{\alpha}{\to} \Delta, \tilde{D} & \tilde{D}, \Pi \overset{\beta}{\to} \Lambda_1, \tilde{D}, \Lambda_2 \end{array}}{\begin{array}{c} \Gamma, \Pi \overset{\omega(m-l, \alpha\#\beta)}{\to} \Delta, \Lambda_1, \tilde{D}, \Lambda_2 \\ \vdots \\ \to \end{array}} J & \begin{array}{c} m \\ l \end{array} \end{array}$$

$< P', d', h' >$ is as follows.

$$\begin{array}{ll} \dfrac{\begin{array}{c} \vdots \\ \Gamma \overset{\alpha}{\to} \Delta, \tilde{D} \end{array}}{\begin{array}{c} \Gamma, \Pi \overset{\omega(m-l, \alpha)}{\to} \Delta, \Lambda_1, \tilde{D}, \Lambda_2 \\ \vdots \\ \to \end{array}} & \begin{array}{c} m \\ l \end{array} \end{array}$$

Note that our definition of height allows such an assignment h' in P' that h' is the restriction of h with respect to P' (cf. Definition 9 (2)). Consequently the reduction sequence from $O(P,d,h)$ to $O(P',d',h')$ corresponds exactly to our Rule 4": $\omega(m-l,\alpha\#\beta) \overset{Rule4''}{\hookrightarrow} \omega(m-l,\alpha)$.

(Preparation 2) The case where the end-piece of P contains a weakening. Let I be a lower most weakening inference in the end-piece. Since the end-sequent is empty, there must exist a cut J below I, whose cut formula is a descendant of the principal formula of I: $< P,d,h >$ is of the following form.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Pi_1 \overset{\alpha}{\rightarrow} \Lambda_1\end{array}}{D,\Pi_1 \overset{\omega(n-m,\alpha)}{\rightarrow} \Lambda_1}\,I \qquad n \qquad\qquad \begin{array}{c}\vdots\\ \Gamma \overset{\beta}{\rightarrow} \Delta,\tilde{D}\end{array} \quad \begin{array}{c}\vdots\\ \tilde{D},\Pi \overset{a[\omega(n-m,\alpha)]}{\rightarrow} \Lambda\end{array}}{\Gamma,\Pi \overset{\omega(l-k,\beta\#a[\omega(n-m,\alpha)])}{\rightarrow} \Delta,\Lambda}\,J \qquad \begin{array}{l} l \\ k \end{array}$$

(Preparation 2.1) The case where no contraction is applied to any descendant of D from the inference I through J. By using *exchanges* if necessary, $< P',d',h' >$ is obtained from $< P,d,h >$ as follows.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Pi_1 \overset{\alpha}{\rightarrow} \Lambda_1\end{array}}{\Pi_1 \overset{\omega(n-m,\alpha)}{\rightarrow} \Lambda_1}\,n}{\Pi \overset{a[\omega(n-m,\alpha)]}{\longrightarrow} \Delta}\,m}{\Gamma,\Pi \overset{\omega(l-k,a[\omega(n-m,\alpha)])}{\longrightarrow} \Delta,\Lambda}\,l \qquad \begin{array}{l} \\ \\ k \end{array}$$

In this case, in the same manner as in Case 2, the reduction sequence from $O(P,d,h)$ to $O(P',d',h')$ corresponds exactly to our Rule 4".

$$\omega(l-k,\beta\#a[\omega(n-m,\alpha)]) \overset{Rule4''}{\hookrightarrow} \omega(l-k,a[\omega(n-m,\alpha)])$$

(Preparation 2.2) The case where a contraction is applied to a descendant of D from the inference I through J. Let $\tilde{I}$ be an uppermost contraction applied to

a descendant of D. $< P', d', h' >$ is obtained by replacing the part of $< P, d, h >$ above $\tilde{D}, \Pi \to \Lambda$ as follows.

$$
\begin{array}{ccc}
< P, d > & & < P', d' > \\
\vdots & & \vdots \\
\dfrac{\Pi_1 \to \Lambda_1}{D, \Pi_1 \to \Lambda_1}\ I & \begin{matrix} n \\ m \end{matrix} & \dfrac{\dfrac{\Pi_1 \to \Lambda_1}{\Pi_1 \to \Lambda_1}\ \begin{matrix} n \\ m \end{matrix}}{\vdots} \\
\vdots & & \vdots \\
\dfrac{D', D', \Pi_1' \to \Lambda_1'}{D', \Pi_1' \to \Lambda_1'}\ \tilde{I} & \begin{matrix} n' \\ m' \end{matrix} & \dfrac{\dfrac{D', \Pi_1' \to \Lambda_1'}{D', \Pi_1' \to \Lambda_1'}\ \begin{matrix} n' \\ m' \end{matrix}}{\vdots} \\
\vdots & & \vdots \\
\tilde{D}, \Pi \to \Lambda & & \tilde{D}, \Pi \to \Lambda
\end{array}
$$

Through the above replacement the ordinal diagram assigned to $\tilde{D}, \Pi \to \Lambda$ remains the same. Hence (a) holds.

(Case 1), (Preparation 1), (Case 2), and (Preparation 2) having been in hand, we can assure that there exists a suitable cut in the end-piece of P (cf. Sublemma 12.9 in Takeuti [16]). Now let J be a suitable cut of $< P, d >$. The essential proof reductions are divided into the following two cases, (Case 3) and (Case 4), according to the outermost logical symbol of the cut formula of J.

(Case 3) The case where the cut formula of J is of the form $A \wedge B$, $\neg A$, $\forall x A(x)$, or non isolated $\forall \phi F(\phi)$.
Each proof reduction is treated similarly. Without loss of generality we consider the case $A \wedge B$. $< P, d, h >$ is of the following form.

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \overset{\alpha}{\to} \Delta_1, A_1} \quad \dfrac{\vdots}{\Gamma_2 \overset{\alpha'}{\to} \Delta_2, B_1}\ k_1}{\Gamma_1, \Gamma_2 \overset{\omega(k_1 - h_1, \alpha \# \alpha')}{\to} \Delta_1, \Delta_2, A_1 \wedge B_1}\ h_1 \quad \dfrac{\dfrac{\vdots}{A_2, \Pi_2 \overset{\beta}{\to} \Lambda_2}\ k_2}{A_2 \wedge B_2, \Pi_2 \overset{\omega(k_2 - h_2, \underline{1} \# \beta)}{\to} \Lambda_2}\ h_2}{\vdots} \\
\dfrac{\Gamma_3 \overset{a[\alpha']}{\to} \Delta_3, A \wedge B \quad l \qquad A \wedge B, \Pi_3 \overset{b[\underline{1}]}{\to} \Lambda_3}{\Gamma_3, \Pi_3 \to \Delta_3, \Lambda_3}\ J\ \begin{matrix} l \\ l \end{matrix} \\
\vdots \\
\dfrac{\cdots\cdots}{\Phi \overset{\omega(l - n, c[a[\alpha'] \# b[\underline{1}]])}{\longrightarrow} \Psi}\ \begin{matrix} l \\ n \end{matrix} \\
\vdots \\
\to
\end{array}
$$

$\Phi \to \Psi$ is the upper most sequent below J
whose height n is less than l, here l denotes the height of
the upper sequents of J.

Then, $< P', d', h' >$ is the following:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \overset{\alpha}{\to} \Delta_1, A_1}\ k_1}{\omega(k_1 - h_1, \alpha)\ \Gamma_1, \Gamma_2 \longrightarrow A_1, \Delta_1, \Delta_2, A_1 \wedge B_1}\ h_1 \quad \dfrac{\vdots}{\Gamma_3 \overset{a[0]}{\to} A, \Delta_3, A \wedge B} \qquad \dfrac{\vdots}{A \wedge B, \Pi_3 \overset{b[\underline{1}]}{\to} \Lambda_3}}{\Gamma_3, \Pi_3 \to A, \Delta_3, \Lambda_3} \qquad \dfrac{\dfrac{\dfrac{\vdots}{A_2, \Pi_2 \overset{\beta}{\to} \Lambda_2}\ k_2}{\omega(k_2 - h_2, \beta)\ A_2 \wedge B_2, \Pi_2, A_2 \longrightarrow \Lambda_2}\ h_2 \quad \dfrac{\vdots}{\Gamma_3 \overset{a[\alpha']}{\to} \Delta_3, A \wedge B} \qquad \dfrac{\vdots}{A \wedge B, \Pi_3, A \overset{b[0]}{\to} \Lambda_3}}{\Gamma_3, \Pi_3, A \to \Delta_3, \Lambda_3}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\cdots\cdots}\ l}{\Phi \overset{\omega(l-m, c[a[0]\#b[\underline{1}]])}{\longrightarrow} A, \Psi}\ m}{\overline{\Phi \to \Psi, A}\ m} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\cdots\cdots}\ l}{\Phi, A \overset{\omega(l-m, c[a[\alpha']\#b[0]])}{\longrightarrow} \Psi}\ m}{\overline{A, \Phi \to \Psi}\ m}}{\Phi, \Phi \overset{\omega(m-n, \omega(l-m, c[a[0]\#b[\underline{1}]])\#\omega(l-m, c[a[\alpha']\#b[0]]))}{\longrightarrow} \Psi, \Psi}\ n}{\overline{\Phi \to \Psi}} \quad \vdots \quad \to$$

As is shown in the above figure, the height h' for $< P', d' >$ is defined as follows.

- $h'(\Phi \to A, \Psi) = \cdots = h'(\Phi \to \Psi, A) = m$
  $h'(\Phi, A \to \Psi) = \cdots = h'(A, \Phi \to \Psi) = m$ with $n \leq m < l$.
- If S is a sequent of $< P', d' >$ except the above, then in $< P, d >$ there exists a sequent $\tilde{S}$, of which S is a copy, then $h'(S) = h(\tilde{S})$.

By virtue of this height assignment h' of $< P', d' >$, the reduction sequence from $O(P, d, h)$ to $O(P', d', h')$ is as follows:

$$\begin{array}{cc} & \omega(l-n, c[a[\alpha']\#b[\underline{1}]]) \\ = & \omega(m-n, \omega(l-m, c[a[\alpha']\#b[\underline{1}]])) \\ = & \omega(m-n, (\boldsymbol{\omega, \underline{1}\#\omega(l-m-1, c[a[\alpha']\#b[\underline{1}]])})) \end{array}$$

$$
\begin{array}{rc}
\overset{Rule1}{\hookrightarrow} & \omega(m-n,(\omega,\omega(l-m-1,c[a[\alpha']\#b[\underline{1}]]))\cdot 2) \\
= & \omega(m-n,(\omega,\omega(l-m-1,c[a[\boldsymbol{\alpha}']\#b[\underline{1}]]))\#(\omega,\omega(l-m-1,c[a[\alpha']\#b[\underline{1}]]))) \\
\overset{Rule4'}{\hookrightarrow} & \omega(m-n,(\omega,\boldsymbol{\omega(l-m-1,c[a[\underline{1}]\#b[\underline{1}]]}))\#(\omega,\omega(l-m-1,c[a[\alpha']\#b[\underline{1}]]))) \\
\overset{Rule5}{\hookrightarrow} & \omega(m-n,(\omega,1\#\omega(l-m-1,c[a[0]\#b[\underline{1}]]))\#(\omega,\boldsymbol{\omega(l-m-1,c[a[\alpha']\#b[\underline{1}]]}))) \\
\overset{Rule5}{\hookrightarrow} & \omega(m-n,(\omega,\underline{1}\#\omega(l-m-1,c[a[0]\#b[\underline{1}]]))\#(\omega,\underline{1}\#\omega(l-m-1,c[a[\alpha']\#b[0]]))) \\
= & \omega(m-n,\omega(l-m,c[a[0]\#b[\underline{1}]])\#\omega(l-m,c[a[\alpha']\#b[0]]))
\end{array}
$$

Here in each step the bold-face letters represent the redex part to be rewritten.

(Case 4) The case where the cut formula of J is of the form isolated $\forall\phi F(\phi)$.
$< P, d, h >$ is as follows.

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \overset{\alpha}{\to} \Delta_1, F(X)\end{array}}{\Gamma_1 \overset{\omega(k_1-h_1,\underline{1}\#\alpha)}{\to} \Delta_1\forall\phi F(\phi)}\begin{array}{c}k_1\\ h_1\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ F(V),\Pi_1 \overset{\beta}{\to} \Lambda_1\end{array}}{\forall\phi F(\phi),\Pi_1 \overset{\omega(k_2-h_2,(\omega,0)\#\beta)}{\to} \Lambda_1}\begin{array}{c}k_2\\ h_2\end{array}}{\begin{array}{c}\vdots\\ \Gamma_2 \overset{a[\underline{1}]}{\to} \Delta_2,\forall\phi F(\phi)\end{array} \qquad \begin{array}{c}\vdots\\ \forall\phi F(\phi),\Pi_2 \overset{b[(\omega,0)]}{\to} \Lambda_2\end{array}}
$$

$$
\dfrac{\Gamma_2,\Pi_2 \to \Delta_2,\Lambda_2}{\begin{array}{c}\vdots\\ \Gamma_3 \overset{c[\gamma]}{\to} \Delta_3\end{array}}
$$

$$
\dfrac{\Gamma_3 \overset{c[\gamma]}{\to} \Delta_3}{\begin{array}{c}\cdots \overset{(j,\omega(1,c[\gamma]))}{\to} \cdots\\ \vdots\\ \to\end{array}}
$$

where $\Gamma_3 \to \Delta_3$ is the i-resolvent of $\Gamma_2,\Pi_2 \to \Delta_2,\Lambda_2$ with $i = d(\forall\phi F(\phi))$, and $\gamma = a[\underline{1}]\#b[(\omega,0)]$.

$< P', d', h' >$ is as follows.

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \overset{\alpha}{\to} \Delta_1, F(X)\end{array}}{\Gamma_1 \overset{\omega(k_1-h_1,\alpha)}{\to} F(X),\Delta_1,\forall\phi F(\phi)}\begin{array}{c}k_1\\ h_1\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ F(V),\Pi_1 \overset{\beta}{\to} \Lambda_1\end{array}}{\forall\phi F(\phi),\Pi_1 \overset{\omega(k_2-h_2,(\omega,0)\#\beta)}{\to} \Lambda_1}\begin{array}{c}k_2\\ h_2\end{array}}{\begin{array}{c}\vdots\\ \Gamma_2 \overset{a[0]}{\to} F(X),\Delta_2,\forall\phi F(\phi)\end{array} \qquad \begin{array}{c}\vdots\\ \forall\phi F(\phi),\Pi_2 \overset{b[(\omega,0)]}{\to} \Lambda_2\end{array}}
$$

$$
\dfrac{\Gamma_2,\Pi_2 \to F(X),\Delta_2,\Lambda_2}{\Gamma_2,\Pi_2 \to F(X),\Delta_2,\Lambda_2}
$$

$$\dfrac{\dfrac{\dfrac{

Here in each step the bold-face letters represent the redex part to be rewritten. See Appendix on the tree interpretation of the above reduction sequence.
□

¿From Theorem 1, if a proof figure $< P, d, h >$ in ISN has the empty sequent " $\rightarrow$ " as the end sequent, then there exists such a strategy of BHG starting with a hydra $O(P, d, h)$ that corresponds to the Takeuti-Arai's cut elimination procedure for $< P, d, h >$. Hence by virtue of Gödel's Incompleteness Theorem, *"All such strategies that correspond to Takeuti-Arai's cut elimination procedures for a proof figure whose end sequent is the empty sequent " $\rightarrow$ " are winning strategies"* is not provable in ISN. Since a cut elimination procedure starting with some proof figure is recursive, we reach Buchholz's Theorem.

Theorem 2 (Buchholz Theorem 3 in [4]) $(\Pi^1_1 - CA) + BI \not\vdash$ *'Every recursive strategy starting with an arbitrary hydra is a winning strategy of BHG'.*

Remark 3: Although our definition of hydras is based on unstructured tree while the original definition of hydras of Buchholz [4] is based on structured trees (Remark 1). Theorem 2 above immediately implies the original independence result of Buchholz, too. This is observed by the fact that our hydras $\mathcal{H}_0$ is embeddable into Buchholz's T_0 by using the natural tree interpretation of the canonical representation with the help of ordinal sum $+$ for each equivalent class modulo $\sim$ of ordinal diagrams.

Remark 4 (Rapid Hydra Game (RHG)): It is at the reduction sequences shown in Case 1.2.2, in Case 2 and in Preparation 2 where our modification of height in Definition 9 works: In Case 1.2.2 the described proof reduction, which Gentzen called VJ Reduktion [7], corresponds exactly to Rule 1 under our modification of height. In Case 2 and in Preparation 2, we are permitted in $< P', d' >$ to assign the same height as in $< P, d >$ to the sequent which is in $< P, d >$ the upper sequent of the eliminated cut, the eliminated weakening, or the eliminated contraction. If we add Rule 6 to the schemata of reduction rules (i.e., Rule 1, Rule 2 and Rule 3) in Definition 7, then with the original assignment of height in [2], we can interpret the proof reduction in terms of this version of Hydra Game:

Rule 6 $(\omega, \alpha) \hookrightarrow \alpha$

We can verify by virtue of Lemma 27.1 in Takeuti [16] combined with $\rho \ll_0 (\omega, \rho)$ that $c[\rho] \ll_0 c[(\omega, \rho)]$ holds for an arbitrary hydra context c, where the notation $\ll_0$ is defined in Def 27.21 in [16]. Tree interpretation of Rule 6 is sketched as follows.
Due to Rule 6, in Case 1.2.2, we do not have to be concerned with the inference J immediately above I, and we can assign the same height as in $< P, d >$ to the sequents in $< P', d' >$ which are in $< P, d >$ the upper sequents of I. In

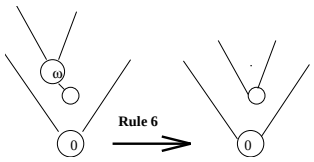

Figure 6: Crushing a ω-node.

Case 2 and in Preparation 2, we are permitted to assign a smaller height than in $< P, d >$ to the sequents in $< P', d' >$ which are in $< P, d >$ the upper sequents of the eliminated cut, the eliminated weakening, or the eliminated contraction. Since a hydra shrinks rapidly in this version of Hydra Game, we shall call this version *Rapid Hydra Game* (RHG). Therefore termination property of RHG and the proposition corresponding to Proposition 1 a fortiori hold. Moreover, the proof reductions under Takeuti's ordinal assignment are interpretable by means of RHG, while these are not interpretable by means of BHG. (Note that we took Arai's ordinal assignment [2], rather than Takeuti's [16], for our interpretation by means of BHG in this section.) With the help of Gödel's Incompleteness Theorem, the corresponding theorem to Theorem 2 still holds for RHG in spite that RHG terminates faster than BHG:

Theorem 3 $(\Pi^1_1 - CA) + BI \not\vdash$ *"Every recursive strategy starting with an arbitrary hydra is a winning strategy of RHG".*

2.3 Note on an Extension to the System of Iterated Inductive Definitions

We shall sketch how to extend our idea in previous subsections to the case of the system of ξ-iterated inductive definitions. Here we assume a primitive recursive ordering $\prec$ on a primitive recursive subset of N with the least element 0 and the largest element ξ. The set $\mathcal{H}(\xi)$ of ordinal diagrams (which corresponds to a certain subset of the domain of Takeuti's system $O(\xi + 1, 1)$ [14]) is defined inductively as follows.

Definition 12 (Sets $\mathcal{H}(\xi)$ of ordinal diagrams)

1. *If* $\gamma \prec \xi$ *and* $\alpha \in \mathcal{H}(\xi) \cup \{0\}$ *then* $(\gamma, \alpha) \in \mathcal{H}(\xi)$.
2. *If* $\alpha_1, \ldots, \alpha_n \in \mathcal{H}(\xi)$, *then* $\alpha_1 \# \cdots \# \alpha_n \in \mathcal{H}(\xi)$.

Definition 13 $\mathcal{H}_0(\xi) = \{(0, \beta) \mid \beta \in \mathcal{H}(\xi)\}$

In this section we call an element of $\mathcal{H}_0(\xi)$ a *hydra*.

Definition 14 (Cf. Extended Hydra Game [13])
$(\mathcal{H}(\xi) \cup \{0\}, R_\xi)$ *denotes a term rewriting system over* $\mathcal{H}(\xi) \cup \{0\}$ *with the set*

R_ξ of the three kind of schemata of reduction rules, Rule 1, Rule 2 of Definition 7, and Rule 3, naturally generalized from Rule 3 of Definition 7, as follows.

Rule 3 $(\eta, 0) \hookrightarrow (\gamma + 1, 0)$ *for an arbitrary* $\gamma \prec \eta$ *for each limit ordinal* $\eta \prec \xi$.

Extended Hydra Game (EHG) [14] is a term rewriting game defined over the terms of $\mathcal{H}_0(\xi) \cup \{0\}$ in $(\mathcal{H}(\xi) \cup \{0\}, R_\xi)$. *Steps* of EHG consist of Rule 1, Rule 2 and Rule 3 of $(\mathcal{H}(\xi) \cup \{0\}, R_\xi)$ described as above. All notions taken in BHG (a strategy, $\hookrightarrow^+$, etc) are carried over to EHG. As in $(\mathcal{H} \cup \{0\}, R)$, in $(\mathcal{H}(\xi) \cup \{0\}, R_\xi)$ if a hydra has a normal form (i.e., irreducible form) it must be 0. As shown in [14], every strategy in EHG is a winning strategy. Moreover, from the fact that accessibility of each ordinal diagram from $O(\xi + 1, 1)$ with respect to $\ll_0$ is provable in ID_ξ (cf. Arai [1]), we can see;

Proposition 2 ([14]) $ID_\xi \vdash$ *"Every recursive strategy starting with a fixed* $\alpha \in \mathcal{H}_0(\xi)$ *is a winning strategy."*

Then, as in BHG, Rule 4 and Rule 5 are derivable using the weak normalizability of $(\mathcal{H}(\xi) \cup \{0\}, R_\xi)$.

Now we shall deal with Arai's System $AI^-_\xi(\mathcal{B})$, which is equivalent to ID_ξ of Feferman [6] in terms of the critical ordinal. Note that $AI^-_\xi(\mathcal{B})$ is a system of iterated Π^1_1-comprehension which extends ID_ξ. Let $\mathcal{B}$ be a set of arithmetical forms; in the language obtained from that of PA by adding a unary predicate variable X, a binary predicate variable Y, and two new individual constants c_0, c_1, we call a formula $B(X, Y.c_0, c_1)$ arithmetical if it has no free number variable. For $\mathcal{B}$ and a primitive recursive ordering $\prec$, $AI^-_\xi(\mathcal{B})$ is introduced in Arai [3]. See Definition 1.4. of Arai [3] for a definition of $AI^-_\xi(\mathcal{B})$. If $\mathcal{B}$ is the set of all arithmetical forms, then we write AI^-_ξ for $AI^-_\xi(\mathcal{B})$. The degree of a semiformula in AI^-_ξ is the same as in Takeuti [16] Def 27.12 except that in AI^-_ξ we do not use a formula of a form $\forall\phi F(\phi)$ nor the notion of implicitness (explicitness); see Definition 2.4 of Arai [3] for a definition of the *degree* of a semiformula. $\xi(n, \mu)$ is defined in the same way as $\omega(n, \mu)$ of Definition 10 in Section 2.2 (by replacing ω by ξ). In the same way as in Definition 9 of Section 2, we introduce the notion of *proof with degree and height* for a proof in AI^-_ξ by modifying Arai's assignment of height; in 2.22.1 $\sim$ 2.22.3 of Definition 2.22 [3], $=$ is modified into $\geq$. Also in the same way as in Definition 11 of Section 2, we modify Arai's assignment of ordinal diagram from $O(\xi + 1, 1)$ to each sequent of a proof with degree and height in AI^-_ξ; in 2.82.5 of Definition 2.8 [3], $(\xi, O(S'))$ is modified into $\xi(1, O(S'))$. Through this modification of Arai's ordinal assignment, as in Section 2 for ISN we can interpret a proof of AI^-_ξ as a hydra $\in \mathcal{H}_0(\xi)$.

Now we shall show that the cut elimination reduction from $< P, d >$ to $< P', d' >$ for AI_{ξ}^{-} in Arai [3] (p236 $\sim$ p239) is interpretable by a finite number of steps of EHG.

Theorem 4 *If $< P, d, h >$ is a proof with degree and height and $< P', d' >$ is the proof with degree resulting from $< P, d >$ by Arai's one step proof reduction in his consistency proof for AI_{ξ} [3], then one can define a height h' for $< P', d' >$ such that $O(P, d, h) \hookrightarrow^{*} O(P', d', h')$.*

Sketch of Proof.
The proof is essentially the same as that of Theorem 2 given in Section 2.2 Hence we only sketch the proof (mostly pointing out the similarity of the proof of Theorem 1 of Section 2.2) according to Proof of Main Lemma of Arai [3]. M1 and M2 of Proof of Main Lemma of [3] are treated in the same way as in (Preparation 0) and (Case 1) respectively of the proof of Theorem 1 of Section 2.2. Each case of M3 of [3] is treated in the same way as in (Preparation 1), (Case 2) and (Preparation 2). M41.1 of [3] is treated exactly the same way as in (Case 4). In M41.2 of [3], the reduction sequence from $O(P, d, h)$ to $O(P', d', h')$ corresponds exactly to Rule 4. In M42.1 of [3], the reduction sequence from $O(P, d, h)$ to $O(P', d', h')$ corresponds to the concatenation of Rule 4' and Rule 5. In M42.2 of [3], the reduction sequence from $O(P, d, h)$ to $O(P', d', h')$ is the one similar to that in (Case 1). M43 $\sim$ M45 of [3] are treated in the same way as (Case 3).
□
¿From Theorem 4, for an arbitrary proof figure $< P, d, h >$ in AI_{ξ}^{-} with the empty sequent " $\rightarrow$ " as the end sequent there exists such a strategy of EHG starting with a hydra $O(P, d, h)$ that corresponds to the Arai's cut elimination procedure for $< P, d, h >$. Hence, by virtue of Gödel's Incompleteness Theorem, *"All such strategies that correspond to the cut-elimination procedure of Arai for any given proof of the empty sequent " $\rightarrow$ " in AI_{ξ}^{-} are winning strategy"* is not provable in AI_{ξ}, from which follows:

Theorem 5 *$ID_{\xi} \not\vdash$ "Every recursive strategy starting with an arbitrary hydra is a winning strategy of EHG."*

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References

[1] T. Arai, An Accessibility Proof of Ordinal Diagrams in Intuitionistic Theories for Iterated Inductive Definitions, *Tsukuba J. Math. 8(1984)*, 209-218.

[2] T. Arai, A Subsystem of Classical Analysis Proper to Reduction Method for Π^1_1-Analysis, *Tsukuba J. Math. 9 (1985)*, 21-29.

[3] T. Arai, A Consistency Proof of a System Including Feferman's ID_ξ by Takeuti's Reduction Methode,*Tsukuba J. Math. 11 (1987)*, 227-239.

[4] W. Buchholz, An Independence Result for $(\Pi^1_1 - CA) + BI$, *Annals of Pure and Applied Logic 33 (1987)*, 131-155.

[5] E. A. Cichon, A Short Proof of Two Recently Discovered Independence Results Using Recursion Theoretic Methods, *Proc. of American Math. Soc. 87 (1983)*. 704-706.

[6] S. Feferman, Formal Theories for Transfinite Iteration of Generalized Inductive Definitions and Some Systems of Analysis, *Intuitionism and Proof Theory*, ed. by Kino, Myhill and Vesley , North Holland, (1970).

[7] G. Gentzen, Neue Fassung des Widerspruchsfreiheitbeweise für die reine Zahlentheorie, *Forschungen zur Logik und zur Grundlegung der exackten Wissenschaften , Neue Folge 4 (1938)*,19-44.

[8] M. Hamano and M. Okada, A Relationship among Gentzen's Proof-Reduction, Kirby-Paris' Hydra Game and Buchholz's Hydra Game, to appear in *Mathematical Logic Quarterly. 43 (1997)*

[9] M. Gardner, Mathematical Games (Tasks You Cannot Help Finishing No Matter How Hard You Try to Block Finishing Them), *Scientific American. August (1983)*, 12-21.

[10] L. Kirby and J. Paris, Accessible Independence Results for Peano Arithmetic, *Bull. London Math. Soc. 14 (1982)*, 285-293.

[11] J. Ketonen and R. Solovay, Rapidly Growing Ramsey Functions, *Ann. of Math. 113 (1981)*, 267-314.

[12] J. W. Klop, Term Rewriting Systems, *Handbook of Logic in Computer Science Vol.2*, ed. by S. Abramsky, D.M. Gabbay and T.S.E. Maibaum, Oxford University Press, (1992), 1-116.

[13] M. Okada, A Simple Relationship between Buchholz's New System of Ordinal Notations and Takeuti's System of Ordinal Diagrams, *J. Symbolic Logic 52 (1987)*, 577-581.

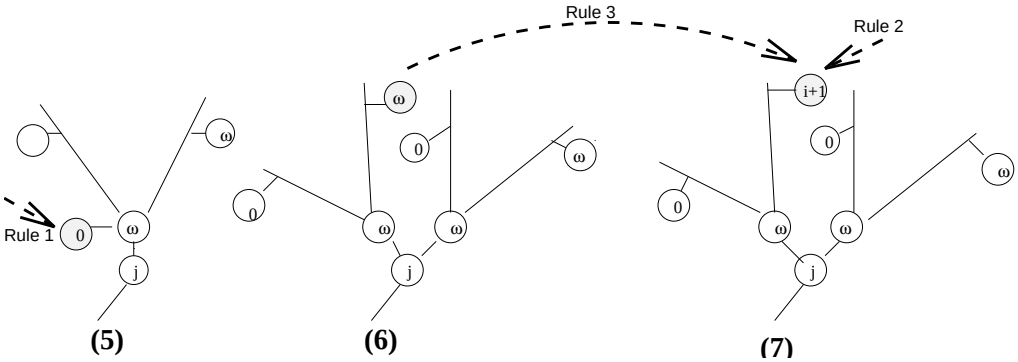
[14] M. Okada, Note on a Proof of the Extended Kirby-Paris Theorem on Labeled Finite Trees, *Europ. J. Combinatorics 9 (1988)*, 249-253.

[15] D. Schmidt, Built-up System of Fundamental Sequences and Hierarchies of Number-Theoretic Functions, *Arch. Math. Logik Grundlag. 18 (1976)*, 47-53.

[16] G.Takeuti, *Proof Theory*, 2nd edition, North Holland, 1987.

3 APPENDIX

In this appendix we shall sketch the tree interpretation of the reduction sequence from $O(P,d,h)$ to $O(P',d',h')$ in Case 4 of Theorem 1: Each tree of the following represents the element of $\mathcal{H}$ described in p.19.



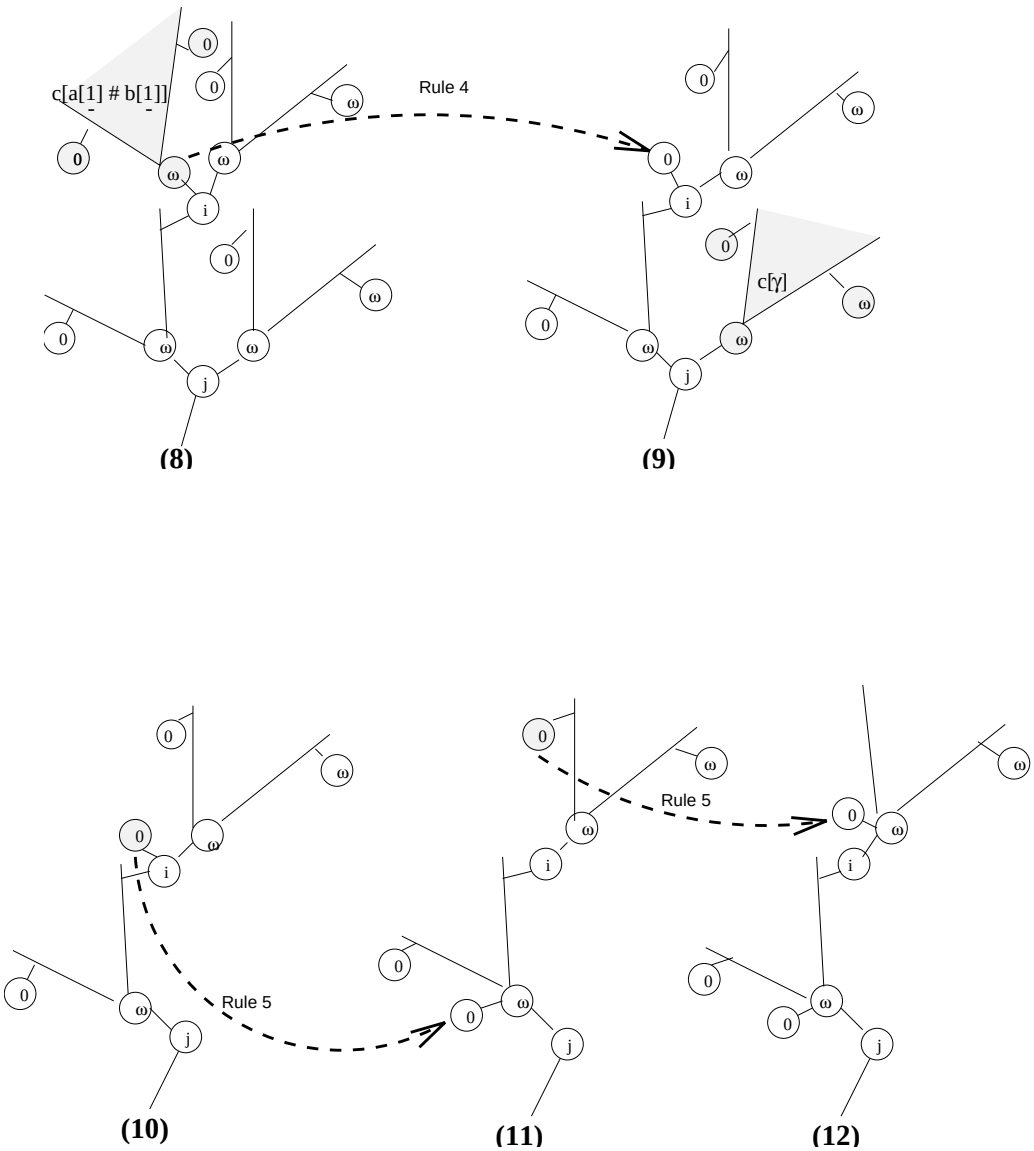


Figure 7: Tree Reduction of Case 4 of Theorem 1