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Functional shifts: hierarchy and self-modification of rules in dynamics

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Abstract

A framework to study evolution of rules of dynamical systems is proposed with a shift-like dynamics called a functional shift. The functional shift is determined by a shift map on a set of bi-infinite sequences of some functions having the same domain and codomain. Considering the bi-infinite sequence of functions corresponding to an iterated function of a dynamical system, the functional shift allows us to analyze the dynamics of a function governing the state change of a dynamical system. Here the function is referred to as a 'rule'. We study the relevance of functional shifts to sofic shifts. We prove that a class of functional shifts of finite type, having bi-infinite sequences of functions given by a shift of finite type, is equivalent to that of sofic shifts. From this theorem, we also prove that the topological entropy of general functional shifts of finite type is computable, and we provide an algorithm to compute it. Next, we show some properties of the topological entropy of rules and meta-rules. A meta-rule is a rule that governs the change of rules. Since the relation between meta-rules and rules corresponds to that of rules and states (namely, a relation of operators and operands), it is clear that states, rules, and meta-rules form a hierarchical relationship. Furthermore, since we can construct rules that govern change of meta-rules, i.e., meta-meta-rules, the hierarchical relationship continues recursively. The significant result is that the topological entropy of meta-rules provides the upper limit for that of rules. This means that the higher the level of rules in the hierarchy is, the stronger the non-linearity of these rules. The hierarchical relationship is discussed in terms of properties of rules and meta-rules. The presented framework can be used to study 'self-modifying' systems in which rules are used to change the rules themselves. We investigate self-modifying systems using a network of operator-operand relations. We argue that the network is invariant under shift maps. Finally we discuss the significance of this result to cognitive development in which the development of rules of self-modifying type can be observed.

Keyword: rule dynamics, functional shifts, topological entropy, self-referential shifts

1. Introduction

This paper studies the dynamics of rules using a formal system. Rules are, in this paper, iterated functions to induce state change, since we concentrate on evolution of functions in describing dynamical systems. In studying dynamical systems, we normally regard a rule governing change of states as invariable. In recent years, however, it is considered that the dynamics of rules is important to reach an intimate understanding of complex systems.

We can identify two possible approaches in attempting to study the dynamics of rules. One is to embed a system into high dimensional dynamics showing chaotic itinerancy [3]. In chaotic itinerancy, an orbit itinerates over quasi-attractors. Since the behavior in a quasi-attractor seems to obey an effective rule, changes among quasi-attractors can be considered as transitions of effective rules. A typical example of this approach is globally coupled chaotic maps (GCM) [4]. While a GCM system is constructed from microscopic mechanisms of objective phenomena and is an effectual method to model complex dynamics in natural systems, to find concrete rules which govern dynamics in quasi-attractors is in general a difficult task. Therefore, this approach is not ideally suited for investigating characteristics of the rule change.

The other approach is to describe dynamic change of rules of a system directly. This approach is more representable than the former one in describing rule dynamics. For instance, Kataoka and Kaneko [5, 6] introduce functional dynamics as an evolution of a function. In these studies a function f_n develops in time according to a functional equation, $f_{n+1} = (1 - \epsilon)f_n + \epsilon f_n \circ f_n$. Considering f_n as a map on the state space, we may regard functional dynamics as a particular model of the evolution of rules. Another example is studied by Kim and Aizawa [1, 2] using cellular automata changing their transition rules in time evolution. Although the system is in itself a large cellular automata with a complex transition rule, they aim at inquiring into the features of the dynamics of rules by seeing the transition rules as changing in time.

In this paper, we take the latter approach for studying the dynamics of rules because of the difficulty of the former approach, mentioned above, and the direct representability of the latter. Our proposal is to use shift-like dynamics on sequences of functions. We call this framework *functional shifts*. Functional shifts are determined by a shift map on a set of bi-infinite sequences of functions. In other words, functional shifts are symbolic dynamics in which each symbol is a function. Considering the bi-infinite sequence of functions corresponding to an iterated function of a dynamical system, the functional shift allows us to analyze the dynamics of a function governing the state change of a dynamical system.

We study the following features of the rule dynamics using functional shifts:

- complexity of dynamics of rules of dynamical systems;
- a hierarchical relationship between rules and meta-rules;
- relations between operators and operands in a self-modifying system.

We measure the complexity of rule dynamics in terms of the topological entropy of functional shifts. A meta-rule is a rule that governs the change of rules. Since the relation between meta-rules and rules corresponds to that of rules and states (namely, a relation of operators and operands), it is clear that states, rules, and meta-rules form a hierarchical relationship.

Self-modifying systems are dynamical systems in which rules can change their own rules. We have difficulty in getting the direct representation of rule dynamics of a self-modifying type. The cause of the difficulty is that operators and operands cannot be perfectly separated in self-modification systems. To describe a dynamical system, we generally decide the act of all operators which determine state change. When we decide them, we need to distinguish between operators and operands. But we cannot decompose operators and operands in a self-modification system, because operators affect, by definition, themselves. Therefore, the operator-operand relation is so complex that it is not treated by means of usual dynamical systems theory. In order to study the self-modifying systems, we propose a minimal model using functional shifts. Because we define functional shifts in a formal style, we can put rules and states in the same level by introducing a correspondence between the rules and the states. And we analyze relationship between operators and operands in the model.

While the framework presented here is a formal system and is used mainly to investigate logical structures of rule dynamics, our underlying motivation is to understand dynamic and complex aspects of nature. In usual scientific method, objects have been treated by being broken down into states and fixed functions governing the state change, particularly when we use dynamical systems to describe them. Here, the decomposability of functions and states, that of operators and operands, or that of rules is assumed beforehand. But, some objects, especially such as biological, cognitive and social phenomena, may sometimes refuse this decomposability or seem to be better understood by not assuming it. Considering how we understand such objects without postulating the decomposability of rules is an important standpoint for the study of complex systems and, as we have mentioned above, shedding light on complex systems from the viewpoint of dynamic rules is crucial for comprehending them.

We can find many instances in which rules describing dynamic behavior of systems change in the course of time. For example, Maynard Smith and Szathmary [9] argue some transitions occurred in history of biological evolution. In the history, behavior and dynamics of biological systems change in time and the rules to describe the dynamics no doubt change along the evolutionary process. The same kind of issues can be found in the developmental process of cognitive systems [8]. Likewise, phenomena showing chaotic itinerancy, which we have identified as dynamics of rules, are found in the models of biological systems and the brain [3]. In social systems, as another example, social rules in a society have been considered to conform individuals behavior to the rules, where the social rules are a kind of operator acting on the individual behavior and the individuals behavior is a kind of operand to be shaped by the social rules. Actually, however, social rules do not exist *a priori*, but are constructed by behavior of individuals in the society and, moreover, are modified under the influence of action of individuals. Namely, the operator-operand relationship is no more able to be clearly decoupled and such social systems as a whole have self-modifying aspect, since actions of constituents of the systems evolve the systems, especially the rules to be obeyed by the constituents.

Unfortunately, despite the ubiquity of such problems, we rarely have tools to describe the change of rules of behavior and to understand such dynamic phenomena. We should recognize the necessity of developing formal devices to treat and analyze them and of constructing a general theory for understanding such dynamic complex systems. Our proposed framework is, on one hand, one such attempt and is an endeavor to enhance the theory of dynamical systems, on the other hand. Studies of dynamical systems have proposed frameworks to describe complex behavior in various natural systems and, *in transit*, the classes of unknown dynamics have been identified. Our framework may lead to a new class of dynamics, such as chaos in functional space.

This paper is organized as follows: we first review some basic definitions of shift space in section 2; we propose a viewpoint of dynamics of rules and shift-like dynamics called functional shifts in section 3; it is proved in section 4 that a class of functional shifts of finite type, which is defined in the section, is equivalent to that of sofic shifts; next, in section 5, we show some properties of the topological entropy in functional shifts; we propose a minimal model of the self-modifying dynamical systems, and investigate features of operator-operand networks in self-modifying systems in section 6; finally, some results are discussed from the viewpoint of the dynamics of rules.

2. Review of Shift Spaces

Since we will study shift-like dynamics, we first give some definitions of shift spaces [7].

If $\mathcal{A}$ is a finite set of symbols, then the full $\mathcal{A}$-shift (simply the full shift) is the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. Such a sequence is denoted by $x = (x_i)_{i\in\mathbb{Z}}$. A block (or word) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. A n-block is simply a block of length n. We write blocks without separating their symbols by commas or other punctuation, so that a typical block over $\mathcal{A} = \{a, b\}$ looks like $aababbabbb$. The shift map σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose ith coordinate is $y_i = x_{i+1}$.

Let $\mathtt{F}$, which we call the forbidden blocks, be a collection of blocks over $\mathcal{A}$. A shift space X is a subset of sequences in $\mathcal{A}^{\mathbb{Z}}$, which does not contain any blocks in $\mathtt{F}$. The set of all n-blocks that occur in points in X is denoted by $\mathcal{B}_n(X)$. The language of X is the collection $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. A shift space X is irreducible if for every ordered pair of blocks $u, v \in \mathcal{B}(X)$ there is a block $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$.

Suppose that X is a shift space over $\mathcal{A}$ and Y is a shift space over $\mathcal{A}'$. A $(m + n + 1)$-block map $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathcal{A}'$ maps from allowed $(m + n + 1)$-blocks in X to symbols in $\mathcal{A}'$. A map $\phi : X \to \mathcal{A}'^{\mathbb{Z}}$ defined by $y = \phi(x)$ with $y_i = \Phi(x_{i-m}x_{i-m+1}\cdots x_{i+n})$ is called a sliding block code induced by Φ. If $\phi(X) \subset Y$, then we write $\phi : X \to Y$. If a sliding block code $\phi : X \to Y$ is onto, ϕ is called a factor code. A shift space Y is a factor of X if there is a factor code from X onto Y. A sliding block code $\phi : X \to Y$ is a conjugacy if it is invertible. Two shift spaces X and Y are conjugate (written $X \cong Y$) if there is a conjugacy from X to Y.

Shift spaces described by a finite set of forbidden blocks are called shifts of finite type. Despite the simplest shifts, these shifts play an essential role in mathematical subjects like dynamical systems. If a dynamical system is hyperbolic, then there is a Markov partition for the system, and furthermore, there is a topological conjugacy from the system to a shift of finite type.

A graph G consists of a finite set $\mathcal{V} = \mathcal{V}(G)$ of vertices (or states) together with a finite set $\mathcal{E} = \mathcal{E}(G)$ of edges. Each edge $e \in \mathcal{E}$ starts at a vertex denoted by $i(e) \in \mathcal{V}(G)$ and terminates at a vertex $t(e) \in \mathcal{V}(G)$ (which can be the same as $i(e)$). Equivalently, the edge e has an initial state $i(e)$ and a terminal state $t(e)$. The edge shift X_G is the shift space over the $\mathcal{A} = \mathcal{E}$ specified by

$$X_G = \{x = (x_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} |\ \forall i \in \mathbb{Z}\ t(x_i) = i(x_{i+1})\}. \tag{1}$$

Since a forbidden blocks $\mathtt{F}$ of an edge shift can be described by $\mathtt{F} = \{ef|\ e, f \in \mathcal{E}, t(e) \neq i(f)\}$, it is easy to see that every edge shift is shift of finite type.

Sofic shifts are defined using graphs whose edges are assigned labels, where several edges may carry the same label. A labeled graph $\mathcal{G}$ is a pair $(G, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ assigns a label $\mathcal{L}(e)$ from the finite set $\mathcal{A}$ to each edge e of G. Let $X_{\mathcal{G}}$ be denoted by

$$X_{\mathcal{G}} = \{x \in \mathcal{A}^{\mathbb{Z}} |\ \exists e \in X_G\ \forall i \in \mathbb{Z}\ x_i = \mathcal{L}(e_i)\}. \tag{2}$$

A subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ is a sofic shift if $X = X_{\mathcal{G}}$ for some labeled graph $\mathcal{G}$. It is known that sofic shifts are shift spaces, and moreover, a shift space is sofic if and only if it is a factor of a shift of finite type [7].

3. Functional Shifts

In this section, we propose a viewpoint of the dynamics of rules based on the inevitably discrete nature of observation. And, we introduce a framework to describe the dynamics of rules in terms of shift maps over sequences of functions.

3.1. From Dynamics of States to Dynamics of Rules

Our ability to observe a system is so limited that the description of the system should be based on the finite precision both in time and space. To construct a rule describing dynamics of the system from such observations with finite precision, we use functions of discrete time and space[1].

In a discretized space, an orbit can show transitions from one state, say state A, to plural distinguished states, states B and C, at a different time, even if the nature of the system is deterministic. If we can observe the state space more precisely, the state A may be divided into two states A_1 and A_2 and the state transitions $A \mapsto B$ and $A \mapsto C$ in the lower precision are $A_1 \mapsto B$ and $A_2 \mapsto C$, respectively[2].

We propose a viewpoint to consider such phenomena that the state change from one to more than one at a different time as a transition of functions governing the state change. In other words, it is our standpoint that in a time period the system obeys a function and it is governed by another function in a different time period and the system shows a change of functions, i.e., dynamics of rules in the terminology of the paper. Thus, the whole description of evolution of the system should be a sequence of functions in addition to a sequence of states. The dynamics of rules is represented by shift operations on the sequence of functions.

3.2. Definition of Functional Shifts

We give the definition of functional shifts based on shift spaces reviewed in the preceding section so as to formally treat dynamics of rules.

Definition 3.1 Let $\mathcal{A}$ be a finite set, and F be a set of maps on $\mathcal{A}$. A *functional shift* $\mathcal{F}$ is a shift space which is a subset of a full shift $F^{\mathbb{Z}}$.

A *generated shift* $X_{\mathcal{F}}$ for $\mathcal{F}$ is defined by

$$X_{\mathcal{F}} = \{x = (x_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} |\ \exists f = (f_i)_{i \in \mathbb{Z}} \in \mathcal{F}\ \forall i \in \mathbb{Z}\ x_{i+1} = f_i(x_i)\}. \tag{3}$$

Although a generated shift is not required by our definition to be a shift space, it is always a shift space.

Theorem 3.2 If $\mathcal{F}$ is a functional shift, $X_{\mathcal{F}}$ is a shift space.

Proof. Let

$$Y_n = \{x \in \mathcal{A}^n \mid \forall f \in \mathcal{B}_n(\mathcal{F})\ \exists i \in \{0, 1, \cdots, n-2\}\ x_{i+1} \neq f_i(x_i)\} \tag{4}$$

and $\mathtt{F} = \bigcup_{n \in \mathbb{N}} Y_n$. Suppose that X is a shift space which can be described by a collection of forbidden blocks $\mathtt{F}$. If $x \in X_{\mathcal{F}}$, then $x \in X$ because every block in $\mathtt{F}$ does not occur in x. Thus $X_{\mathcal{F}} \subset X$. Conversely if

[1] Note that making a continuous function on the continuous space from observations can be done only in some limit of an approximation.

[2] In chaotic systems, since orbits can approach arbitrary neighbor of some states, such a nondeterministic state change is possible in a discrete state space, even if the degree of discretization is fine as far as the precision is not infinite.

$x \in X$, then $x \in X_{\mathcal{F}}$ because $\mathtt{F}$ is a set of blocks never occurring in points in $X_{\mathcal{F}}$. Therefore $X \subset X_{\mathcal{F}}$. Hence $X = X_{\mathcal{F}}$ and $X_{\mathcal{F}}$ is a shift space. ∎

Since $X_{\mathcal{F}}$ is a shift space, we can construct a functional shift $\mathcal{F}'$ such that $\mathcal{F}' \cong X_{\mathcal{F}}$. It means that functional shifts allow us to describe a hierarchy of rules and meta-rules recursively. Here, a meta-rule is a rule that governs the change of rules, and meta-rules are points in $\mathcal{F}$ if rules are in $X_{\mathcal{F}}$.

4. Functional Shifts of Finite Type

In this section, we study the case in which a functional shift is described by a shift of finite type. Firstly, we define functional shifts of finite type.

Definition 4.1 A *functional shift of finite type* $\mathcal{F}$ is a functional shift that can be described by a finite set of forbidden blocks.

In the following example, we give a functional shift of finite type whose generated shift is equal to the even shift. The even shift is a set of all binary sequences having even number of 0's between any two 1's.

Example 4.2 Let $\mathcal{A} = \{0, 1\}$, and $F = \{f_i | f_i : \mathcal{A} \to \mathcal{A}, i \in \{0, 1, 2\}\}$ be a set of functions such that $f_0(0) = 1$, $f_0(1) = 1$, $f_1(0) = 0$, $f_1(1) = 0$, $f_2(0) = 0$, and $f_2(1) = 1$. If a functional shift $\mathcal{F}$ can be described by a set of forbidden blocks $\mathtt{F} = \{f_0 f_2, f_1 f_0, f_1 f_1, f_2 f_2\}$, the generated shift $X_{\mathcal{F}}$ is the even shift.

Proof. For the definition of $\mathtt{F}$, it is necessary that $(f_1 f_2)^{n/2} f_0$ occurs in points in $\mathcal{F}$ if $10^n 1 \in \mathcal{B}(X_{\mathcal{F}})$, because $f(0) = 1$ iff $f = f_0$ and $f(1) = 0$ iff $f = f_1$. Since $(f_1 f_2)^m f_0 \in \mathcal{B}(\mathcal{F})$ for $m \geq 1$, $10^{2n-1} 1 \notin \mathcal{B}(X_{\mathcal{F}})$ and $10^{2n} 1 \in \mathcal{B}(X_{\mathcal{F}})$ for $n \geq 1$. Furthermore, since $f_0^n f_1 \in \mathcal{B}(\mathcal{F})$ for $n \in \mathbb{N}$, $10^{2n} 1$ occurs in $\mathcal{B}(X_{\mathcal{F}})$ for $n \geq 0$. Thus, $X_{\mathcal{F}}$ is the even shift. ∎

Since the even shift does not have finite type, all generated shifts for a functional shift of finite type are not shifts of finite type. In later paragraphs, we will prove that the class of generated shifts for a functional shift of finite type (simply the class of functional shifts of finite type) is equivalent to that of sofic shifts.

Theorem 4.3 If $\mathcal{F}$ is a functional shift of finite type, $X_{\mathcal{F}}$ is sofic.

Proof. Let

$$T(\mathcal{F}) = \{\langle x, f \rangle \in (\mathcal{A} \times F)^{\mathbb{Z}} | \ \exists f \in \mathcal{F} \wedge \forall i \in \mathbb{Z} \ x_{i+1} = f_i(x_i)\} \tag{5}$$

be a set of elements which are sequences of pairs $(\cdots, \langle x_{-1}, f_{-1} \rangle, \langle x_0, f_0 \rangle, \langle x_1, f_1 \rangle, \cdots)$. We first prove that $T(\mathcal{F})$ has finite type if $\mathcal{F}$ is a functional shift of finite type. Since $\mathcal{F}$ has finite type, there is a finite set $\mathtt{F}$ of forbidden blocks. Then

$$\mathtt{F}_{T(\mathcal{F})} = \{(\langle x_0, f_0 \rangle \langle x_1, f_1 \rangle) \in (\mathcal{A} \times F)^2 | \ x_1 \neq f_0(x_0)\} \cup \{\langle x, f \rangle \in \mathcal{B}((\mathcal{A} \times F)^{\mathbb{Z}}) | \ f \in \mathtt{F}\} \tag{6}$$

is a finite collection of forbidden blocks of $T(\mathcal{F})$. Consequently, $T(\mathcal{F})$ has finite type.

We next consider a 1-block map $\Phi : \mathcal{B}_1(\mathcal{A} \times F) \to \mathcal{A}$ such that $\Phi(\langle x, f \rangle) = x$. Since a sliding block code $\phi : T(\mathcal{F}) \to X_{\mathcal{F}}$ induced by Φ is onto, ϕ is a factor code. If a shift space is a factor of a shift of finite type, then it is sofic. Hence $X_{\mathcal{F}}$ is a sofic shift. ∎

Theorem 4.4 Every sofic shift is a generated shift for a functional shift of finite type.

Proof. Suppose that X is a sofic shift over $\mathcal{A}$, and let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph such that $X = X_{\mathcal{G}}$. If a is an edge of G, then $f_a : \mathcal{A} \cup \mathcal{E} \cup \{d\} \to \mathcal{A} \cup \mathcal{E} \cup \{d\}$ (where $\mathcal{A} \cap \mathcal{E} = \emptyset$ and $d \notin \mathcal{A} \cup \mathcal{E}$) is defined by

$$f_a(x) = \begin{cases} d & \text{if } x \in \mathcal{E} \text{ and } x = a, \\ \mathcal{L}(a) & \text{otherwise.} \end{cases} \tag{7}$$

Let $F = \{f_a | \ a \in \mathcal{E}\}$ and $\mathtt{F} = \{f_a f_b \in F^2 | \ t(a) \neq i(b)\}$. Recall that $i(a)$ is an initial state and $t(a)$ is a terminal state of a. If a functional shift $\mathcal{F}$ over F can be described by $\mathtt{F}$, then $X_{\mathcal{F}} = X$. Since $\mathtt{F}$ is a finite set, $\mathcal{F}$ is a functional shift of finite type. Thus every sofic shift is a generated shift for a functional shift of finite type. ∎

We next show that the class of functional shifts of finite type contains all generated shifts for a functional shift in the collection.

Theorem 4.5 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be functional shifts, and $\mathcal{F}_2$ be a generated shift for $\mathcal{F}_1$, namely $\mathcal{F}_2 = X_{\mathcal{F}_1}$. If $\mathcal{F}_1$ is a functional shift of finite type, there is a functional shift of finite type $\mathcal{F}_3$ satisfying $X_{\mathcal{F}_3} = X_{\mathcal{F}_2}$.

Proof. Let $\mathcal{A}_i$ be a finite set of symbols of $\mathcal{F}_i$ and F_i be a set of all maps on $\mathcal{A}_i$ for $i = 1, 2$. For each $\langle g, h\rangle \in \mathcal{A}_1 \times F_1$, a map $f_{\langle g,h\rangle} : F_1 \cup \mathcal{A}_2 \cup \{d\} \to F_1 \cup \mathcal{A}_2 \cup \{d\}$ is defined by

$$f_{\langle g,h\rangle}(x) = \begin{cases} d & \text{if } x \in F_1 \text{ and } x = h, \\ g(x) & \text{otherwise,} \end{cases} \tag{8}$$

where $d \notin F_1 \cup \mathcal{A}_2$.

In the proof of theorem 4.3, we explain that $T(\mathcal{F})$ given by equation (5) is a shift of finite type if $\mathcal{F}$ has finite type. Hence there is a finite set $\mathtt{F}$ of forbidden blocks of $T(\mathcal{F}_1)$. Let $F_3 = \{f_{\langle g,h\rangle} | \ \langle g, h\rangle \in \mathcal{A}_1 \times F_1\}$ and

$$\mathtt{F}_3 = \{f_{\langle g,h\rangle} \in F_3 | \ \langle g, h\rangle \notin \mathcal{B}_1(T(\mathcal{F}_1))\} \cup \{f \in \mathcal{B}(F_3^{\mathbb{Z}}) | \ \exists \langle x, y\rangle \in \mathtt{F} \ f_i = f_{\langle x_i, y_i\rangle} \text{ for all } i\}. \tag{9}$$

Here, $\mathtt{F}_3$ is a finite set because $\mathtt{F}$ is also finite. Suppose that $\mathcal{F}_3$ is a functional shift which can be described by the forbidden blocks $\mathtt{F}_3$. Then $X_{\mathcal{F}_3} = X_{\mathcal{F}_2}$. Since $\mathtt{F}_3$ is a finite set, $\mathcal{F}_3$ is a functional shift of finite type. ∎

As the class of functional shifts of finite type contains all sofic shifts (also contains shifts of finite type), this class is the smallest collection of shift spaces that contains all shifts of finite type and also contains all generated shifts for a functional shift in the collection.

5. Topological Entropy of Functional Shifts

The topological entropy of a dynamical system is a conjugacy invariant that measures the variety and nonlinearity of orbits in the system. In this section, by examining the topological entropy of functional shifts, we study the complexity of them.

We consider a distance function d of a shift space X such that

$$d(x, y) = \begin{cases} 2^{-|k|} & \text{if } x_k \neq y_k \text{ and } x_i = y_i \text{ for } -|k| < i < |k|, \\ 0 & \text{if } x = y. \end{cases} \tag{10}$$

The metric space (X, d) is compact.

Let σ be a shift map on X. The topological entropy $h(\sigma)$ is equal to

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(X)|, \tag{11}$$

where $h(X)$ is called the entropy of shift space X [7]. Recall that $\mathcal{B}_n(X)$ is the set of all n-blocks that occur in points in X.

We prove that the entropy $h(X_{\mathcal{F}})$ is computable if $\mathcal{F}$ is a functional shift of finite type.

Theorem 5.1 Let $\mathcal{F}$ be a functional shift of finite type. There is an algorithm to compute the entropy $h(X_{\mathcal{F}})$.

Proof. In the proof of theorem 4.3, we show that every generated shift for a functional shift of finite type is a sofic shift. Furthermore, there is an algorithm to compute the entropy under sofic shifts[3]. Hence there is an algorithm to compute the entropy $h(X_{\mathcal{F}})$. ∎

We next turn to the behavior of the entropy under functional shifts and generated shifts.

Theorem 5.2 If $\mathcal{F}$ is a functional shift, then $h(X_{\mathcal{F}}) \leq h(\mathcal{F})$.

[3] If a labeled graph $\mathcal{G}$ is irreducible right-resolving presentation, $h(X_{\mathcal{G}})$ equals the maximum eigenvalue of adjacency matrix of $\mathcal{G}$. Every sofic shift X has a right-resolving presentation, and there is an algorithm to find the right-resolving presentation $\mathcal{G}$ such that $X = X_{\mathcal{G}}$. Furthermore we can find irreducible subgraphs of a sofic shift X, and the entropy of X is equal to the maximal entropy of subshifts given the irreducible subgraphs. Thus, the entropy under sofic shifts is computable[7].

Proof. If $\varphi_n : \mathcal{B}_n(\mathcal{F}) \to 2^{\mathcal{B}_n(X_\mathcal{F})}$ is defined by

$$\varphi_n(f) = \{x \in \mathcal{B}_n(X_\mathcal{F}) | \ \exists a \in \mathcal{A} \ x_0 = f_0(a) \wedge x_i = f_i(x_{i-1})\}. \tag{12}$$

then it is clear that $|\varphi_n(f)| \leq |\mathcal{A}|$ for all $f \in \mathcal{B}_n(\mathcal{F})$ and

$$\mathcal{B}_n(X_\mathcal{F}) \subset \bigcup_{f \in \mathcal{B}_n(\mathcal{F})} \varphi_n(f). \tag{13}$$

Thus $|\mathcal{B}_n(X_\mathcal{F})| \leq |\mathcal{B}_n(\mathcal{F})||\mathcal{A}|$. Accordingly,

$$\begin{aligned} h(X_\mathcal{F}) &= \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(X_\mathcal{F})| \\ &\leq \lim_{n\to\infty} \frac{1}{n} \log(|\mathcal{B}_n(\mathcal{F})||\mathcal{A}|) \\ &= \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(\mathcal{F})| \\ &= h(\mathcal{F}). \end{aligned} \tag{14}$$

Hence $h(X_\mathcal{F}) \leq h(\mathcal{F})$. ∎

Our result is discussed from the viewpoint of hierarchy of rules in section 7.

6. Self-referential Shifts

In this section we consider self-modifying systems, in which rules governing the system are used to change the rules themselves. To describe such systems, we must take the self-referential nature of a dynamical system into account. A fundamental idea for this end is to adopt a code which is one-to-one correspondence between operators and operands, where an operator is a bi-infinite sequence in a functional shift and an operand in a generated shift. Supposing that each operator is equal to an operand corresponding to the operator under the code, we regard the system as self-modifying.

As a minimal model of self-modifying dynamical systems, we introduce a self-referential shift. This model is represented by a functional shift which is conjugate to a generated shift for itself, where a conjugacy is a 'code' between operators and operands. This section is devoted to studying some characteristics of self-referential shifts.

We give the following definition of self-referential shifts.

Definition 6.1 A *self-referential shift* $\mathcal{S}$ is a pair $(\mathcal{F}, \phi)$, where $\mathcal{F}$ is a functional shift satisfying $\mathcal{F} \cong X_\mathcal{F}$, and $\phi : X_\mathcal{F} \to \mathcal{F}$ is a conjugacy from $X_\mathcal{F}$ to $\mathcal{F}$.

Sometimes we refer to a conjugacy ϕ as a code of $\mathcal{S}$. We now present a simple example of self-referential shifts.

Example 6.2 Let $\mathcal{A} = \{0, 1\}$, and $F = \{f, g\}$ be a set of functions such that $f(0) = 0$, $f(1) = 1$, $g(0) = 1$, and $g(1) = 0$. If $\mathcal{F} = F^\mathbb{Z}$, and ϕ is induced by Φ satisfying $\Phi(0) = f$ and $\Phi(1) = g$, then $\mathcal{S} = (\mathcal{F}, \phi)$ is a self-referential shift.

Proof. For $x \in \mathcal{A}^\mathbb{Z}$, there is a $f \in \mathcal{F}$ such that $x_{i+1} = f_i(x_i)$ for $i \in \mathbb{Z}$. For example, $\cdots 0010111 \cdots$ corresponds to $\cdots fgggff \cdots$. Thus $X_\mathcal{F}$ is a full 2-shift. Consequently, ϕ is a conjugacy and $\mathcal{S}$ is a self-referential shift. ∎

We next define an operator-operand relation on $\mathcal{F}$, in which we regard $f \in \mathcal{F}$ as the operator of $x \in X_\mathcal{F}$ if $x_{i+1} = f_i(x_i)$ for all $i \in \mathbb{Z}$.

Definition 6.3 Let $\mathcal{S} = (\mathcal{F}, \phi)$ be a self-referential shift. An operator-operand relation $\rhd$, which is a binary relation over $\mathcal{F}$, is defined by

$$f \rhd g \ \Leftrightarrow \ \exists x \in X_\mathcal{F} \ \phi(x) = g \wedge \big(\forall i \in \mathbb{Z} \ x_{i+1} = f_i(x_i)\big). \tag{15}$$

The operator-operand relation is invariant under a shift map. We now prove it as follows.

Theorem 6.4 Let $\mathcal{S} = (\mathcal{F}, \phi)$ be a self-referential shift and σ be a shift map on $\mathcal{F}$. Then $f \rhd g$ if and only if $\sigma(f) \rhd \sigma(g)$.

Proof. The followings are then equivalent:

$$\begin{aligned} f \rhd g \quad &\Leftrightarrow \quad \exists x \in X_{\mathcal{F}} \ \phi(x) = g \wedge \left(\forall i \in \mathbb{Z} \ x_{i+1} = f_i(x_i)\right) \\ &\Leftrightarrow \quad \exists y \in X_{\mathcal{F}} \ \phi(y) = \sigma(g) \wedge \left(\forall i \in \mathbb{Z} \ y_i = f_i(y_{i-1})\right) \\ &\Leftrightarrow \quad \exists y \in X_{\mathcal{F}} \ \phi(y) = \sigma(g) \wedge \left(\forall i \in \mathbb{Z} \ y_{i+1} = \sigma(f)_i(y_i)\right) \\ &\Leftrightarrow \quad \sigma(f) \rhd \sigma(g). \end{aligned}$$

∎

For theorem 6.4, the operator-operand relation is described as a relation on the quotient set $\mathcal{F}/\sim$, where $\sim$ is an equivalence relation on $\mathcal{F}$ such that $f \sim g$ iff $i \in \mathbb{Z}\ f = \sigma^i(g)$. An example of the operator-operand relation is displayed in Figure 1. In this figure, an allow from f to g means that $f \rhd g$ is satisfied.

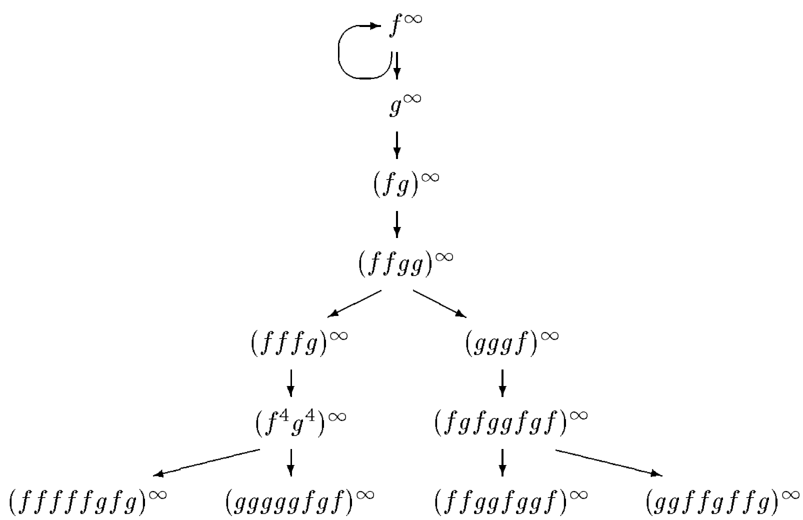

Figure 1: A part of the operator-operand relation of a self-referential shift determined in example 6.2.

7. Discussion

Let us bring the evolution of rules into focus in the present results. Since a sequence of functions is regarded as an iterated function of a dynamical system, a functional shift $\mathcal{F}$ is a set of rules which govern the change of $X_{\mathcal{F}}$. Thus, we will refer to a point in $\mathcal{F}$ as a 'rule'.

We present, in section 5, that the topological entropy of a functional shift $\mathcal{F}$ provides the upper limit for that of $X_{\mathcal{F}}$. If $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_n$ is a collection of functional shifts such that $\mathcal{F}_i = X_{\mathcal{F}_{i+1}}$, namely the collection forms a hierarchy of rules, then $h(\mathcal{F}_i) \leq h(\mathcal{F}_{i+1})$ for $1 \leq i \leq n$. This would provide a support that the higher the level in the hierarchy of rules is, the stronger the non-linearity of the rules at the level.

In section 6, we propose the model of self-modifying systems. In our model rules change themselves, but dynamic change of relation between operators and operands does not appear, as shown by theorem 6.4. The reason is that self-referential shifts are closed systems. If we think of an open system, we may observe change of relations of operators and operands. For instance, if we consider a shift map σ' which rewrites a part of a sequence additional to σ and substitute σ' for σ in theorem 6.4, then the theorem is clearly not satisfied. Therefore, dynamic change of relation between the operator and operand will appear in an open system to extend a self-referential shift.

In studying cognitive development, the development of rules of the self-modifying type is important. For example, Karmiloff-Smith[8] states that the self-modification of internal representations is necessary in the developmental process of cognitive ability of children. The representation can be considered as the basis of rules to describe behavior or dynamics of cognitive individuals. The development of rules has characteristics of self-modifying systems in our framework. Furthermore, the interaction with environment is requisite to understand behavior of cognitive systems, namely cognitive systems are open systems. Accordingly, if we apply self-referential shifts to study the cognitive development, we need to extend the definition of self-referential shifts.

8. Conclusion

In the present paper, we have introduced a shift-like dynamics, called a *functional shift*, to study the evolution of rules, by focusing on the dynamics of a function governing the state change of a dynamical system. We prove that the class of functional shifts of finite type is equivalent to that of sofic shifts. Furthermore, from this theorem, we also prove that the topological entropy under functional shifts of finite type is computable, and we provide an algorithm to compute it. We show that the topological entropy of a functional shift gives the upper limit for that of a generated shift for the functional shift, where the functional shift corresponds to a meta-rule of a rule represented by the generated shift. This means that the non-linearity is strengthened in the higher level in a hierarchy of rules. As a natural extension of our framework, a minimal model of self-modifying systems is proposed. We call it a self-referential shift. Since each state has properties of the operator and operand in self-referential shifts, we define the operator-operand relation on the state space. We prove that operator-operand relation is invariant under shift map.

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