Overview of Task 4

establish complexity analysis of constrained rewriting

- task 1 defines complexity of constrained rewriting
- in plan we start from 2013 but HZ, GM & NH work in this September

Q

- how to measure complexity of conditional part?
- how to handle built-in data types?
- how to automate analysis?

MEMBERS

Zankl, Moser, Cynthia — Innsbruck

Gramlich, DC — Vienna

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Complexity Analysis for Ordinary Rewriting

Example of TRS

TRS $\mathcal{R}$

$$\mathsf{add}(x, 0) \to x \qquad \mathsf{mul}(x, 0) \to 0$$
$$\mathsf{add}(x, \mathsf{s}(y)) \to \mathsf{s}(\mathsf{add}(x, y)) \qquad \mathsf{mul}(x, \mathsf{s}(y)) \to \mathsf{add}(x, \mathsf{mul}(y, x))$$

rewriting

$$\begin{aligned} \mathsf{mul}(\mathsf{s}(0), \mathsf{s}(0)) &\to \mathsf{add}(\mathsf{s}(0), \mathsf{mul}(\mathsf{s}(0), 0)) \\ &\to \mathsf{add}(\mathsf{s}(0), 0) \\ &\to \mathsf{s}(\mathsf{add}(0, 0)) \\ &\to \mathsf{s}(0) \end{aligned}$$

let $\mathbf{n} := \mathsf{s}^n(0)$. how many steps do we need to compute

$$\mathsf{mul}(\mathbf{m}, \mathbf{n}) \quad ?$$

Runtime Complexity

Definition

for TRS $\mathcal{R}$ over signature $\mathcal{F}$

- $\mathcal{D} := \{\mathrm{root}(\ell) \mid \ell \to r \in \mathcal{R}\}$ defined symbols
- $\mathcal{C} := \mathcal{F} \setminus \mathcal{D}$ constructor symbols

Definition runtime complexity

$\mathsf{rc}_{\mathcal{R}}(n) = \max\{\mathsf{dh}(t, \to_{\mathcal{R}}) \mid t \text{ is basic term of size up to } n\}$, where

- $\mathsf{dh}(t, \to) = \max\{k \mid t \to^{k} u \text{ for some } u\}$
- $f(t_1, \ldots, t_n)$ is basic term if $t_1, \ldots, t_n$ contain no defined symbols

Example

$\mathsf{add}(\mathsf{s}(\mathsf{s}(0)), \mathsf{s}(x))$ is basic but $\mathsf{mul}(\mathsf{s}(\mathsf{add}(0, \mathsf{s}(0)), \mathsf{s}(x))$ is not basic

Quiz

TRS $\mathcal{R}$

$$\begin{aligned} x - \mathsf{0} &\to x \\ \mathsf{s}(x) - \mathsf{s}(y) &\to x - y \end{aligned}$$

e.g.

$$\begin{aligned} \mathbf{10} - \mathbf{3} &\to \mathbf{9} - \mathbf{2} \\ &\to \mathbf{8} - \mathbf{1} \\ &\to \mathbf{7} - \mathbf{0} \\ &\to \mathbf{7} \end{aligned}$$

QUESTIONS

- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^3)$ (cubic) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^2)$ (quadratic) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n)$ (linear) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(1)$ (constant) ?! — no

Quiz

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$

$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

QUESTIONS

- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^3)$ (cubic) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^2)$ (quadratic) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n)$ (linear) ? — no:

$$t_n := \mathbf{n} \times \mathbf{n}$$

Quiz

TRS $\mathcal{R}$

$$\mathsf{append}(\mathsf{nil}, ys) \to ys$$
$$\mathsf{append}(x : xs, ys) \to x : \mathsf{append}(xs, ys)$$

e.g.

$$\begin{aligned}\mathsf{append}(1 : 2 : 3 : \mathsf{nil}, 4 : 5 : \mathsf{nil}) &\to 1 : \mathsf{append}(2 : 3 : \mathsf{nil}, 4 : 5 : \mathsf{nil}) \\ &\to 1 : 2 : \mathsf{append}(3 : \mathsf{nil}, 4 : 5 : \mathsf{nil}) \\ &\to 1 : 2 : 3 : \mathsf{append}(\mathsf{nil}, 4 : 5 : \mathsf{nil}) \\ &\to 1 : 2 : 3 : 4 : 5 : \mathsf{nil}\end{aligned}$$

QUESTIONS

- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^3)$ (cubic) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^2)$ (quadratic) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n)$ (linear) ? — yes

Quiz

TRS $\mathcal{R}$

$$\mathsf{append}(\mathsf{nil}, ys) \to ys \qquad \mathsf{rev}(\mathsf{nil}) \to \mathsf{nil}$$
$$\mathsf{append}(x : xs, ys) \to x : \mathsf{append}(xs, ys) \qquad \mathsf{rev}(x : xs) \to \mathsf{append}(\mathsf{rev}(xs), x : \mathsf{nil})$$

e.g.

$$\begin{aligned}\mathsf{rev}(1 : 2 : 3 : \mathsf{nil}) &\to \mathsf{append}(\mathsf{rev}(2 : 3 :), 1 : \mathsf{nil})\\ &\to \mathsf{append}(\mathsf{append}(\mathsf{rev}(3 : \mathsf{nil}), 2 : \mathsf{nil}), 1 : \mathsf{nil})\\ &\to \mathsf{append}(\mathsf{append}(\mathsf{append}(\mathsf{rev}(\mathsf{nil}), 3 : \mathsf{nil}), 2 : \mathsf{nil}), 1 : \mathsf{nil})\\ &\to \mathsf{append}(\mathsf{append}(\mathsf{append}(\mathsf{nil}, 3 : \mathsf{nil}), 2 : \mathsf{nil}), 1 : \mathsf{nil})\\ &\to \cdots \to 3 : 2 : 1 : \mathsf{nil}\end{aligned}$$

QUESTIONS

- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^3)$ (cubic) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^2)$ (quadratic) ? — yes
- $\mathsf{rc}_{\mathcal{R}}(n) \in O(n)$ (linear) ? — no

Is Runtime Complexity Adequate Measure?

- consider TRS $\mathcal{R}$ of Ackerman function:

$$\mathsf{ack}(\mathsf{s}^m(0), \mathsf{s}^n(0)) \to \mathsf{s}^{ack(n,m)}(0)$$

  for all $n, m \in \mathbb{N}$

- $\mathsf{rc}_{\mathcal{R}}(n) \in O(1)$ ...

Dal Lago & Martini, 2009; Avanzini & Moser, 2010

THEOREM

for finite TRS

polynomial (innermost) runtime complexity

induces

polytime computability

Complexity of Conditions

how to measure complexity of conditional part?

Example

consider $\mathcal{R}$

$$\mathsf{f}(x, y) \to \mathsf{c} \quad \text{if } x \times x < y$$

we have

$$\mathsf{f}(\mathbf{5}, \mathbf{100}) \to_{\mathcal{R}} 1$$

- all basic terms reach normal form in O(1) steps, but
- computational costs should not be O(1)

☞ we have to count $x \times x < y$ in some way

Base k Integer (Bignum Arithmetic) TRSs

for instance $k = 10$, TRS $\mathcal{B}_k$ is

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

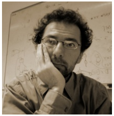
$$
\begin{array}{ll}
x + (y : z) \to y : (x + z) & 0 : x \to x \\
(x : y) + z \to x : (y + z) & x : (y : z) \to (x + y) : z
\end{array}
$$

CONJECTURE

$\mathsf{irc}_{\mathcal{R} \cup \mathcal{B}_k}(n) \in \mathbf{P} \iff \mathsf{irc}_{\mathcal{R} \cup \mathcal{B}_\ell}(n) \in \mathbf{P}$ for all $k, \ell \geqslant 2$

Constructor-Restricted Polynomial Interpretations

Bonfante & Cichon & Marion & Touzet, JFP'01



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Definition

constructor-restricted polynomial interpretation of degree $\boxed{d}$ is algebra $\mathcal{A}$ such that

- carrier is $\mathbb{N}$
- $f_{\mathcal{A}}(x_1, \dots, x_n)$ is polynomial of at most degree d if $f \in \mathcal{D}$
- $f_{\mathcal{A}}(x_1, \dots, x_n) = x_1 + \cdots + x_n + c_f$ if f is constructor

$\ell >_{\mathcal{A}} r$ if $[\alpha]_{\mathcal{A}}(\ell) > [\alpha]_{\mathcal{A}}(r)$ for all $\alpha : \mathcal{V} \to \mathbb{N}$

Theorem

let $\mathcal{A}$ be monotone constructor-restricted polynomial interpretation

$$\mathcal{R} \subseteq >_{\mathcal{A}} \quad \Longrightarrow \quad \mathsf{rc}_{\mathcal{R}}(n) \in O(n^d)$$

Runtime Complexity Analysis

TRS $\mathcal{R}$

$$\begin{array}{lcl} x + \mathsf{0} & \to & x \\ x \times \mathsf{0} & \to & \mathsf{0} \end{array} \qquad \begin{array}{lcl} x + \mathsf{s}(y) & \to & \mathsf{s}(x + y) \\ x \times \mathsf{s}(y) & \to & x \times y + x \end{array}$$

constructor-restricted polynomial interpretation $\mathcal{A}$ of degree 2

$$\begin{array}{rl} \mathsf{0}_{\mathcal{A}} = 1 & \times_{\mathcal{A}}(X, Y) = 2XY + 2X + 2Y + 2 \\ \mathsf{s}_{\mathcal{A}}(s) = X + 1 & +_{\mathcal{A}}(X, Y) = X + 2Y \end{array}$$

$$\bigwedge \left\{ \begin{array}{llll} [\alpha]_{\mathbb{N}}(x + \mathsf{0}) & = \alpha(x) + 2 & > x & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = \alpha(x) + 2\alpha(y) + 2 & > \alpha(x) + 2\alpha(y) + 1 & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ [\alpha]_{\mathbb{N}}(x \times \mathsf{0}) & = 4\alpha(x) + 4 & > 1 & = [\alpha]_{\mathbb{N}}(\mathsf{0}) \\ [\alpha]_{\mathbb{N}}(x \times \mathsf{s}(y)) & = \ldots & > \ldots & = [\alpha]_{\mathbb{N}}(x \times y + x) \end{array} \right\}$$

for all $\alpha : \mathcal{V} \to \mathbb{N}$. hence $\mathsf{rc}_{\mathcal{R}}(n) \in O(n^2)$

how to find interpretation? ☞ **QF_NIA SMT sovler**

Example

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

take well-founded strictly monotone interpretations

$$0_{\mathbb{N}} = a \qquad +_{\mathbb{N}}(X, Y) = dXY + eX + fY + g$$
$$\mathsf{s}_{\mathbb{N}}(X) = bX + c \qquad \times_{\mathbb{N}}(X, Y) = iXY + jX + kY + l$$

then $\mathcal{R} \subseteq >_{\mathbb{N}}$ corresponds to

$$\models \bigwedge \left\{ \begin{array}{lll} [\alpha]_{\mathbb{N}}(x + 0) & = bdX + eX + af + g & \\ & > \; X & = [\alpha]_{\mathbb{N}}(x) \\ [\alpha]_{\mathbb{N}}(x + \mathsf{s}(y)) & = bXY + eX + bfY + cf + g & \\ & > \; bdXY + beX + bfY + bg + c & = [\alpha]_{\mathbb{N}}(\mathsf{s}(x + y)) \\ & \dots & \end{array} \right\}$$

$$\wedge \; \bigwedge\{b, e, f, j, k > 0, \quad a, c, g, l \geq 0\}$$

Shifting Method

TRS $\mathcal{R}$

$$x + 0 \to x \qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$
$$x \times 0 \to 0 \qquad x \times \mathsf{s}(y) \to x \times y + x$$

take well-founded strictly monotone interpretations

$$0_{\mathbb{N}} = a \qquad +_{\mathbb{N}}(X, Y) = dXY + eX + fY + g$$
$$\mathsf{s}_{\mathbb{N}}(X) = bX + c \qquad \times_{\mathbb{N}}(X, Y) = iXY + jX + kY + l$$

$\mathcal{R} \subseteq >_{\mathbb{N}}$ if

$$\models \bigwedge \left\{ \begin{array}{l} bd + e \geqslant 1, af + g > 0 \\ b \geqslant bd, e \geqslant be \\ bf \geqslant bfcf + g > bg + c \\ \dots \end{array} \right\}$$
$$\wedge \bigwedge \{b, e, f, j, k > 0, \quad a, c, g, l \geq 0\}$$

Questions about Automation

usually we employ

- interpretations on $\mathbb{N}$
- shifting method to eliminate quantifiers
- QF_NIA SMT solver to find satisfiable coefficients

however,

- interpretations on $\mathbb{R}$ seem more powerful

  Neurauter et al. 2010, Middeldorp et al. 2011
- shifting method misses several examples

  Neurauter et al. 2010
- actually we want to find minimal interpretations

QUESTION

experiments by Giesl (1995) showed ineffectiveness of QE-CAD

is this still valid observation?

Travel Plans

- Hirokawa visits Innsbruck from Sep 2 (Sun) to Sep 15 (Sat)

- probably Moser visits Kanazawa in March to attend
  PR 2013: 3rd Workshop on Proof Theory and Rewriting