Completion-like Procedures Based on Constrained Equations

(and an overview of task 3)

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Outline

Task 3: Completion

- Current State
- Goals

Completion-like Procedures Based on Constrained Equations

- Preliminaries
- Completion
  - Standard Completion
  - Ordered Completion
- Inductive Theorem Proving
  - Rewriting Induction
- Automation

Overview

Task 3: Completion

Who is Involved?

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Current State

Completion with Termination Tools

- combining multi-completion with the use of termination tools (H,I)
- implemented in mkb_{TT} (H,I)
- variants for ordered completion and AC-completion (I)

Maximal Completion (J)

- completion as satisfaction problem based on MaxSAT/MaxSMT
- implemented in Maxcomp

Multi-Context Rewriting Induction with Termination Tools (H)

- based upon rewriting induction with termination tools
- adapt multi-completion approach to use multiple contexts

Goals (1)

Completion

- enhance understanding of differences between mkb_{TT}/Maxcomp
- develop combined approach

Completion Modulo

- extend Maxcomp to deal with AC- and S-normalized completion
- efficient AC-matching and unification
- optimizations: AC-term indexing and critical pair criteria

$\mathcal{E}$-Unification

- given $s \approx t$, find all substitutions σ such that $s\sigma \leftrightarrow_{\mathcal{E}}^{*} t\sigma$
- idea: tool can apply completion + basic/LSE narrowing

Goals (2)

Rewriting Induction

- Maxcomp-like approach for rewriting induction
- lemma generation techniques, handle non-orientable equations

Completion Competition

- competition of different approaches for completion
  (+ ordered completion, completion modulo, rewriting induction?)
- benchmark collection for different categories
- clarifications required:
  - what is a correct answer? ... for ordered completion?
  - is there a meaningful way to compare two answers?

Progress Report

Constrained Equations for Completion-Like Procedures

Outline

Knuth-Bendix Completion

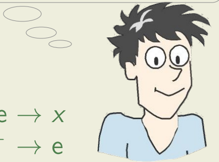
$$\underset{\text{reduction ordering}}{\succ} + \underset{\text{equations}}{\mathcal{E}_0} \longrightarrow_{KB} \underset{\text{rewrite system}}{\mathcal{R}}$$

$\mathcal{R}$ is convergent, i.e., confluent and terminating, and $\approx_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R}}$

Example (Group Theory)

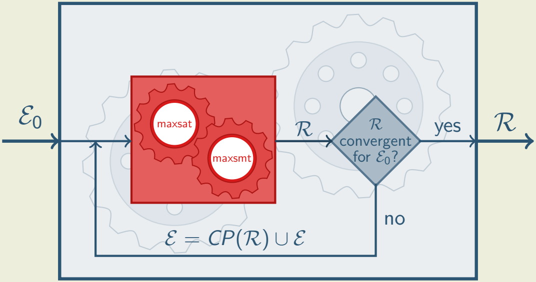
$$y \leftarrow^!_{\mathcal{R}} (x^- \cdot x)^- \cdot (\mathsf{e} \cdot (y \cdot \mathsf{e})) \overset{?}{\approx} y \cdot \mathsf{e} \rightarrow^!_{\mathcal{R}} y$$

$$\mathcal{E}_0 : \quad \begin{aligned} \mathsf{e} \cdot x &\approx x \\ x^- \cdot x &\approx \mathsf{e} \\ (x \cdot y) \cdot z &\approx x \cdot (y \cdot z) \end{aligned}$$



$$\mathcal{R} : \quad \begin{aligned} \mathsf{e} \cdot x &\rightarrow x \\ x^- \cdot x &\rightarrow \mathsf{e} \\ (x \cdot y) \cdot z &\rightarrow x \cdot (y \cdot z) \\ \mathsf{e}^- &\rightarrow \mathsf{e} \\ x^- \cdot (x \cdot y) &\rightarrow y \end{aligned} \qquad \begin{aligned} x \cdot \mathsf{e} &\rightarrow x \\ x \cdot x^- &\rightarrow \mathsf{e} \\ x^{--} &\rightarrow x \\ (x \cdot y)^- &\rightarrow y^- \cdot x^- \\ x \cdot (x^- \cdot y) &\rightarrow y \end{aligned}$$

Maximal Completion



+ fast, simple correctness proof, easy to implement
– no interreduction – limited scalability, restricted termination power

Aims

- adapt to ordered/AC/normalized completion, rewriting induction
- enhance efficiency by incorporating interreduction

Outline

Definition (termination constraint)

$$C ::= \ell \to r \mid \top \mid \bot \mid \neg C \mid C \vee C \mid C \wedge C$$

for TRS $\mathcal{R}$ define $\mathcal{R} \models C$ inductively:

$\mathcal{R} \models \ell \to r$ iff $\ell \to r \in \mathcal{R}$	$\mathcal{R} \models \top$
$\mathcal{R} \not\models \bot$	$\mathcal{R} \models C_1 \vee C_2$ iff $\mathcal{R} \models C_1$ or $\mathcal{R} \models C_2$
$\mathcal{R} \models \neg C$ iff $\mathcal{R} \not\models C$	$\mathcal{R} \models C_1 \wedge C_2$ iff $\mathcal{R} \models C_1$ and $\mathcal{R} \models C_2$

Definition (constrained equalities)

- constrained equality $(s \approx t, C)$ is pair of equality $s \approx t$ and termination constraint C
- constrained equation system (CES) $\mathbb{C}$ is set of constrained equalities

Let $\mathcal{R}$ be terminating.

Notation

$$\mathcal{E}^\top = \{(s \approx t, \top) \mid s \approx t \in \mathcal{E}\}$$

$$\mathbb{C}[\![\mathcal{R}]\!] = \{s \approx t \mid (s \approx t, C) \in \mathbb{C} \text{ and } \mathcal{R} \models C\} \qquad \mathcal{R}\text{-projection}$$

$$\mathbb{C} \ominus \mathcal{R} = \left\{(s \approx t, C \wedge \neg \bigwedge \mathcal{R} \mid (s \approx t, C) \in \mathbb{C}\right\}$$

$$\mathbb{C}{\downarrow_\mathcal{R}} = \left\{(s{\downarrow_\mathcal{R}} \approx t{\downarrow_\mathcal{R}}, C \wedge \bigwedge \mathcal{R}) \mid (s \approx t, C) \in \mathbb{C} \text{ and } s{\downarrow_\mathcal{R}} \neq t{\downarrow_\mathcal{R}}\right\}$$

Example

$\mathcal{E}$: 1: $\mathsf{s}(\mathsf{p}(x)) \approx x$ 2: $\mathsf{p}(\mathsf{s}(x)) \approx x$ 3: $\mathsf{s}(x) + y \approx \mathsf{s}(x + y)$

$\mathcal{R}$: $\mathsf{s}(\mathsf{p}(x)) \to x$ $\mathsf{s}(x + y) \to \mathsf{s}(x) + y$

$$\mathcal{E}^\top = \{(1, \top), (2, \top), (3, \top)\}$$

$$\mathcal{E}^\top[\![\mathcal{R}]\!] = \{1, 2, 3\}$$

$$\mathbb{C} = \{(1, \mathsf{s}(\mathsf{p}(x)) \to x), (2, \mathsf{s}(x) + y \to \mathsf{s}(x + y)), (3, \mathsf{s}(x + y) \to \mathsf{s}(x) + y)\}$$

$$\mathbb{C}[\![\mathcal{R}]\!] = \{1, 3\}$$

Definition

- $\mathcal{E}$ is $\mathcal{R}$-joinable if $s \downarrow_{\mathcal{R}} t$ for all $s \approx t \in \mathcal{E}$
- $\mathcal{E}$ is ground $\mathcal{R}$-joinable if $s\sigma \downarrow_{\mathcal{R}} t\sigma$ for all $s \approx t \in \mathcal{E}$ and ground σ

Definition

mapping S from CESs to CESs is (ground) reduction if $\forall$ CES $\mathbb{C}$, TRS $\mathcal{R}$

$$S(\mathbb{C})[\![\mathcal{R}]\!] \text{ (ground) } \mathcal{R}\text{-joinable} \Longrightarrow \mathbb{C}[\![\mathcal{R}]\!] \text{ (ground) } \mathcal{R}\text{-joinable}$$

Definition

$$S_{\mathcal{R}}(\mathbb{C}) = (\mathbb{C} \ominus \mathcal{R}) \cup \mathbb{C}{\downarrow_{\mathcal{R}}} \cup F(\mathcal{R})^{\top}{\downarrow_{\mathcal{R}}}$$

Lemma

S is a (ground) reduction.

Outline

Definition

$$S_{KB}(\mathbb{C}) = (\mathbb{C} \ominus \mathcal{R}) \cup \mathbb{C}{\downarrow_{\mathcal{R}}} \cup F(\mathcal{R})^{\top}{\downarrow_{\mathcal{R}}}$$

$F(\mathcal{R}) = \mathrm{CP}(\mathcal{R})$ and $\mathcal{R}$ is terminating with $\mathcal{R} \subseteq \leftrightarrow^*_{\mathcal{E}}$ (for fixed $\mathcal{E}$)

Theorem

If $\mathbb{C} = S^n_{KB}(\mathcal{E}^{\top})$ *and* $\mathbb{C}[\![\mathcal{R}]\!] = \varnothing$ *then* $\mathcal{R}$ *is convergent for* $\mathcal{E}$*.*

Proof. easy!

- $n = 0$: if $\mathbb{C}[\![\mathcal{R}]\!] = \varnothing$ then $\mathcal{E} = \varnothing$, thus $\mathcal{R} = \varnothing$
- $n > 0$:
  - $\leftrightarrow^*_{\mathcal{R}} = \leftrightarrow^*_{\mathcal{E}}$, since $\mathcal{R} \subseteq \leftrightarrow^*_{\mathcal{E}}$ by assumption and $\mathcal{E} \subseteq {\downarrow_{\mathcal{R}}}$ as S^n_{KB} is reduction (by induction on n)
  - $\mathcal{R}$ is terminating by assumption
  - $\mathcal{R}$ is confluent because for $\mathbb{C}' = S^{n-1}_{KB}(\mathcal{E}^{\top})$ the set $S_{KB}(\mathbb{C}')[\![\mathcal{R}]\!] = \varnothing$, so $\mathrm{CP}(\mathcal{R}) \subseteq {\downarrow_{\mathcal{R}}}$

□

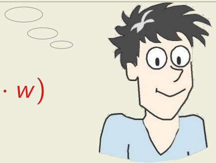
Ordered Completion

$\succ$ ground-total reduction ordering $+$ $\mathcal{E}_0$ equations $\xrightarrow{} OKB$ $\mathcal{E}, \mathcal{R}$ equations + rewrite system

$\mathcal{E}_\succ \cup \mathcal{R}$ is ground-confluent, terminating, and $\approx_{\mathcal{E}_0} = \leftrightarrow^*_{\mathcal{R}\cup\mathcal{E}}$

Example (Entropic Groupoid)

$$(a \cdot a) \cdot b \leftarrow^!_{\mathcal{R}\cup\mathcal{E}_\succ} (a \cdot b) \cdot (a \cdot b) \overset{?}{\approx} (a \cdot b) \cdot (b \cdot a) \rightarrow^!_{\mathcal{R}\cup\mathcal{E}_\succ} a$$



$a \prec b$

$\mathcal{E}_0$:

$$(x \cdot y) \cdot x \approx x$$
$$(x \cdot y) \cdot (z \cdot w) \approx (x \cdot z) \cdot (y \cdot w)$$

$\mathcal{E}$:

$$(x \cdot y) \cdot z \approx (x \cdot w) \cdot z$$

$\mathcal{R}$:

$$(x \cdot y) \cdot x \rightarrow x$$
$$x \cdot (y \cdot z) \rightarrow x \cdot z$$
$$((x \cdot y) \cdot z) \cdot w \rightarrow x \cdot w$$

Ordered Completion

Definition

- $\mathcal{R}$ is ground convergent if $\mathcal{R}$ is terminating and for all ground peaks … h that $s \to_{\mathcal{R}}^{*} u \leftarrow_{\mathcal{R}}^{*} t$

  set of $\succ$-oriented ground instances of $\mathcal{E}$

- $(\mathcal{E}, \mathcal{R})$ is ground convergent with respect to total reduction order $\succ$ if $\mathcal{E}_{\succ} \cup \mathcal{R}$ is ground convergent

Definition

$$S_O(\ldots) \ldots \mathbb{C}{\downarrow_{\mathcal{R}}} \;\cup\; F(\mathcal{R},\mathbb{C})^{\top}{\downarrow_{\mathcal{R}}}$$

extended critical pairs

- $F(\mathcal{R},\mathbb{C}) = \mathsf{CP}_{\rhd}(\mathcal{R} \cup \mathbb{C}[\![\mathcal{R}]\!])$
- $\mathcal{R}$ is totally terminating with $\mathcal{R} \subseteq \leftrightarrow^*_{\mathcal{E}}$

Theorem

If $\mathbb{C} = S^n_O(\mathcal{E}^{\top})$ *and* $S_O(\mathbb{C})[\![\mathcal{R}]\!] = \mathbb{C}[\![\mathcal{R}]\!]$ *then* $(\mathbb{C}[\![\mathcal{R}]\!], \mathcal{R})$ *is ground convergent for* $\mathcal{E}$

Example

$\mathcal{E}$:

$1: (-x + x)\cdot 1 \approx 0$	$2: (x + -x)\cdot 1 \approx x + -x$	$3: -x + x \approx y + -y$
$4: (x + -x)\cdot 1 \approx 0$	$5: 0 \approx x + -x$	$6: 0 \cdot 1 \approx 0$
$7: (-x + x) \approx 0$		

$\mathbb{C}_0 = \{(1,\top),(2,\top),(3,\top)\}$ $\qquad \mathcal{R}_1 = \{1,2\}$

$\mathbb{C}_1 = S_O(\mathbb{C}_0) = \{(1,\neg R_1),(2,\neg R_1),(3,\top),(4,R_1),(5,R_1)\}$ $\qquad \mathcal{R}_2 = \{1,2,4,5'\}$

Rewriting Induction

$\succ$ reduction ordering $+$ $\mathcal{R}_0$ rewrite system $+$ $\mathcal{E}$ equations $\longrightarrow_{RI}$ yes

such that $\mathcal{R}_0$ is quasi reducible and if we obtain yes then $\mathcal{R}_0 \vdash_i \mathcal{E}$
i.e. $s\sigma \leftrightarrow^*_{\mathcal{R}_0} t\sigma$ for all $s \approx t \in \mathcal{E}$ and ground substitutions σ

Example (Addition)

$\mathcal{R}_0 \overset{?}{\vdash_i} \quad (x+y)+z \approx x+(y+z)$

$$\mathcal{R}_0 : \quad x + 0 \approx x$$
$$x + \mathsf{s}(y) \approx \mathsf{s}(x+y)$$

$$\mathcal{E} : \quad (x+y)+z \approx x+(y+z)$$

yes

Rewriting Induction

Definition

Given TRS $\mathcal{R}_0$,

- defined symbols $\mathcal{D} = \{f \mid f$ is root symbol of ℓ for $\ell \to r \in \mathcal{R}_0\}$
- constructor symbols $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$
- term $t = f(t_1, \ldots, t_n)$ is basic if $f \in \mathcal{D}$ and all $t_i \in \mathcal{T}(\mathcal{C}, \mathcal{V})$
- basic positions $\mathcal{B}(t) = \{p \in \mathcal{P}\mathrm{os}(t) \mid t|_p$ is basic$\}$

$\mathcal{R}_0$ is quasi-reducible if no basic term is in normal form

Definition

For $\mathcal{R}_0$ quasi-reducible,

- TRS $\mathcal{R}$ is $\mathcal{R}_0$-expandable if every ℓ for $\ell \to r \in \mathcal{R}$ has basic position
- $\mathrm{Expd}(\mathcal{R}_0, \mathcal{R})$ is set of CPs from overlaps $(\ell_1 \to r_1, p, \ell_2 \to r_2)_\mu$ where $\ell_1 \to r_1 \in \mathcal{R}_0$, $\ell_2 \to r_2 \in \mathcal{R}$, and p is basic in ℓ_2

Definition

for fixed $\mathcal{R}_0$ and $\mathcal{E}$

$$S_{RI}(\mathbb{C}) = (\mathbb{C} \ominus \mathcal{R}) \cup \mathbb{C}{\downarrow_\mathcal{R}} \cup F(\mathcal{R})^\top{\downarrow_\mathcal{R}}$$

- $F(\mathcal{R}) = \mathsf{Expd}(\mathcal{R}_0, \mathcal{R})$
- $\mathcal{R}$ is terminating and $\mathcal{R}_0$-expandable such that $\mathcal{R}_0 \subseteq \mathcal{R}$ and $\ell\sigma \leftrightarrow^*_{\mathcal{R}_0 \cup \mathcal{E}} r\sigma$ for all $\ell \to r$ in $\mathcal{R}$ and ground substitutions σ

Theorem

If $\mathbb{C} = S^n_{RI}(\mathcal{E}^\top)$ *and* $\mathbb{C}[\![\mathcal{R}]\!] = \varnothing$ *then* $\mathcal{R}_0 \vdash_i \mathcal{E}$

Example

$\mathcal{R}_0$: $1 : x + 0 \to x$ $\quad$ $2 : x + \mathsf{s}(y) \to \mathsf{s}(x + y)$

$\mathcal{E}$: $3 : (x + y) + z \approx x + (y + z)$

$4 : x + z \approx x + (0 + z)$ $\quad$ $5 : \mathsf{s}(x + y) + z \approx x + (\mathsf{s}(y) + z)$

$\mathbb{C}_2[\![\mathcal{R}_2]\!] = \varnothing$, so $\mathcal{R}_0 \vdash_i (x + y) + z \approx x + (y + z)$

$\mathbb{C}_0 = \{(3, \top)\}$ $\quad$ $\mathcal{R}_1 = \{1, 2, 3\}$

$\mathbb{C}_1 = S_{RI}(\mathbb{C}_0) = \{(3, \neg R_1), (4, R_1), (5, R_1)\}$ $\quad$ $\mathcal{R}_2 = \{1, 2, 3'\}$

Outline

Completion

Definition

$$S_{\mathcal{R}}(\mathbb{C}) = (\mathbb{C} \ominus \mathcal{R}) \;\cup\; \mathbb{C}{\downarrow_{\mathcal{R}}} \;\cup\; F(\mathcal{R})^{\top}{\downarrow_{\mathcal{R}}}$$

Theorem

If $\mathbb{C} = S^n_{KB}(\mathcal{E}^{\top})$ *and* $\mathbb{C}[\![\mathcal{R}]\!] = \varnothing$ *for* $\mathcal{R} \in \mathfrak{R}(\mathbb{C})$ *then* $\mathcal{R}$ *is convergent for* $\mathcal{E}$.

Procedure

how to find $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \ldots$?

$$\mathbb{C}_0 = \mathcal{E}^{\top} \quad \mathbb{C}_1 = S_{\mathcal{R}_1}(\mathbb{C}_0) \quad \mathbb{C}_2 = S_{\mathcal{R}_2}(\mathbb{C}_1) \quad \mathbb{C}_3 = S_{\mathcal{R}_3}(\mathbb{C}_2) \quad \ldots$$

Maximal Completion Approach

to obtain assignment α

SAT/SMT encoding of $>_{\mathsf{kbo}}$, $>_{\mathsf{lpo}}$ or $>_{\mathsf{mpo}}$

maximize $\displaystyle\bigvee_{(s \approx t, C) \in \mathbb{C}_k} C \Rightarrow, (\ulcorner s > t \urcorner \vee \ulcorner t > s \urcorner)$ *subject to* $\displaystyle\bigwedge_{i=1}^{k} \neg \bigwedge \mathcal{R}_i$

and let $\mathcal{R}_k = \{ s \to t \mid (s \simeq t, C) \in \mathbb{C}_k$ and $\alpha \models \ulcorner s > t \urcorner \}$

Preliminary Results

Completion

115 systems in Maxcomp distribution

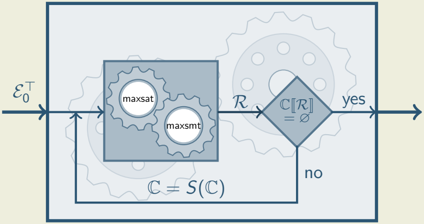
	Maxcomp		mkb_{TT}		cKB	
	LPO	KBO	LPO	KBO	LPO	KBO
completed	86	69	70	67	84	56
failure	6	3	6	3	0	0
timeout	23	43	39	45	31	59

Rewriting Induction

103 systems from *Dream Corpus of Inductive Conjectures*

	cRI	MRIt
	LPO	?
success	30	35
timeout	73	98

Summary



- generalize maximal completion to constrained equality framework: ordered & AC-completion, inductionless & rewriting induction
- simple correctness proofs, easy to implement
- restricted maximization, allows for interreduction

Further Work

- cover S-normalized completion, tools for ordered & AC-completion
- improve termination power: DPs + ?
- ...