

# Proving Confluence of Conditional Term Rewriting Systems via Unravelings

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June 28, 2013

# Example: Conditional TRS even/odd

## Example (CTRS representing Even/Odd)

$$\mathcal{R}_{\text{even}} = \left\{ \begin{array}{l} \text{even}(0) \rightarrow \text{true} \\ \text{odd}(0) \rightarrow \text{false} \\ \text{even}(s(x)) \rightarrow \text{true} \Leftarrow \text{odd}(x) \rightarrow^* \text{true} \\ \text{even}(s(x)) \rightarrow \text{false} \Leftarrow \text{odd}(x) \rightarrow^* \text{false} \\ \text{odd}(s(x)) \rightarrow \text{true} \Leftarrow \text{even}(x) \rightarrow^* \text{true} \\ \text{odd}(s(x)) \rightarrow \text{false} \Leftarrow \text{even}(x) \rightarrow^* \text{false} \end{array} \right\}$$

# Outline

- 1 Transformations of CTRSs
- 2 Soundness and Confluence
- 3 Yet another transformation
- 4 Conclusion

# Conditional term rewriting

## CTRSs

- Conditional rule:  $l \rightarrow r \Leftarrow s_1 = t_1, \dots, s_k = t_k$
- Oriented CTRS: “=” is interpreted as “ $\rightarrow^*$ ”
- Conditions cause problems

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- Terminating
- Non-overlapping
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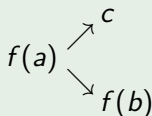
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# Transformation of CTRSs into unconditional TRSs

## Sequential unraveling of [Ohlebusch 2002]

$$l \rightarrow r \Leftarrow s_1 \rightarrow^* t_1, \dots, s_k \rightarrow^* t_k \implies \begin{cases} l \rightarrow U_1(s_1, \vec{X}_1) \\ U_1(t_1, \vec{X}_1) \rightarrow U_2(s_2, \vec{X}_2) \\ \vdots \\ U_k(t_k, \vec{X}_k) \rightarrow r \end{cases}$$

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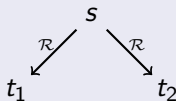
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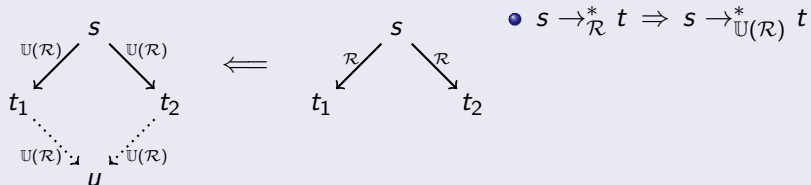
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How to prove confluence of CTRS  $\mathcal{R}$  via  $\mathbb{U}(\mathcal{R})$ ?



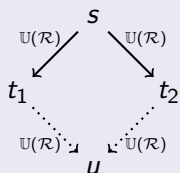
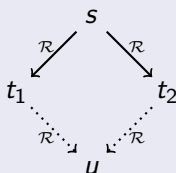
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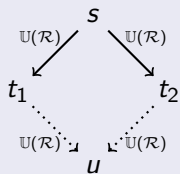
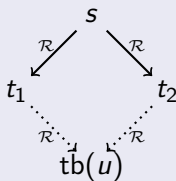
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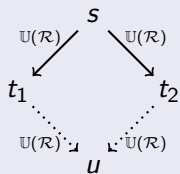
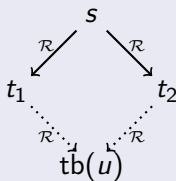
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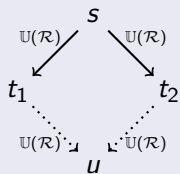
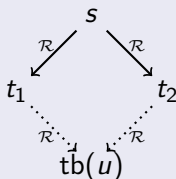

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- $s \rightarrow_{\mathbb{U}(\mathcal{R})} t \Rightarrow s \rightarrow_{\mathcal{R}} \text{tb}(t)$  for weakly left-linear CTRSs.
- Implies soundness for joinability  $s \downarrow_{\mathbb{U}(\mathcal{R})} t \Rightarrow s \downarrow_{\mathcal{R}} t$
- CR of  $\mathbb{U}(\mathcal{R})$  + Soundness for joinability  $\Rightarrow$  CR of  $\mathcal{R}$ .

New proof for result of [Suzuki, Middeldorp, Ida, RTA 1995]

- Orthogonal properly oriented right-stable 3-CTRSs are confluent.



# The unraveling $\mathbb{U}_{conf}$

## Example (even/odd-CTRS)

$$\begin{array}{l} \text{even}(s(x)) \rightarrow \text{true} \Leftarrow \text{odd}(x) \rightarrow^* \text{true} \\ \text{even}(s(x)) \rightarrow \text{false} \Leftarrow \text{odd}(x) \rightarrow^* \text{false} \end{array} \quad \left\{ \begin{array}{l} \text{even}(s(x)) \rightarrow U_1(\text{odd}(x), x) \\ U_1(\text{true}, x) \rightarrow \text{true} \\ \text{even}(s(x)) \rightarrow U_2(\text{odd}(x), x) \\ U_2(\text{false}, x) \rightarrow \text{false} \end{array} \right.$$

$$\begin{array}{l} \text{even}(s(0)) \begin{cases} \nearrow U_1(\text{odd}(0), 0) \rightarrow U_1(\text{false}, 0) \\ \searrow U_2(\text{odd}(0), 0) \rightarrow U_2(\text{false}, 0) \rightarrow \text{false} \end{cases} \end{array}$$

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- Idea: Pick labels for  $U$ -symbols based on terms in rule

$$l \rightarrow r \Leftarrow s_1 \rightarrow^* t_1, \dots, s_k \rightarrow^* t_k \quad \left\{ \begin{array}{l} l \rightarrow U_{l, s_1}(s_1, \vec{X}_1) \\ U_{l, s_1}(t_1, \vec{X}_1) \rightarrow U_{l, s_1, t_1, t_2}(s_2, \vec{X}_2) \\ \vdots \\ U_{l, s_1, t_1, \dots, s_k}(t_k, \vec{X}_k) \rightarrow r \end{array} \right.$$

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## Example (even/odd-CTRS using $\mathbb{U}_{conf}$ )

$$\begin{array}{l} \text{even}(s(x)) \rightarrow \text{true} \Leftarrow \text{odd}(x) \rightarrow^* \text{true} \left\{ \begin{array}{l} \text{even}(s(x)) \rightarrow U_{\text{even}(s(x)),\text{odd}(x)}(\text{odd}(x), x) \\ U_{\text{even}(s(x)),\text{odd}(x)}(\text{true}, x) \rightarrow \text{true} \end{array} \right. \\ \text{even}(s(x)) \rightarrow \text{false} \Leftarrow \text{odd}(x) \rightarrow^* \text{false} \left\{ \begin{array}{l} \text{even}(s(x)) \rightarrow U_{\text{even}(s(x)),\text{odd}(x)}(\text{odd}(x), x) \\ U_{\text{even}(s(x)),\text{odd}(x)}(\text{false}, x) \rightarrow \text{false} \end{array} \right. \end{array}$$

## even/odd-CTRS

## Example

$$\mathbb{U}_{conf}(\mathcal{R}) = \left\{ \begin{array}{l} \text{even}(0) \rightarrow \text{true} \\ \text{even}(s(x)) \rightarrow U_{\text{even}(s(x)), \text{odd}(x)}(\text{odd}(x), x) \\ U_{\text{even}(s(x)), \text{odd}(x)}(\text{true}, x) \rightarrow \text{true} \\ U_{\text{even}(s(x)), \text{odd}(x)}(\text{false}, x) \rightarrow \text{false} \\ \text{odd}(0) \rightarrow \text{false} \\ \text{odd}(s(x)) \rightarrow U_{\text{odd}(s(x)), \text{even}(x)}(\text{even}(x), x) \\ U_{\text{odd}(s(x)), \text{even}(x)}(\text{true}, x) \rightarrow \text{true} \\ U_{\text{odd}(s(x)), \text{even}(x)}(\text{false}, x) \rightarrow \text{false} \end{array} \right\}$$

- $\mathbb{U}_{conf}(\mathcal{R})$  is left-linear  $\implies \mathbb{U}_{conf}$  is sound for joinability
- $\mathbb{U}_{conf}(\mathcal{R})$  is SN and non-overlapping  $\implies \mathbb{U}_{conf}(\mathcal{R})$  is confluent
- $\implies \mathcal{R}$  is confluent.

# Conclusion and Perspectives

## What we have shown

- Unravelings can be used to prove confluence of CTRSs
- The unraveling  $\mathbb{U}_{conf}$  has good properties for overlay CTRSs.

## What we might show in the future

- Reachability analysis using tree automata
- Different transformations
- Tools proving confluence of CTRSs