Basics of CafeOBJ and Peano Style Natural Numbers

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Topics

- Basic concepts for modeling, specification, verification in CafeOBJ
- Basics of CafeOBJ language system: module, signature, equation, expression/term, reduce, parse
- Specification and verification of Peano style natural numbers

Modeling, Specifying, and Verifying in CafeOBJ

1. By understanding a problem to be modeled/ specified, determine several sorts of objects (entities, data, agents, states) and operations (functions, actions, events) over them for describing the problem

2. Define the meanings/functions of the operations by declaring equations over expressions/terms composed of the operations

3. Write proof scores for properties to be verified

Natural Numbers -- Signature --

0 0+1 0+1+1 0+1+1+1 0+1+1+1+1 …

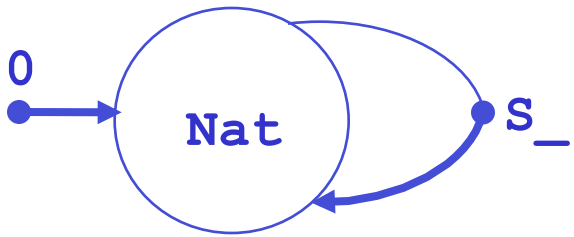
0 s(0) s(s(0)) s(s(s(0))) s(s(s(s(0)))) …

objects: Nat

operations: 0 : returns zero without arguments

s : if given a natural number *n*, returns the next natural number (s *n*) of *n*

```
-- sort
[ Nat ]
--constructor operators
op 0 :  -> Nat {constr}
op s_:  Nat -> Nat {constr}
```



Natural Number
-- Expressions/terms composed of operators

1. 0 is a natural number
2. If *n* is natural number then (s *n*) is a natural number
3. An object which is to be a natural number by 1 and 2 is only a natural number

Peano's definition of natural numbers (1889), Giuseppe Peano (1858-1932)

```
Nat = {0, s(0), s(s(0)), s(s(s(0))), s(s(s(s(0)))) … }
```

```
Nat = {0, s 0, s s 0, s s s 0, s s s s 0, … }
```

Describe a concept in expressions/terms!

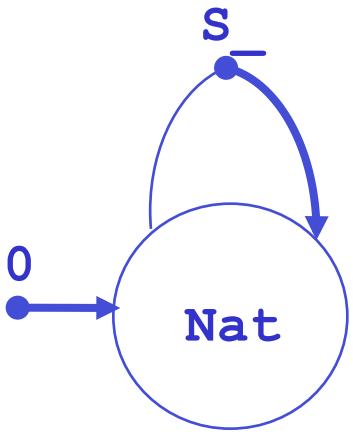
CafeOBJ module specifying PNAT -- Peano Style natural numbers

```
mod! PNAT {
   [ Nat ]
   op 0 : -> Nat {constr} .
   op s_ : Nat -> Nat {constr} .

   -- op _=_ : Nat Nat -> Bool {comm} .
   -- eq (N:Nat = N) = true .
   eq (0 = s(N2:Nat)) = false .
   eq (s(N1:Nat) = s(N2:Nat)) = (N1 = N2) .
}
```

Constructors (indicated by {constr}) define recursively the set of terms which constitute a sort.

Natural numbers
-- signature and expressions/terms

```
-- sort
[ Nat ]
-- operations
op 0 :   -> Nat {constr}
op s_:   Nat -> Nat {constr}
```



$$\mathrm{Nat} = \{ 0 \} \cup \{ s\ n \mid n \in \mathrm{Nat} \}$$

Mathematical Induction over Natural Numbers (1)

The recursive structure defined by two constructors of sort Nat induces the following induction scheme.

Goal: Prove that a property $P(n)$ is true for any natural number $n \in \{0, s\ 0, s\ s\ 0, \ldots\}$

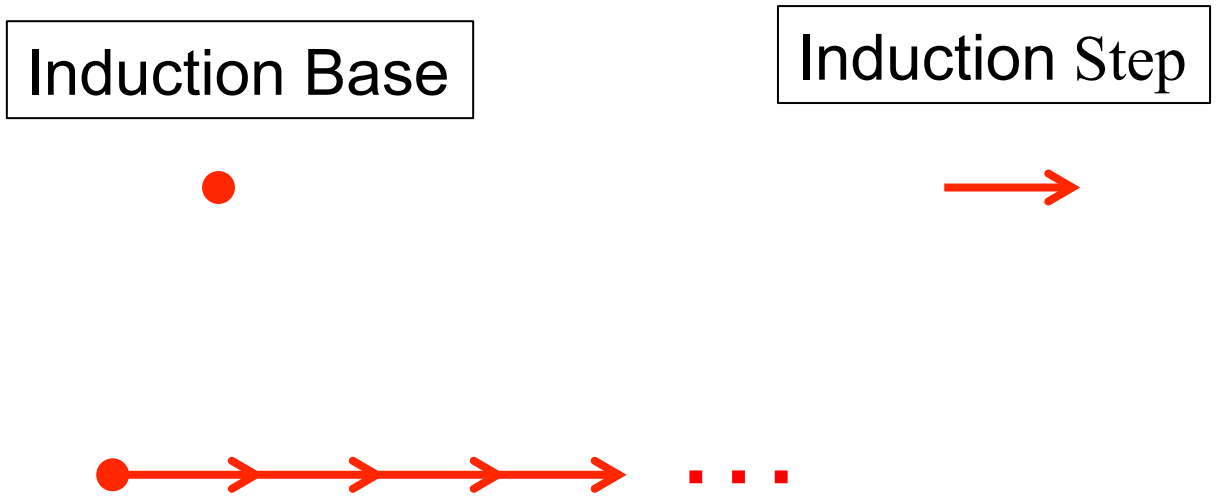
Induction Scheme:

$$\frac{P(0) \qquad \forall n \in \mathbf{N}.[P(n) \Rightarrow P(s\ n)]}{\forall n \in \mathbf{N}.P(n)}$$

Concrete Procedure: (induction with respect to n)

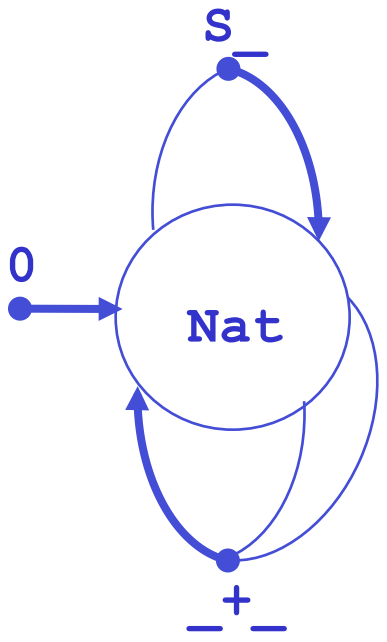
1. Prove $P(0)$ is true
2. Assume that $P(n)$ holds, and prove that $P(s\ n)$ is true

Mathematical Induction over Natural Numbers (2)



Natural numbers with addition operation -- signature and expressions/terms

```
-- sort
[ Nat ]
-- operations
op 0 :   -> Nat {constr}
op s_:   Nat -> Nat {constr}
op _+_: Nat Nat -> Nat
   -- _+_ is a defined operator
```



$$\mathrm{Nat} = \{ 0 \} \cup \{ s\ n \mid n \in \mathrm{Nat} \}$$

$$\mathrm{NatExp} = \{ 0 \} \cup \{ s\ n \mid n \in \mathrm{Nat} \} \cup \{ n1 + n2 \mid n1 \in \mathrm{Nat} \wedge n2 \in \mathrm{Nat} \}$$

Natural numbers with addition -- equations define meaning/function

CafeOBJ module PNAT+ defining Peano Natural numbers with addition

```
mod! PNAT+ {
  pr(PNAT)
  op _+_ : Nat Nat -> Nat .
  vars N1 N2 : Nat .
  -- equations
  eq 0 + N2 = N2 .
  eq (s N1) + N2 = s(N1 + N2) .
}
```

Defined operator (_+_) is erased by the two equations.

Sufficient Completeness

Computation/inference with the equations

```
   (s s 0) + (s 0)
= s((s 0) + (s 0))
= s s(0 + (s 0))
= s s s 0
```

```
CafeOBJ> select PNAT+
PNAT+> red s s 0 + s 0 .
PNAT+> -- reduce in PNAT+ :
((s (s 0)) + (s 0)):Nat
(s (s (s 0))):Nat
(0.000 sec for parse,
3 rewrites(0.000 sec),
5 matches)
```

Natural numbers with addition
-- expressions/terms composed by operators

```
NatExp = {
0, s 0, s s 0, s s s 0, ... ,
0 + 0, 0 + (s 0), 0 + (s s 0), 0 + (s s s 0), ...,
(s 0) + 0, (s 0) + (s 0), (s 0) + (s s 0),
                          (s 0) + (s s s 0), ...,
(s s 0) + 0, (s s 0) + (s 0), (s s 0) + (s s 0),
                        (s s 0) + (s s s 0), ...,
... ...
0 + (0 + 0), 0 + (0 + (s 0)), ...
...
(0 + 0) + 0, (0 + (s 0)) + 0, ...
...
. }
```

Because _+_ is a defined operator, any _+_ operator is supposed to be eliminated. That is, NatExp ==> Nat .

Reduction of CafeOBJ is honest to equational reasoning

- The **basic mechanism of CafeOBJ verification is equational reasoning. Equational reasoning is to deduce an equation (a candidate of a theorem) from a given set of equations (axioms of a specification).**

- **The CafeOBJ system supports an automatic equational reasoning based on term rewriting.**

- **“`reduce`” or “`red`” command of CafeOBJ helps to do equational reasoning by term rewriting.**

What can be done with `red` (reduction) command?

Let us fix a context M (a module M in CafeOBJ), and let (**`t1 =*M> t2`**) denote that **`t1`** is reduced to **`t2`** in the context. That is, (**`red in M : t1 .`**) returns **`t2`** . Let (**`t1 =M t2`**) denote that **`t1`** is equal to **`t2`** in the context **`M`**. That is (**`t1 = t2`**) can be inferred by equational reasoning in **`M`**. It is important to notice:

(**`t1 =*M> t2`**) implies (**`t1 =M t2`**)

but

(**`t1 =M t2`**) does not implies (**`t1 =*M> t2`**)

Proof score for right zero property: (N:Nat + 0 = N)

```
-- proof by induction with respect to N:Nat
-- induction base case:
-- opening module PNAT+ to make use of all its contents
open PNAT+
red 0 + 0 = 0 .
close
-- induction step case:
open PNAT+
-- declare that the constant n stands for any Nat value
op n : -> Nat .
-- induction hypothesis:
eq n + 0 = n .
-- induction step proof for (s n):
red s n + 0 = s n .
close
```

Declaring constants and equations then reduce

While a module is opened, declaring constants and equations represents assumptions for equational reasoning done by **red**.

```
%PNAT+> op n : -> Nat .
...
%PNAT+> **> induction hypothesis:
%PNAT+> eq n + 0 = n .
%PNAT+> **> induction step proof for (s n):
**> induction step proof for (s n):
%PNAT+> red s n + 0 = s n .
*
-- reduce in %PNAT+ : (((s n) + 0) = (s n)):Bool
(true):Bool
```

This is a proof of
$\forall$ ***N*:Nat.** [**(*N* + 0) = *N*** implies ((**s *N*) + 0) = (s *N***)].

Proof score for associativity of (_ + _)
(N1:Nat + N2:Nat) + N3:Nat = N1 +(N2 + N3)

```
**> induction base case:
open PNAT+
red 0 + (`n2:Nat + `n3:Nat) = (0 + `n2) + `n3 .
Close
**> induction step case:
open PNAT+
**> declare that the constant n1 stands for any Nat value
op n1 : -> Nat .
**> induction hypothesis:
eq (n1 + N2:Nat) + N3:Nat = n1 + (N2 + N3) .
**> induction step proof for (s n1):
red ((s n1) + `n2:Nat) + `n3:Nat = (s n1) + (`n2 + `n3) .
close
```

Comments

A line beginning with “--” (or “”) is ignored, and**
A line beginning with “-->” (or “>”) is echoed back.**

```
CafeOBJ> -- this is a comment
CafeOBJ>
```

```
CafeOBJ> ** this is a comment
CafeOBJ>
```

```
CafeOBJ> --> this is a comment
--> this is a comment
CafeOBJ>
```

```
CafeOBJ> **> this is a comment
**> this is a comment
CafeOBJ>
```

It is very important to write as much appropriate comments as possible for explaining specifications and proof scores (verifications/proofs).

Three kinds of modules

CafeOBJ specification is composed of modules.
There are three kinds of modules.

```
mod! <module_name> {
  <modlue_element> *
}
```

```
mod* <module_name> {
  <modlue_element> *
}
```

```
mod <module_name> {
  <modlue_element>*
}
```

mod! declares that the module denotes tight denotation
mod* declares that the module denotes loose denotation
mod does not declare any semantic denotation

[Naming convention] module name starts with two successive upper case characters
(example: **TEST, NAT, PNAT+, ACCOUNT-SYS,**…)

A module is composed of signature and axioms/equations

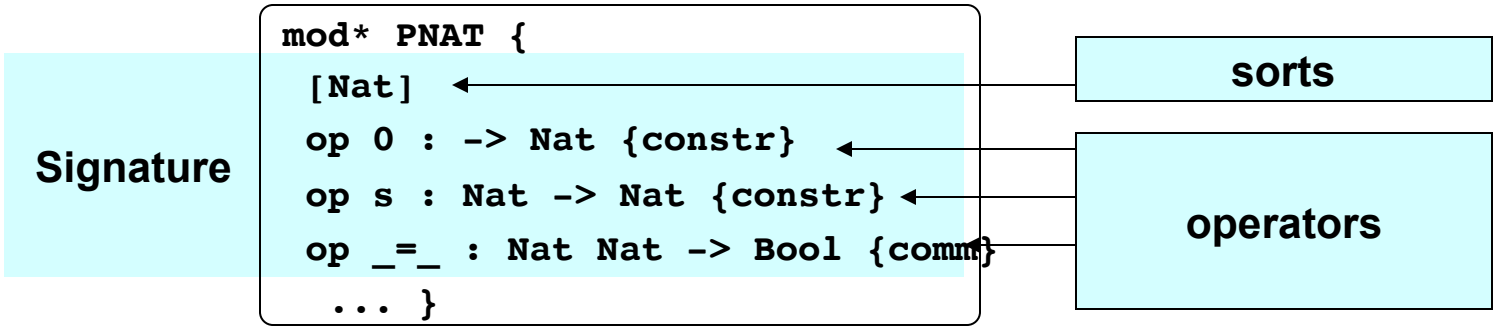
```
mod! PNAT {
  [ Nat ]
  op 0 : -> Nat {constr} .
  op s_ : Nat -> Nat {constr} .
  op _=_ : Nat Nat -> Bool {comm} .

  eq (N:Nat = N) = true .
  eq (0 = s(N2:Nat)) = false .
  eq (s(N1:Nat) = s(N2:Nat))
          = (N1 = N2) .
}
```

signature

axioms/equations

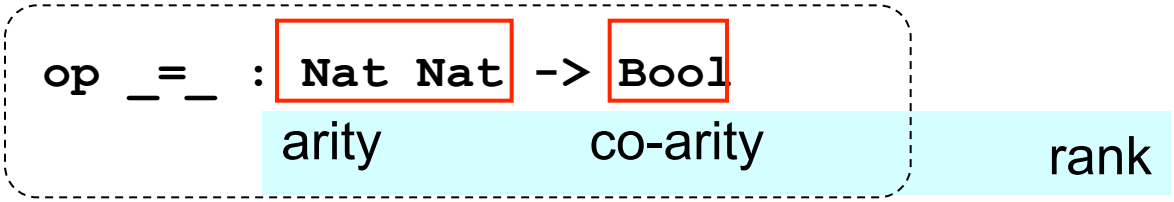
Signature:
sort name, operator name, arity, co-arity, rank

A signature is a pair of a set of sorts and a set of operations.



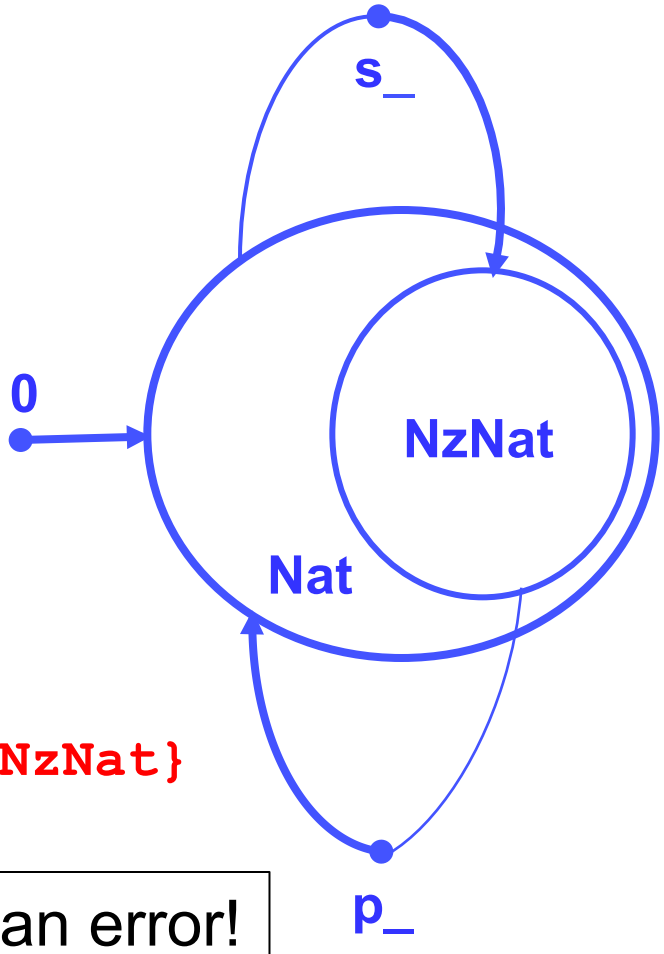
[Convention] The first and second letter of a sort name is written in a upper case and lower case letter respectively. (E.g. `Nat, Set`)

[Convention] The first letter of an operation name is written in a lowerl case letter or a non-alphabet letter. (E.g. `0, s, +`)



Order sorted signature and sorted terms -- Natural numbers with predecessor function

```
-- signature
-- sorts
[ NzNat < Nat ]
-- operators
op 0 :   -> Nat {constr}
op s_:   Nat -> NzNat {constr}
op p_:   NzNat -> Nat
eq p s N:Nat = N .
```



Sorted terms

NzNat = {s *n* | *n* ∈ Nat}

Nat = {0} ∪ NzNat ∪ {p *n* | *n* ∈ NzNat}

`(p 0)` is handled as an error!

Recursive definition of terms
- term is also called expression or tree

For a given signature, `t` is a term of a sort `S` if and only if `t` is

- a variable `X:S`,
- a constant `c` declared by "`op c : -> S`", or
- a term $f(t_1,\ldots,t_n)$ for "`op f :` $S_1 \ldots S_n$ `-> S`" and a term t_i of a sort S_i (i =1,…,n).
- a term of a sort `S'` which is a sub-sort of `S` (Example: Since `NzNat < Nat`, a term (`s 0`)`of sort NzNat` is also a term of sort `Nat`)

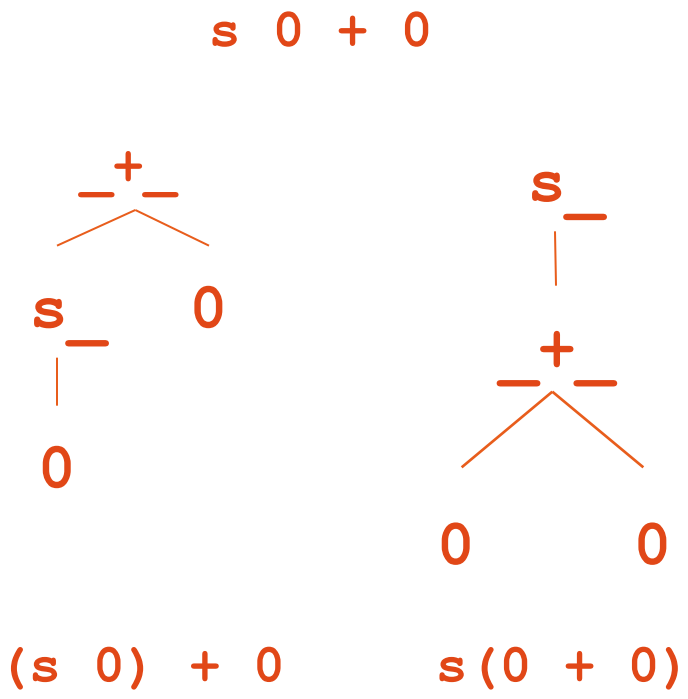
Several forms of function application: standard, prefix, infix, postfix, distfix

```
op f : Nat Nat -> Nat .
    f(2,3)   standard
op f__ : Nat Nat -> Nat .
    (f 2 3) prefix
op _+_ : Nat Nat -> Nat .
    (2 + 3) infix
op  _! : Nat -> Nat .
    (5 !)    postfix
op if_then_else_fi : Bool Nat Nat -> Nat .
    (if 2 < 3 then 4 else 5 fi)  distfix
```

“(“ and “)” are meta-charactors for grouping expressions in CafeOBJ and can not be used for any other purpose.

Parsing – precedence of operators

```
s 0 + 0 represents (s 0) + 0, because the
operator s_ has high precedence than the
operator _+_
```



The preceedences of the operators can be checked by the commands

```
describe op s_
describe op _+_
```

Equation

An equation is a pair of terms of a same sort, and written as:

```
eq l = r .
```

in CafeOBJ. Where `l` is called the left-hand side (LHS) of the equation and `r` is the right-hand side (RHS). An equation can have a condition (COND, a Boolean term) `c` like:

```
ceq l = r if c .
```

- **Properties to be verified are also expressed as equations.**

Conditions for an equation to be a rewriting rule

For an equation to be used as a rewriting rule for doing reductions, the following conditions must be satisfied.

(1) LHS is not a variable.

an example violating this condition:

```
eq N:Nat = N:Nat + 0 .
```

(2) All variables in RHS are in LHS.

an example violating this condition:

```
eq 0 = N:Nat * 0 .
```

Two way of declaring variables
- use appropriate one based on the situation

Variable can be declared in an equation directly. The scope of the variable ends at the end of the equation.

```
mod! PNAT+ { [Nat] …
  eq      0        + N2:Nat = N2 .
  eq (s N1:Nat) + N2:Nat = s(N1 + N2) . }
```

Variables can be declared before equations. This is just abbreviation for saving many variable declarations in the equations. N2 in the first eq has nothing to do with N2 in the second eq .

```
mod! PNAT+ { [Nat] …
  vars N1 N2 : Nat .
  eq      0  + N2 = N2 .
  eq (s N1) + N2 = s(N1 + N2) . }
```

Constant v.s. variable

Using a variable in an equation instead of a constant makes a drastic change of meaning of the proof score. Be careful!

- **The scope of a constant is to the end of a open-close session assuming that the declared constants are fresh.**
- **The scope of a variable is inside of the equation.**

```
open PNAT+
op n : -> Nat .
eq n + 0 = n .
red (s n) + 0 = s n .
close
```

```
open PNAT+
var N : Nat .
eq N + 0 = N .
red (s `n:Nat) + 0 = s `n .
close
```

Constant: $\forall N{:}\mathtt{Nat}.\ [(N + 0) = N \Rightarrow ((\mathtt{s}\ N) + 0) = (\mathtt{s}\ N)]$

Variable: $\forall N{:}\mathtt{Nat}.[(N + 0) = N] \Rightarrow \forall N{:}\mathtt{Nat}.[((\mathtt{s}\ N) + 0) = (\mathtt{s}\ N)]$

Two equality predicates _=_ and _==_

Assume that (**t1 =*> t1'**) and (**t2 =*> t2'**) in any context
then
if (**t1'** and **t2'** are the same term)
then (**red t1 = t2 .**) returns **true**
and
(**red t1 == t2 .**) returns **true**
if (t1' and t2' are different terms)
then (**red t1 = t2 .**) returns (**t1' = t2'**)
but
(**red t1 == t2 .**) returns **false**

If reduction/rewriting is not complete w.r.t. a set of equations, _==_ may returns false even if two terms may have a possibility of being equal w.r.t. the set of equations.