Analysis of Convolutional Codes on the Erasure Channel

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I. Introduction

The erasure channel is perhaps the simplest communications channel model. A symbol from a q-ary alphabet is input into the channel, and either the symbol is received correctly with probability $1-p$, or is converted to an erasure symbol with probability p. In this paper, we compare the maximum likelihood (ML) sequence decision and the maximum *a posteriori* (MAP) symbol decision for codes which are transmitted over the erasure channel.

II. Equivalence of ML and MAP

When decoding convolutional codes transmitted over an AWGN channel, it is widely known that the probability of symbol error for the Viterbi algorithm (which is a *sequence* ML decoder) is generally higher than that for the more complex BCJR algorithm (which is a *symbol* MAP decoder). The Viterbi algorithm is often preferred in practice because of its lower complexity, even though it is suboptimal in the symbol error sense. However, when decoding on the erasure channel, the two algorithms have the same probability of symbol error. In fact, we show for linear codes in general:

Theorem 1 *When a codeword from a linear error-correcting code with elements from the field* GF(q) *is transmitted over a q-ary erasure channel, the symbol error rate of the maximum likelihood (ML) sequence decision is the same as that of the symbol maximum* a posteriori *(MAP) probability decision.*

Theorem 1 is distinct from the well-known result that for equally-likely sequences, the sequence ML and the sequence MAP decisions are the same.

III. Viterbi and BCJR State Metrics

The similarities between the Viterbi algorithm's forward recursion and the BCJR algorithm's forward recursion have often been noted, recently in [1]. The state metrics of the Viterbi algorithm are often logarithms of probabilities, but here we use probabilities. At time t, the metric for state m of the Viterbi algorithm is $S_t(m)$. For the forward recursion of the BCJR algorithm, the state metrics are $A_t(m)$, which are also probabilities.

Theorem 2 *For the decoding of convolutional codes transmitted over the erasure channel:*

1. *The state metrics of the Viterbi and BCJR algorithm are identical, that is $S_t(m) = A_t(m)$,*
2. *The state metrics take on only one of two values, either a constant λ_t, or 0,*
3. *The number of state metrics with value λ_t is q^{i_t}, i_t a non-negative integer.*

Statement 1 assumes that the state metrics are normalized such that $\sum_m S_t(m) = \sum_m A_t(m) = 1$. If the state metrics are not normalized, the Viterbi state metrics will be proportional to the BCJR state metrics, and the remaining statements hold. The constant λ_t depends on time, but at any instant in time, all the non-zero state metrics have the same value. Statement 3 implies $\lambda_t = 1/q^{i_t}$, where the integer i_t depends upon the erasure pattern.

For decoding on general channels, the state metric recursion $A_1, A_2, \ldots$ forms a first-order Markov chain, where A_t is the vector of state metrics at time t. For decoding on the erasure channel, Theorem 2 establishes that the number of state metric vectors for the BCJR algorithm is finite and thus the number of states in this Markov chain is also finite. This property has been exploited to write exact analytic expressions for both the thresholds of turbo codes [2], and the bit-error rate of the BCJR algorithm decoding convolutional codes [3].

IV. Probability of First Error Event

Theorem 2 implies that it is also possible to construct a finite-state Markov chain for the state metric recursion of the Viterbi algorithm. The Markovian state describes not only the value of the state metrics at time t, but also which trellis states, if ultimately included in the traceback path, correspond to an error event beginning at some time t_0. Using this Markov chain, it is possible to find an exact analytic expression for the probability of an event error. For the rate-1/2 systematic convolutional code with parity generator polynomial $1+D$, the probability of an error event P_{EV} beginning at some time t_0 is given by:

$$P_{\text{EV}} = \frac{1}{4} \cdot \frac{2p^3 - 2p^4 + p^5}{(p^2-p+1)^2} = \frac{1}{2}p^3 + \frac{1}{2}p^4 - \frac{1}{4}p^5 + \cdots.$$

At low probability of channel erasure, the error event probability is dominated by the leading term $1/2\,p^3$, which corresponds to the code's minimum distance of three.

References

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