Power-Constrained Communications
Using LDLC Lattices

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Abstract—An explicit code construction for using low-density lattice codes (LDLC) on the constrained power AWGN channel is given. LDLC lattices can be decoded in high dimension, so that the code relies on the Euclidean distance between codepoints. A sublattice of the coding lattice is used for code shaping. Lattice codes are designed using the continuous approximation, which allows separating the contribution of the shaping region and coding lattice to the total transmit power. Shaping and lattice decoding are both performed using a belief-propagation decoding algorithm. At a rate of 3 bits per dimension, a dimension 100 code which is 3.6 dB from the sphere bound is found.

I. INTRODUCTION

In his 1959 paper, Shannon used random Euclidean-space code constructions to prove that the AWGN channel capacity is $\frac{1}{2}\log\left(1 + \mathrm{SNR}\right)$ [1]. Lattices, however, are appealing for their regular structure, and it is now known that codes based upon lattices can also achieve the channel capacity [2] [3]. But to use lattices on the AWGN channel with a power constraint, a subset of the lattice points must be selected, or shaped, to construct the lattice code. When designing codes for high SNRs, the problem of shaping can be separated from that of coding, that is, the design of the lattice [4].

Channel capacity cannot be practically approached using conventional lattices alone because the decoding complexity grows quickly as the lattice dimension increases. However recently, Sommer, Feder and Shalvi described LDLC lattices, which have a sparse inverse generator matrix, making them amenable to belief-propagation decoding with complexity which is linear in the dimension. As such, it is computationally tractable to decode a dimension 10^6 LDLC lattice, which showed a noise threshold within 0.6 dB of the *coding* capacity [5]. However, *shaping* aspects, required to use LDLC lattices on AWGN channels, have not received attention.

Conway and Sloane suggested using "self-similar" lattices for shaping, where the shaping region is a Voronoi region of a shaping lattice, and the shaping lattice is a sublattice of the coding lattice [11]. This scheme has an important practical aspect: a one-to-one mapping between information bits and lattice code points is easy to find. However, it requires a lattice quantizer, although it does not need to be an optimal quantizer.

In this paper, a construction of LDLC-based lattice codes for the AWGN channel is given. A self-similar lattice is used for shaping, and quantization is based on a belief-propagation decoder. Whereas the belief-propagation decoder usually converges when decoding on the unconstrained-power channel, it usually does not converge when used for quantization. A simple lattice quantizer is proposed which employs multiple hard decisions made by the belief-propagation decoder. This quantizer is not optimal, but its performance is good enough to provide sufficient shaping performance. As a result, the constructed code depends both upon the coding lattice (since the shaping lattice is a sublattice of the coding lattice) and the quantization algorithm.

For an LDLC lattice characterized by a parameter α, with $\alpha \geq 0$, it is known that the coding loss decreases as alpha increases towards one [5]. However, using the proposed quantizer, it is found that the shaping loss increases for increasing alpha. Thus, there is trade-off between shaping loss and coding loss as α varies. For a specific dimension $n = 100$ LDLC lattice, the value of $\alpha \approx 0.64$ minimized the sum of the coding loss and shaping loss. At a rate of 3 bits/dimension, this loss is 3.6 dB with respect to the sphere bound. At a rate of 2 bits/dimension, the loss is 3.8 dB.

In the coding literature, it is common to express the gain of shaping and coding schemes with respect to a cube. However, this paper instead expresses losses with respect to a sphere; this facilitates comparisons with sphere bounds.

II. LATTICES FOR COMMUNICATIONS

A. Lattices

An n-dimensional lattice Λ is defined by an n-by-n generator matrix G. The lattice consists of the discrete set of points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for which

$$\mathbf{x} = G\mathbf{b}, \tag{1}$$

where $\mathbf{b} = (b_1, \ldots, b_n)$ is from the set of all possible integer vectors, $b_i \in \mathbb{Z}$. Lattices are a linear subspace of the n-dimensional real space $\mathbb{R}^n$. The volume of the Voronoi region is $V(\Lambda)$, found as the determinant of G:

$$V(\Lambda) = |\det G|. \tag{2}$$

Shortest-distance quantization finds the lattice point $\widehat{\mathbf{x}} \in \Lambda$ which is closest to a given point $\mathbf{y}$, in the Euclidean distance sense:

$$\widehat{\mathbf{x}} = \arg\min_{\mathbf{x} \in \Lambda} ||\mathbf{y} - \mathbf{x}||^2. \tag{3}$$

Shortest-distance quantizers are difficult to implement. Instead, quantizers based upon belief-propagation decoding will be used, described in the next section; such a quantizer is denoted $Q_\Lambda(\mathbf{y})$.

B. Unconstrained Power Communications Using Lattices

Although impractical, the *unconstrained power* communications channel is a useful theoretical device for considering the coding aspects of lattices, separately for any shaping region. An arbitrary point of a coding lattice Λ_c is transmitted over the AWGN channel. Since there is no transmit power constraint, the system is constrained by the lattice density measured by $V(\Lambda_c)$.

An arbitrary point of Λ_c is transmitted by n uses of an AWGN channel with Gaussian variance σ^2. The SNR is defined as

$$\mathrm{SNR}_c = \frac{V(\Lambda_c)^{2/n}}{\sigma^2}. \tag{4}$$

Maximum-likelihood decoding can be realized by shortest-distance quantization, and a decoder error occurs if the quantizer output is different from the transmitted lattice point.

Somewhat analogous to the Shannon limit, there exist lattices such that the probability of decoder error becomes arbitrary small, if and only if

$$\mathrm{SNR}_c \geq 2\pi e, \tag{5}$$

provided the lattice dimension is allowed to grow without bound [6].

For lattices of finite dimension n, Tarokh, Vardy and Zeger developed a universal lower bound on the probability of decoder error by approximating Voronoi regions as spheres [7]. This sphere bound is:

$$P_e \geq e^{-z}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \frac{z^{n/2-1}}{(\frac{n}{2}-1)!}\right), \tag{6}$$

where

$$z = \frac{\Gamma(\frac{n}{2}+1)^{2/n}}{2\pi}\mathrm{SNR}_c, \tag{7}$$

with $\Gamma(\frac{n}{2}+1) = (n/2)!$ the Gamma function. There exist lattices for which this bound is tight, in dimensions as low as 16.

C. Constrained Power Communications using Lattice Codes

A lattice may also form the basis of a lattice code for use on a *constrained power* communications channel. A lattice code is the intersection of a shaping region B and a coding lattice Λ_c. Information is encoded to one of M levels per dimension, so that the code rate is $R = \log_2 M$ bits per dimension. The number of codepoints is M^n. Thus, $V(B) = M^n V(\Lambda_c)$, if

$V(B)$ is the volume of the shaping region. All codepoints are equally likely, so the average transmit power is given by:

$$P_{av} = \frac{1}{M^n}\sum_{\mathbf{x} \in \Lambda_c \cap B}||\mathbf{x}||^2. \tag{8}$$

We distinguish between two types of decoders for lattice codes. The *lattice code decoder* decodes to an element of the lattice code $\Lambda_c \cap B$. On the other hand, a *lattice decoder* ignores the shaping region, and decodes to an element of the lattice Λ_c. Thus it is possible that the lattice decoder will decode to a lattice point which is not a member of the codebook.

When lattice coding decoding is used, the Shannon limit $R \leq \frac{1}{2}\log_2\left(1 + \frac{P_{av}}{\sigma^2}\right)$ can be achieved [2] [3]. However, when lattice decoding only is used, then the channel capacity is strictly lower, and is given by, [8] [9], $R \leq \frac{1}{2}\log_2\left(\frac{P_{av}}{\sigma^2}\right)$. However, at high rates, the missing 1 is not significant and lattice decoding capacity approaches maximum-likelihood decoding.

Tarokh, et al. also developed a universal lower bound on the probability of error for finite-dimensional codes under lattice decoding (rather than lattice code decoding). This bound is also given by (6), evaluated instead with:

$$z = \left(\frac{n}{2}+1\right)\frac{1}{2^{2R}-1}\frac{P_{av}}{\sigma^2}. \tag{9}$$

This bound is also expected to be tight for lattice codes in dimensions of interest.

Once the lattice code is determined by selection of the coding lattice Λ_c and shaping region B, the average transmit power is fixed. Because lattices points are regularly distributed, the average power P_{av} of $\Lambda_c \cap B$ maybe be approximated by the average power of a probability distribution uniform over B [4]. This is called the continuous approximation. Under this assumption, the contribution of Λ_c and B to the average power can be separated as:

$$P_{av} \approx G(B)V(B)^{2/n}, \tag{10}$$

where,

$$G(B) = \frac{\int_B ||\mathbf{x}||^2 d\mathbf{x}}{nV(B)^{\frac{2}{n}+1}}. \tag{11}$$

is the normalized second moment of B. In particular, $V(B) = M^n V(\Lambda_c)$, so that once the codebook size M^n has been fixed, $V(B)^{2/n}$ depends only on the lattice Λ_c. On the other hand, the normalized second moment of the shaping region $G(B)$ depends only on the shape of B, but is not changed by scaling of B.

Over all possible shaping regions, spheres minimize the average transmit power. The normalized second moment of an n-dimensional sphere S_n is.

$$G(S_n) = \frac{\Gamma\left(\frac{n}{2}+1\right)^{2/n}}{\pi(n+2)}. \tag{12}$$

Definition The *shaping loss* $\gamma_s(B)$ of a shaping region B with respect to a sphere is:

$$\gamma_s(B) = \frac{G(B)}{G(S_n)} = \frac{\pi(n+2)G(B)}{\Gamma\left(\frac{n}{2}+1\right)^{2/n}} \quad (13)$$

The shaping loss is greater than or equal to 1.

D. LDLC Lattice Construction

A low-density lattice code is a dimension n lattice with a non-singular generator matrix G, for which $H = G^{-1}$ is sparse with constant row and column weight d. For a given $V = |\det G|$ and a parameter w, the inverse generator H is designed as follows. Let

$$h = [1, w, w, \ldots, w, 0, \ldots, 0] \quad (14)$$

be a row vector with a single one, followed by $d-1$ w's ($w \geq 0$), followed by $n-d$ zeros. The matrix H can be written as permutations π_i of h, followed by a random sign change S_i, followed by scaling by $k > 0$:

$$H = k \begin{bmatrix} S_1 \cdot \pi_1(h) \\ S_2 \cdot \pi_2(h) \\ \vdots \\ S_n \cdot \pi_n(h) \end{bmatrix} \quad (15)$$

such that the permutations result in H having exactly one 1 in each column, and exactly $d-1$ w's in each column. The sign-change matrix S_i is a square, diagonal matrix, where the diagonal entries are +1 or -1 with probability one-half. Then, k is selected to normalize the determinant to V:

$$k = V^{1/n} \left| \det \begin{bmatrix} S_1 \cdot \pi_1(h) \\ S_2 \cdot \pi_2(h) \\ \vdots \\ S_n \cdot \pi_n(h) \end{bmatrix} \right|^{-1/n}. \quad (16)$$

The above is a special case of the standard LDLC constructions, which are characterized by a parameter $\alpha \geq 0$. Belief-propagation decoding of LDLC lattices will converge exponentially fast if and only if $\alpha \leq 1$ [5, Theorem 1]. For the construction considered here, $\alpha = (d-1)w^2$, or,

$$w = \sqrt{\frac{\alpha}{(d-1)}}. \quad (17)$$

Thus, in this paper, LDLC lattice constructions characterized by the parameters (n, d, α, V) are considered.

III. PROPOSED CODE AND ENCODER

A. BP Quantization of LDLC Lattices

On the unconstrained power channel, LDLC lattices are well-suited for belief-propagation decoding. This decoding algorithm has some similarities to the decoding of low-density parity-check codes, but an important difference is that the decoder messages are functions, rather than real numbers (such as LLRs). This function may be quantized [5] or approximated by a mixture of Gaussians [10]. On the unconstrained power channel, the belief-propogation decoder usually converges, but
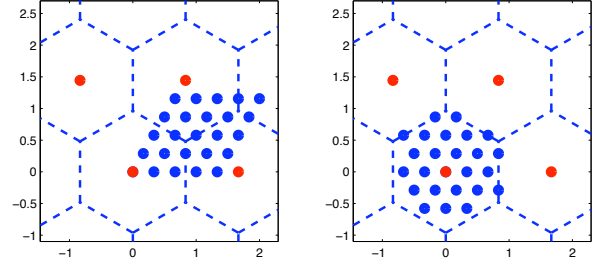

Fig. 1. (a) Encoding $G_c\mathbf{b}$ before shaping. (b) Lattice code, $G_c\mathbf{b} \mod \Lambda_s$, after shaping.

does not converge for quantization — that is, when the input is an arbitrary point in Euclidean space $\mathbb{R}^n$.

A suboptimal quantizer could be implemented by letting the decoder iterate for a fixed number of iterations I_{max}, generate a hard decision $\widehat{\mathbf{b}}$, and producing $\widehat{\mathbf{x}} = G\widehat{\mathbf{b}}$ as the quantizer output. However, the quantizer error can be improved by searching over combinations of the less reliable integer positions. A candidate list $\mathcal{B}$ is constructed, and for each $\mathbf{b} \in \mathcal{B}$, the lattice point closest to $\mathbf{y}$:

$$\widehat{\mathbf{x}} = \min_{\mathbf{b} \in \mathcal{B}} ||\mathbf{y} - G\mathbf{b}||^2, \quad (18)$$

is selected as the output $\widehat{\mathbf{x}} = Q_\Lambda(\mathbf{y})$.

The candidate list $\mathcal{B}$ is constructed as follows. After each iteration i, the decoder produces an estimated integer sequence $\mathbf{b}^{(i)}$. If the belief-propagation decoder does not converge, there are some integer positions which are unstable as iterations progress, that is, the estimated decisions change frequently (most positions are stable, however). For each position $t = 1, \ldots, n$, the set of integers observed over I_{max} iterations one or more times is U_t (with $U_t \subset \mathbb{Z}$). Associated with each element in U_t, is the frequency $f_{t,i}$, such that $\sum_{i=1}^{|U_t|} f_{t,i} = 1$. The maximum frequency is $m_t = \max_i f_{t,i}$. The list $m_1, \ldots, m_n$ is sorted, and the least Q reliable positions form an index set $\mathcal{Q}$. The set of candidate vectors $\mathcal{B}$ consists of integers vectors for which index positions $\mathcal{Q} = \{q_1, q_2, \ldots, q_Q\}$ are drawn from the set $U_{q_1} \times U_{q_2} \times \cdots \times U_{q_Q}$, and the values in the remaining $n - Q$ positions are fixed, with the same values as in $\mathbf{b}^{(I_{max})}$.

In practice, it was found that the cardinality of U_t, that is $|U_t|$, was usually no more than 2, that is, unstable positions oscillated between one of two possible solutions.

B. Lattice Code

For the lattice code under consideration, the shaping region is based upon the origin-centered Voronoi region of the shaping lattice. The shaping lattice Λ_s is generated by $G_s = MG_c$, and Λ_s is a sublattice of the coding lattice Λ_c. If shortest-distance quantization were feasible, then B would be equal the origin-centered Voronoi region. First this idealized code is described, and then a variation which allows for a sub-optimal quantizer is described.

Voronoi-Cell Bounding Region [11]. If shortest-distance quantization is possible, then the bounding region B is equal to the Voronoi region of the shaping lattice centered on the origin. The column vector $\mathbf{b} = (b_1, \ldots, b_n)$ is the information sequence, with $b_i \in \{0, 1, \cdots, M - 1\}$. First, encode the information to the lattice point $G_c\mathbf{b}$ and then:

$$\mathbf{x} = G_c\mathbf{b} - G_s\mathbf{c} \tag{19}$$

where $\mathbf{c}$ is a vector of integers such that $G_s\mathbf{c}$ is the element of the shaping lattice closest to $G_c\mathbf{b}$. This quantization operation may be expressed as $G_s\mathbf{c} = Q_{\Lambda_s}(G_c\mathbf{b})$, or equivalently,

$$\mathbf{x} = G_c\mathbf{b} \mod \Lambda_s, \tag{20}$$

so $\mathbf{x}$ is inside the origin-centered Voronoi region. This is illustrated in Fig. 1. Voronoi regions are not spheres, but may be sufficiently sphere-like to have good shaping gain.

Best-Effort Quantizer Shaping Region. Here, a shortest-distance quantizer is not required. Minimum-distance quantization is computationally difficult, but the belief-propogation quantizer which does not necessarily find the element of Λ_c closest to $G_c\mathbf{b}$, may be used instead. Let

$$G_s\mathbf{c} = Q_{\Lambda s}(G_c\mathbf{b}) \tag{21}$$

be such a quantizer for Λ_s.

This encoding scheme generates a code with M^n points which are a finite subset of Λ_c, and accordingly there exists a shaping region B such that $B \cap \Lambda_c$ generates this code.

If the transmitted sequence is correctly received, that is, $\widehat{\mathbf{x}} = G_c\mathbf{b} - G_s\mathbf{c}$, then the original data can be recovered by first observing that:

$$G_c\mathbf{b} - G_s\mathbf{c} = G_c(\mathbf{b} - M\mathbf{c}) \tag{22}$$

Then, the decoder find the information sequence $\mathbf{b}$ by applying the modulo M operation to each element of $(\mathbf{b} - M\mathbf{c})$, since:

$$b_i \equiv (b_i - Mc_i) \mod M. \tag{23}$$

The suboptimal belief-propagation quantizer can be used for code shaping; the penalty is losses due to poor shaping. Additionally, losses may also be attributed to the non-spherical nature of the shaping region.

C. Ineffective Shaping Approaches

Here, we consider two ineffective approaches for shaping a coding lattice Λ_c. This allows us to argue that the practical shaping scheme presented here is a suitable scheme.

First, a cubical power constraint could be enforced by finding the vector $G\mathbf{b}$, and component-wise transmitting each component modulo M. This guarantees that the vector lies inside an n-dimensional cube of volume M^n. However, the resulting codebook is not a subset of the coding lattice Λ_c, and thus cannot be readily decoded.

Second, if the shaping region B is selected to be an n-dimensional cube (or an n-dimensional sphere) of volume M^n centered on the origin, then $B \cap \Lambda_c$ will be a codebook which satisfies the cubical (spherical) power constraint. However,

there is no simple method to index the codebook points to an information sequence. Further, while the size of the codebook is approximately M^n, the exact size depends upon truncation effects at the boundary of the cube (sphere).

D. Numerical Evaluation of $G(B)$

Evaluation of the normalized second moment $G(B)$ is difficult, but can be estimated by Monte Carlo integration. Sample points uniformly distributed over the integration region are provided by the equally probable codepoints. Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ be N points uniformly distributed over B. Then,

$$\int_B ||\mathbf{x}||^2 d\mathbf{x} \approx \frac{V(B)}{N} \sum_{i=1}^{N} ||\mathbf{x}_i||^2. \tag{24}$$

Note that $V(B) = M^n$, and $V(\Lambda_c) = 1$ is assumed. Then,

$$G(B) \approx \frac{\sum_{i=1}^{N} ||\mathbf{x}_i||^2}{NnM^2}. \tag{25}$$

IV. CODE DESIGN

In this section, the design LDLC-based lattice codes for use on the power-constrained channel is described. The lattice dimension n and code rate $R = \log_2 M$ are fixed. For convenience, the density of the coding lattice is assumed to be 1, that is $V(\Lambda_c) = 1$. Since the shaping lattice Λ_s is a sublattice of Λ_c, specifically $G_s = MG_c$,

$$\det G_s = M^n \det G_c = M^n. \tag{26}$$

The coding lattice Λ_c is $(n, d, \alpha, 1)$ and the shaping lattice Λ_s is (n, d, α, M^n). Thus, the proposed lattice code has two design parameters, d and α.

In particular, our design technique follows the principle of separation of coding loss and shaping loss, and is illustrated in Fig. 2. The shaping loss is found using the Monte Carlo integration technique described earlier, and is shown on the lower y-axis of Fig 2-(b). The coding loss is found by comparing the transmit power of the LDLC lattice on the unconstrained power channel with the corresponding lattice sphere bound, (6) and (7), for a fixed probability of error, as shown in Fig. 2-(a). This loss, as a function of α, is illustrated in Fig. 2-(b), on the upper y-axis. From Fig. 2-(b), it can be seen that the coding loss decreases for increasing α, but the shaping loss increases. Thus, the code design should select the value of α which minimizes the sum of the coding loss and shaping loss. This sum is illustrated in Fig. 2-(c), and occurs around $\alpha = 0.6$ to 0.64.

The value $\alpha = 0.64$ was selected for rates corresponding to both $M = 4$ and $M = 8$. For the case of $M = 8$, the combined loss is about 3.65 dB. The resulting lattice code for $M = 8$ is then applied to the AWGN channel. The decoder is the standard belief-propagation decoder for the lattice Λ_c, resulting in a lattice decoder (rather than a lattice code decoder) The performance is characterized by the SNR loss of the lattice code relative to the corresponding sphere bound (6) and (9), at a fixed error rate. In simulations, the performance loss at error rate of 10^{-3} is about 3.6 dB,

Fig. 2. Design of LDLC lattice codes with $n = 100, d = 3$. (a) Coding loss for Λ_c, on unconstrained power channel. (b) Coding loss (from (a) at error rate 10^{-3}) and shaping loss (based on Monte Carlo integration; $Q = 10$ was used) versus lattice parameter α. (c) Sum of coding loss and shaping loss, that is, the gap in (b), and a minimum is observed around $\alpha = 0.6$.

which is consistent with the predicted value. For the case of $M = 4$ (rate 2 bits/dimension), the estimated loss is 3.5 dB (Fig. 2-(b)), but the loss observed in the simulation is 3.8 dB. The difference of 0.3 dB may be attributed to the continuous approximation, which is less accurate at lower rates.

V. DISCUSSION

This paper demonstrated an explicit scheme for using LDLC lattices to construct lattice codes for the AWGN channel. Even though a shortest-distance quantizer for the lattice was not feasible, it was shown that by using belief-propagation decoding to implement a sub-optimal quantizer, that reasonable shaping gain and coding gain could be obtained. Lattices were designed to trade-off shaping loss and coding loss to minimize the overall performance loss.

When LDLC lattices are used for coding alone, performance improves as the design parameter α approaches 1. However, for the problem of performing shaping and coding together, we found the best parameter to be $\alpha \approx 0.6$, which is noticeably distinct. Note however, that this choice of α depended not only on the lattice design, but on the performance of the quantizer as well. If the quantizer is changed, the ideal value of α may also change.

The shaping losses are close to the well-known shaping loss of 1.53 dB associated with cubic shaping. However, the coding lattice Λ_c cannot easily be shaped into an n-cube while remaining easy to decode, as was discussed in Section III-C. For similar reasons, using a spherical shaping region is difficult. So we argue that it is reasonable to accept moderate shaping losses in exchange for low coding losses provided by LDLC lattices, so long as the sum of the losses is reasonable.

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[11] J. H. Conway and N. J. A. Sloane, "A fast encoding method for lattice codes and quantizers," *IEEE Transactions on Information Theory*, vol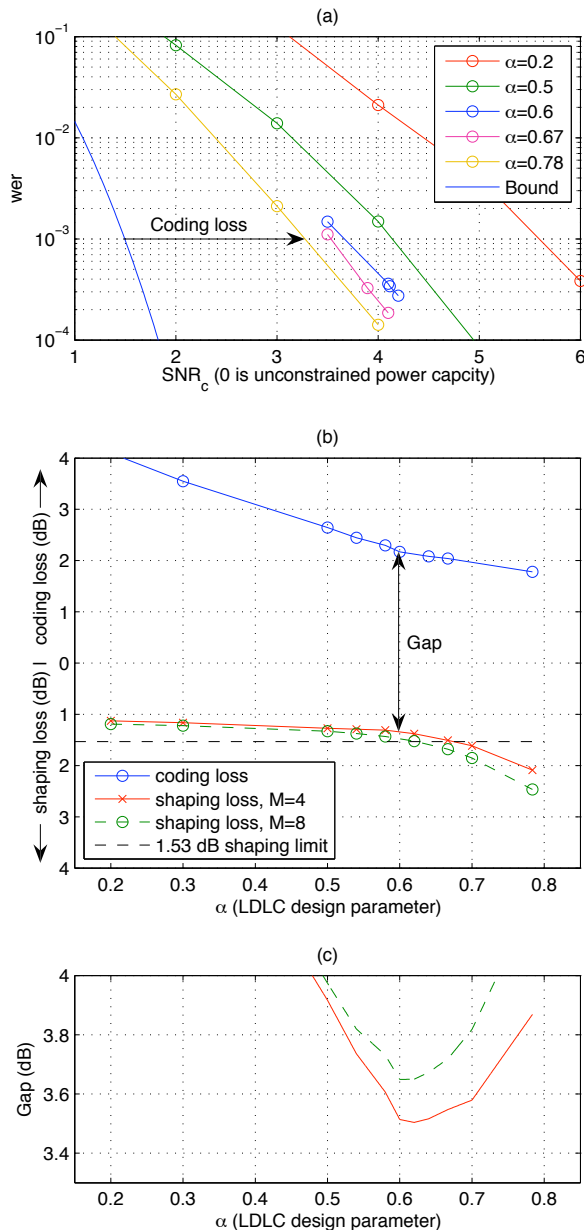. 29, pp. 820–824, November 1983.