Concatenation of a Discrete Memoryless Channel and a Quantizer

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***Abstract*— The concatenation of an arbitrary discrete memoryless channel with binary input followed by a quantizer is considered. For a restricted quantizer alphabet size, it is shown that the maximum of the mutual information between the channel input and the quantizer output can be found by dynamic programming. Numerical examples are given to illustrate the results. This problem is shown to be an example of concave programming.**

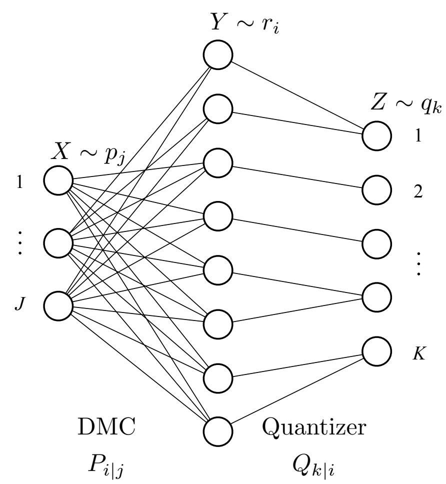

Fig. 1. A discrete memoryless channel followed by a quantizer. Given p_j, $P_{i|j}$ and K, find $Q_{k|i}$ which maximizes $I(X;Z)$.

I. Introduction

For a communication system consisting of a discrete memoryless channel (DMC) followed by a quantizer, the problem of maximizing the mutual information between the channel input and the quantizer output is considered. For binary input channels, this paper gives a method to find the quantizer which maximizes mutual information, among all discrete quantizers. This method uses a dynamic programming approach.

The DMC-quantizer problem is of interest in channel coding. In particular, many implementations of belief-propagation decoders for LDPC codes and turbo codes quantize messages. If the message alphabet is discrete, then the iterative decoder can be viewed as a sequence of discrete decoding mappings [1]. One technique for designing such decoding mappings is to find a quantizer for a discrete memoryless channel derived during density evolution [2]. Since these codes can communicate near channel capacity, a reasonable selection criteria for this quantizer is maximization of mutual information. Indeed, LDPC decoders have been designed considering mutual information [3]; these result in non-uniform message quantization. For both the AWGN and the binary symmetric channels, such decoders using only four bits per message can come quite close to unquantized performance. Conventional uniform quantization requires about six bits per message.

Designing efficient LDPC decoders is tied to the problem of quantizing channel outputs, and there are many prior studies on quantizing the channel output. The cutoff rate can easily be computed, and has been suggested as a criteria for designing channel quantizers [4, Sec. 6.2], and this was done in conjunction with decoding of turbo codes [5].

On the other hand, it is more difficult to maximize mutual information, and a limited amount of prior work has concentrated on continuous-to-discrete quantization of the binary-input AWGN channel. For the special case of three outputs, it is straightforward to select a single parameter which maximizes mutual information [6]. However, for a larger number of outputs, global optimization is difficult, although local optimization methods are better than uniform quantization [7].

This paper considers arbitrary DMCs, rather than specific continuous-output channels. The problem setup, summarized in Fig. 1, is given in detail in Section II. Then, Section III describes a dynamic programming method which can find a quantizer and the associated mutual information, when the channel input is binary. Under a suitable assumption, this method gives the globally optimal quantizer among all discrete quantizers. Section IV gives some numerical results. Section V has comments on the convexity of the mutual information, and it is shown that the problem is distinct from the well-known computation of channel capacity of Arimoto and Blahut [8] [9]. The paper concludes with discussion in Section VI.

II. Quantization of the Output of a DMC

The problem setup is as follows: a discrete memoryless channel has input X and output Y. This Y is then quantized to Z, as shown in Fig. 1. The sequence $X \rightarrow Y \rightarrow Z$ forms

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a Markov chain. For convenience, define:

$$\begin{aligned}
p_j &= \Pr(X=j), \\
&\qquad \text{with } j=1,\ldots,J, \\
r_i &= \Pr(Y=i), \\
&\qquad \text{with } i=1,\ldots,I, \\
q_k &= \Pr(Z=k), \\
&\qquad \text{with } k=1,\ldots,K, \\
P_{i|j} &= \Pr(Y=i|X=j), \\
Q_{k|i} &= \Pr(Z=k|Y=i), \text{ and} \\
T_{k|j} &= \Pr(Z=k|X=j) = \sum_i Q_{k|i}P_{i|j}.
\end{aligned}$$

The alphabet sizes of X, Y and Z are J, I and K, respectively. The quantizer is given by $Q_{k|i}$. This concatenation of the discrete memoryless channel P and quantizer Q can be regarded as a single discrete memoryless channel with input distribution p_j and transition probabilities $T_{k|j}$. It is clear that the maximum possible rate for this concatenation is:

$$C(P,Q) = I(X;Z), \tag{1}$$

where the mutual information is:

$$I(X;Z) = \sum_k \sum_j p_j T_{k|j} \log \frac{T_{k|j}}{\sum_{j'} p_{j'} T_{k|j'}}. \tag{2}$$

The problem statement is as follows: given an input distribution p_j, a channel $P_{i|j}$ and an integer K, find the maximum possible rate $C(P)$,

$$C(P) = \max_Q C(P,Q) = \max_Q I(X;Z),$$

and, find the quantizer Q^* which achieves this maximum,

$$Q^* = \arg\max_Q I(X;Z). \tag{3}$$

Here, the mutual information $I(X;Z)$ is given explicitly by

$$I(X;Z) = \sum_k \sum_j p_j \sum_i Q_{k|i} P_{i|j} \log \frac{\sum_{i'} Q_{k|i'} P_{i'|j}}{\sum_{j'} p_{j'} \sum_{i'} Q_{k|i'} P_{i'|j}}.$$

Clearly, if $K \geq I$, then a trivial quantizer which maps each channel output to a unique quantizer output will result in no loss of information and can achieve the maximum possible rate of $C(P) = I(X;Y)$.

Accordingly, the focus of this paper is on non-trivial quantizers for which $K < I$. In particular, note that the data processing inequality [10] gives,

$$I(X;Y) - I(X;Z) \geq 0. \tag{4}$$

An alternative perspective of this problem is to find a quantizer with fixed K that minimizes this difference. Also, with K levels allowed, it is not meaningful to consider fewer levels $K' < K$. If Z' denotes an optimal quantizer with K' outputs, then $\max I(X;Z) \geq \max I(X;Z')$ [11, Ch. 2]. That is, there can be no benefit to considering fewer than K quantizer levels.

III. Dynamic Programming for Binary Case

This section describes a method to find the optimal quantizer when the channel input is binary, that is $J = 2$. The method is a dynamic programming algorithm. Only discrete quantizers are considered; this is a reasonable restriction from an engineering perspective. Also, it is assumed that when the channel outputs are ordered according to log-likelihood ratios, that the optimal quantizer will only combine adjacent channel outputs.

A. Supporting Proposition

Begin by noting that for a deterministic quantizer $Q_{k|i}$, for any fixed i, there is exactly one value k^*, for which $Q_{k^*|i} = 1$, and for all other values of k, $Q_{k|i} = 0$. For a given k, let the set $\mathcal{A}_k$ contain the values i for which $Q_{k|i} = 1$. Then, the mutual information objective function can be written as:

$$I(X;Z) = \sum_k \sum_j p_j \sum_{i \in \mathcal{A}_k} P_{i|j} \log \frac{\sum_{i' \in \mathcal{A}_k} P_{i'|j}}{\sum_{j'} p_{j'} \sum_{i' \in \mathcal{A}_k} P_{i'|j'}}.$$

Next, assume that the indices of Y are ordered according to their log likelihood ratios, that is,

$$\log \frac{P_{i|1}}{P_{i|2}} < \log \frac{P_{i'|1}}{P_{i'|2}} \quad \text{if and only if} \quad i < i'. \tag{5}$$

Then, we make the following assertion, which is given without proof.

Proposition 1 For the quantizer which maximizes mutual information, each set $\mathcal{A}_k$ consists of adjacent index values. Thus, $\mathcal{A}_1$ is the set $\{1, 2, \ldots, a_1\}$, and $\mathcal{A}_2$ is the set $\{a_1 + 1, \ldots, a_2\}$, etc. and A_K is the set $\{a_{K-1} + 1, \ldots, I\}$ (assume $a_0 = 1$ and $a_K = I$). Each $\mathcal{A}_k$ has at least one element.

For an arbitrary set $\mathcal{A}$ the partial sum of the mutual information, $g(\mathcal{A})$, is defined as,

$$g(\mathcal{A}) = \sum_j p_j \sum_{i \in \mathcal{A}} P_{i|j} \log \frac{\sum_{i' \in \mathcal{A}_k} P_{i'|j}}{\sum_{j'} p_{j'} \sum_{i' \in \mathcal{A}_k} P_{i'|j'}}.$$

For notational convenience, $\mathcal{A}_k$ may be written as $a_{k-1} \to a_k$, and so the mutual information may be written as:

$$I(X;Z) = \sum_k g(a_{k-1} \to a_k).$$

B. Dynamic Programming

Dynamic programming can be used to find a quantizer. Under the adjacent-index proposition, it is sufficient to find the set boundaries $a_1, \ldots, a_{K-1}$.

The dynamic program iteratively computes partial sums of the mutual information. In particular, the dynamic program has a state value $S_k(a)$ which is the maximum partial sum of mutual information for the first k groups, and the state a represents the last index of group k, in particular,

$$S_k(a_k) = \max_{a_1, \cdots, a_{k-1}} \sum_{k'=1}^{k} g(a_{k'-1} \to a_{k'}). \tag{6}$$

The dynamic program computes the following recursively. For each $k = 1, \ldots, K-1$, compute the state metric for $a = a_k, a_{k+1}, \ldots, a_{I-K+k}$:

$$S_k(a) = \max_{a'} S_{k-1}(a') + g(a' \to a). \tag{7}$$

The recursion is initialized with $S_0(1) = 0$.

Finally, the dynamic program computes $C(P)$ as,

$$\begin{aligned} C(P) &= \max_{a_1, \cdots a_{K-1}} \sum_k g(a_{k-1} \to a_k), &(8)\\ &= S_K(I), &(9)\end{aligned}$$

where $S_K(I)$ is found by evaluating (7) with $k = K$ and $a = I$.

The quantizer values may be found by:

$$a_k = \arg\max_{a'} S_{k-1}(a') + g(a' \to a). \tag{10}$$

This recursion may be thought of being analogous to the Viterbi algorithm, where the group index k is time, $S_k(a)$ are the state metrics and $g(a' \to a)$ are the transition metrics. The quantizer value is analogous to the traceback operation.

Under Proposition 1, this dynamic program produces the optimal quantizer.

IV. Numerical Examples

This section gives some numerical examples to illustrate the results.

Symmetry plays a role in the decoding of LDPC codes. For both analytical and practical purposes, it is desirable that if an input distribution is symmetrical then the output distribution should also be symmetrical. However, the following is an example of an optimal quantizer which produces an output distribution which is not symmetrical.

The following definition of symmetry is is used. Assume a binary input X with $p_1 = p_2 = 1/2$, and let,

$$L(y) = \sum_i \delta\big(y - \log \frac{P_{i|2}}{P_{i|1}}\big), \tag{11}$$

where $\delta(\cdot)$ is the Dirac delta function. Then the distribution is symmetric if

$$L(y) = L(-y). \tag{12}$$

Example 1 Let a $J = 2, I = 6$ channel be given by the following, which produces a symmetrical distribution,

$$P = \begin{bmatrix} 0.001 & 0.01 & 0.02 & 0.04 & 0.2 & 0.729 \\ 0.729 & 0.2 & 0.04 & 0.02 & 0.01 & 0.001 \end{bmatrix},$$

which has $I(X;Y) = 0.8759$. When $K = 4$, two optimal quantizer are found:

$$Q^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

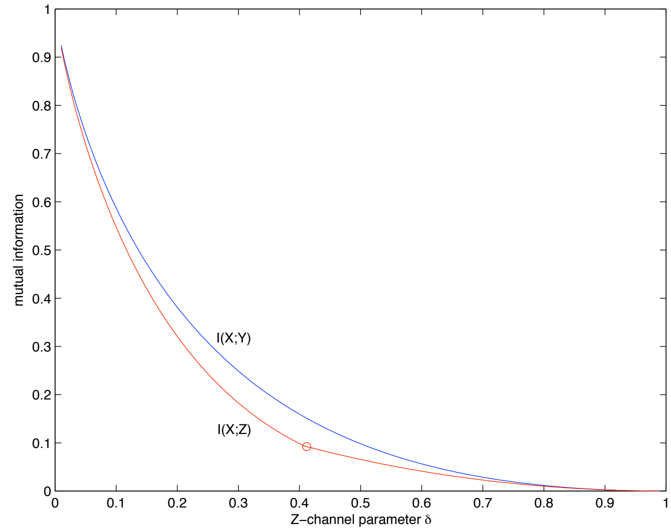

Fig. 2. Mutual information for a variation on the Z-channel.

Both quantizers result in $I(X;Z) = 0.8623$, a loss of 0.0136. As can be seen, each quantizer is asymmetric, and produces an asymmetric output distribution q_k. However, if the corresponding output distributions are $q_k^{(1)}$ and $q_k^{(2)}$, then $\frac{1}{2}q_k^{(1)} + \frac{1}{2}q_k^{(2)}$, *is* symmetric. Thus, by using time-sharing, a symmetric output distribution is produced. This is distinct from using a single quantizer $\frac{1}{2}Q^{(1)} + \frac{1}{2}Q^{(2)}$, which is suboptimal.

Also, when $K = 3$ or $K = 5$ a single optimal quantizer was found, which were both symmetrical.

Example 2. The binary Z-channel with error probability δ has the conditional output distribution,

$$\begin{bmatrix} 1-\delta & \delta \\ 0 & 1 \end{bmatrix}. \tag{14}$$

Consider two parallel Z-channels, with inputs X_1 and X_2 and corresponding output Y_1 and Y_2. Construct a derived channel which has as input the mod-2 sum $X = X_1 + X_2$ and output the pair (Y_1, Y_2). This model is of interest when considering the implementation of LDPC codes over the Z-channel. The conditional channel output distribution is:

$$P = \frac{1}{2}\begin{bmatrix} (1-\delta)^2 & \delta(1-\delta) & \delta(1-\delta) & 1+\delta^2 \\ 0 & 1-\delta & 1-\delta & 2\delta \end{bmatrix},$$

which has $I = 4$ corresponding to the four possible received sequences.

When quantizing to $K = 2$ levels, the optimal quantizer depends on δ. When $\delta < 0.407$, an optimal quantizer is:

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

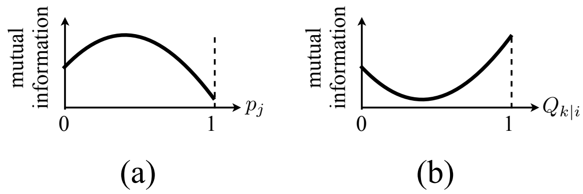

Fig. 3. Illustration of extrema problems. (a) Mutual information is concave in p_j; for the channel capacity problem, the maximum mutual information is found using Lagrange multipliers. (b) Mutual information is convex in $Q_{k|i}$; for the rate-distortion problem, the minimum is found using Lagrange multipliers but for the present problem, the maximum is at the boundary.

otherwise, an optimal quantizer is:

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \tag{16}$$

The mutual information vs. δ is shown in Fig. 2, and the crossover of 0.407 is shown by a circle.

V. Comments on the Convexity of $I(X;Z)$

Superficially, the information maximization problem, given by eqn. (3), appears similar to either the computation of the rate-distortion function or the computation of DMC capacity [10, Sec. 13.7], but it is distinct. This section discusses the distinctions.

The convexity of mutual information plays an important role in these extremum problems. Mutual information, as given in eqn. (2), is concave in p_j, but convex in $T_{k|j}$, which is illustrated in Fig. 3. Note that it is easy to show that $I(X;Z)$ is convex in $Q_{k|i}$ as well as $T_{k|j}$. The relationship $T_{k|j} = \sum_i Q_{k|i} P_{i|j}$ is an affine transform, and affine transformations do not change the convexity of functions [12].

The rate-distortion function of the quantization from Y to Z is found using a two-step alternating *minimization*. Mutual information is convex in $Q_{k|i}$, and Largrange multipliers are used in one of the two steps to find the $Q_{k|i}$ which minimizes mutual information, with a fixed output distribution. However, in the present problem, mutual information is *maximized* over $Q_{k|i}$. Further, the rate-distortion problem is subject to a distortion measure, which does not exist in the current problem. Instead, the quantizer restriction is through the alphabet size K.

In the computation of the capacity of the DMC from X to Y, the channel is fixed and the capacity-achieving input distribution p_j is found by a two-step alternating maximization. Although mutual information is concave in p_j, since the objective is to maximize mutual information, this is a convex programming problem. Again, Lagrange multipliers are used in one of the two steps to find the p_j which maximizes mutual information, for a fixed $\Pr(X|Y)$. However, for the present problem the input distribution p_j is fixed and quantizer $Q_{k|i}$ is found by an optimization method.

Now, the information maximization problem, given by eqn. (3) can be seen as maximization of a convex function, which is equivalent to minimization of a concave function. Such problems can be solved by concave programming. Concave programming (also known as concave minimization or concave optimization) is a class of mathematical programming problems which has the general form: $\min f(\mathbf{x})$, subject to $\mathbf{x} \in S$, where $S \subseteq \mathbb{R}^n$ is a feasible region and $f(\mathbf{x})$ is a concave function [13] [14]. As distinct from linear programming and convex programming, where the local minimum is the global minimum, in concave programming problems, there are multiple local minima. While there are numerous concave programming approaches, this type of global optimization problem is known to be NP-complete.

Also, another information extremum problem is the information bottleneck method, from the field of machine learning [15]. The problem setup is identical, using the same Markov chain $X \rightarrow Y \rightarrow Z$. However, the problem statement is distinct:

$$\min_Q I(Y;Z) - \beta I(X;Z), \tag{17}$$

where $\beta \geq 0$ is a Lagrange multiplier that sweeps a kind of rate-distortion curve. The information bottleneck method also uses alternating minimization. In the limit $\beta \rightarrow -\infty$, the two problems agree, but $\beta < 0$ is undefined, and these two problems are also distinct.

VI. Discussion

This paper considered the problem of quantizing the output of a DMC in order to maximize mutual information between the channel input and quantizer output. Besides theoretical interest, this problem has practical application, because the quantizer can be used to find non-uniform message quantizers for belief-propagation decoders.

For a binary-input channel, we gave an explicit method that finds the optimal quantizer among all deterministic quantizers. It was assumed that the optimal quantizer always combines adjacent channel outputs, when the log-likelihood ratios are sorted. This assumption clearly holds for DMC's derived from the binary-input AWGN channel. However, the extension to non-binary cases may be difficult, which lack a convenient ordering of log-likelihood ratios.

In the numerical results section, we were concerned about the symmetry of the quantizer. Symmetry is important both for analysis, for example, designing irregular LDPC codes, and for practical implementations which can exploit symmetry to simplify designs. It was found that optimal quantizers did not produce a symmetric output, even if the input was symmetrical.

We also showed that this problem is distinct from the computation of the rate-distortion function and computation of DMC capacity. These algorithms of Arimoto and Blahut are basically convex programming problems. However, the problem in this paper is a concave programming problem.

Unfortunately, concave programming is NP-complete in general, which also indicates that extension to higher-order input alphabets may be difficult.

REFERENCES

[1] T. J. Richardson and R. L. Urbanke, "The capacity of low-density parity check codes under message-passing decoding," *IEEE Transactions on Information Theory*, vol. 47, pp. 599–618, February 2001.

[2] B. Kurkoski, K. Yamaguchi, and K. Kobayashi, "Noise thresholds for discrete LDPC decoding mappings," in *Proceedings IEEE Global Telecommunications Conference*, (New Oreleans, LA, USA), pp. 1–5, IEEE, November-December 2008.

[3] J.-S. Lee and J. Thorpe, "Memory-efficient decoding of LDPC codes," *Information Theory, 2005. ISIT 2005. Proceedings. International Symposium on*, pp. 459–463, September 2005.

[4] J. M. Wozencraft and I. M. Jacobs, *Principles of Communication Engineering*. Prospect Heights, IL, USA: Waveland Press Inc., 1990. Originally Published 1965 by Wiley.

[5] G. Jeong and D. Hsia, "Optimal quantization for soft-decision turbo decoder," in *Proceedings IEEE Vehicular Technology Conference*, pp. 1620–1624, IEEE, September 1999.

[6] X. Ma, X. Zhang, H. Yu, and A. Kavcic, "Optimal quantization for soft-decision decoding revisited," in *International Symposium on Information Theory and its Applications*, (Xian, China), October 2002.

[7] A. D. Liveris and C. N. Georghiades, "On quantization of low-density parity-check coded channel measurements," in *Proceedings IEEE Global Telecommunications Conference*, (San Francisco, CA, USA), pp. 1649–1653, December 2003.

[8] S. Arimoto, "An algorithm for computing the capacity of arbitrary discrete memoryless channels," *IEEE Transactions on Information Theory*, vol. 18, pp. 14–20, January 1972.

[9] R. E. Blahut, "Computation of channel capacity and rate-distortion functions," *IEEE Transactions on Information Theory*, vol. 18, pp. 460–473, July 1972.

[10] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley, 1991.

[11] R. G. Gallager, *Information Theory and Reliable Communication*. New York, NY, USA: Wiley, 1968.

[12] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[13] R. Horst and P. M. Pardalos, eds., *Handbook of Global Optimization*. Kluwer Academic Publishers, 1995.

[14] R. Horst, P. M. Pardalos, and N. V. Thoai, *Introduction to Global Optimization*. Kluwer Academic Publishers, 1995.

[15] N. Tishby, F. C. Pereira, and W. Bialek, "The information bottleneck method," in *Proceedings 37th Annual Allerton Conference on Communication, Control, and Computing*, (Monticello, IL, USA), September 1999.