Low Complexity Construction of Low Density Lattice Codes Based on Array Codes

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***Abstract*—Recently a variety of lattices called low density lattices codes (LDLC) have been studied because they can be decoded efficiently using belief propagation, and can be seen as a Euclidean space codes analogue to low density parity check codes (LDPC). Previous LDLC lattice designs, like Latin square, are based on high-complexity computer search to eliminate 4-cycles. Array codes have been used to construct LDPC codes efficiently. This work describes the design of LDLC based on array codes. This construction is 4-cycle free and a systematic construction of the parity check matrix. For all cases considered, the LDLC based on array codes have a better symbol error rate performance than the latin square. For example, there is a 0.5 dB gain for dimension $n = 91$.**

I. Introduction

Lattice codes have potential to become an efficient and practical coding scheme for the AWGN channel and particularly for multi-terminal Gaussian networks because the encoder and the channel use the same real algebra, which is natural for the AWGN channel. Shannon showed that codes with very long random Gaussian codewords can achieve capacity [1], and now it is known that lattices codes can also achieve capacity [2][3][4]. Lattice codes can be seen as the Euclidean space analogue to linear finite-field codes.

Sommer et al. introduced high dimensional lattices called low density lattices codes (LDLC) [5], characterized by a sparse parity check matrix $H = G^{-1}$, where G is the lattice generator matrix. The advantage that H is sparse is to develop linear-time iterative decoding schemes using belief propagation (BP) [6][7]. The absence of cycles of length 4 has been shown to improve the performance of the BP algorithm [8].

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For practical applications a power restriction is needed. The codebook must be shaped in order to satisfy the power constraint, and make sure that only lattices points that lie inside of a specific region $B \subset \mathbb{R}^n$ are used. AWGN channel capacity can be reached by choosing a spherical shaping region [3]. A shaping algorithm using iterative LDLC lattice decoding was suggested in [9].

A desired property for a LDLC parity check matrix is to have a triangular structure in order to aid the shaping operation. Uchikawa et al. [10] showed that spatially-couple LDLC presents a quasi-triangular parity check matrix. In [11] a method to construct a triangular parity check matrix for LDLC lattices was given. First a Latin square LDLC lattice matrix was generated, and then by row and column permutations the triangular structure of the parity check matrix was generated. But this construction presents a high computational complexity because they searched for a matrix with no 4-cycles and as the dimension increases, the complexity grows exponentially. Also, this pseudo-random construction requires storage of all positions of the non-zero elements.

On the other hand, finite fields codes based on array code were introduced by Fan [12], and have been widely studied, having a deterministic and low computational complexity construction. Continued by Eleftheriou and Olcer [13], they introduced a triangular-structured parity check matrix based on array codes, adding benefits of easy encoding and maintaining a low computational complexity and deterministic construction.

Following these principles we propose in this work a construction of LDLC based on array codes. The proposed construction presents two benefits:

1) A low complexity construction to obtain a parity

check matrix with a triangular structure, sparse and 4-cycle free, that make it suitable for BP decoding and shaping.
2) Reduced encoding complexity because no storage of pseudo-random elements is needed.

Also we show by simulations the performance is 0.5dB better than Latin square construction [11] at $n = 91$, where n is the lattice dimension.

The structure of this paper is as follows. In Section II we describe the construction of array codes and modified array codes. Section III gives a definition of lattices and lattice codes and discusses LDLC lattices. Section IV describes the construction of the LDLC based on array codes and its properties. Section V presents simulation results of the performance of the LDLC based on array codes in terms to the gap from capacity. Finally Section VI summarizes the paper and gives some open questions related to this work.

II. Array Codes

Array codes [14] refer to a general class of algebraic error correcting codes defined on arrays for detecting and correcting burst of errors. If we see array codes as binary codes, their parity check matrix show sparseness, which can be used for decoding in an iterative manner [12].

In binary codes, an array code parity check matrix is defined by three parameters: a prime number p and two integers j and k such that $j \leq k \leq p$, where j and k represent the number of nonzero elements by row and column respectively, called the column and row weight of the parity check matrix.Then the parity check matrix of an array code is:

$$H = \begin{bmatrix} P^{(0\times0)} & P^{(1\times0)} & \dots & P^{((k-1)\times0)} \\ P^{(0\times1)} & P^{(1\times1)} & \dots & P^{(k-1)\times1)} \\ P^{(0\times2)} & P^{(1\times2)} & \dots & P^{((k-1)\times2)} \\ \vdots & \vdots & \ddots & \vdots \\ P^{(0\times(j-1))} & P^{(1\times(j-1))} & \dots & P^{((k-1)\times(j-1))} \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} I & I & \dots & I \\ I & P^{(1\times1)} & \dots & P^{((k-1)\times1)} \\ I & P^{(1\times2)} & \dots & P^{((k-1)\times2)} \\ \vdots & \vdots & \ddots & \vdots \\ I & P^{(1\times(j-1))} & \dots & P^{((k-1)\times(j-1))} \end{bmatrix}, \quad (2)$$

where P is a $p \times p$ permutation matrix and I is the $p \times p$ identity matrix. The matrix P^n represents an n left cyclic shift. For example, for $p = 5$,

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (3)$$

and

$$P^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad (4)$$

represents three left shift by P. By construction, H is 4-cycle free, since there are not two rows where a 1 is overlapping in two columns.

But a parity check matrix in triangular form is more desirable. We could use Gaussian elimination to obtain the desirable triangular form but it results in increasing the processing complexity, making this method unattractive.

Elefteriou and Olcer [13] defined a modified array code by cyclically shifting the rows of the H matrix in a blockwise manner. The number of cyclic shifts for each block row is such that it contains the identity matrix I along the diagonal. We define this new matrix H_s as:

$$\begin{bmatrix} I & I & I \cdots \cdots & I \\ P^{k-1} & I & P \cdots \cdots & P^{k-2} \\ P^{2(k-2)} & P^{2(k-1)} & I \cdots \cdots & P^{2(k-3)} \\ \vdots & \vdots & \vdots & \vdots \\ P^{(j-1)(k-j+1)} & P^{(j-1)(k-j+2)} & \cdots \cdots & P^{(j-1)(k-1)} \end{bmatrix} \quad (5)$$

The matrix H_s is 4-cycle free and has the same column and row weight as H.

To make this matrix in a triangular form, the lower triangular elements of H_s are set to zero, obtaining:

$$H_p = \begin{bmatrix} I\,I\,I \cdots & I & I & \cdots & I \\ 0\,I\,P \cdots & P^{(j-2)} & P^{(j-1)} & \cdots & P^{(k-2)} \\ 0\,0\,I \cdots & P^{(2(j-3))} & P^{(2\times(j-2))} & \cdots & P^{(2(k-3))} \\ \vdots\,\vdots\,\vdots & \vdots & \vdots & & \vdots \\ 0\,0\,0 \cdots & I & P^{(j-1)} & \cdots & P^{(j-1)(k-1)} \end{bmatrix}, \quad (6)$$

where 0 is the $p \times p$ null matrix.

III. Lattices and Lattice Codes

A lattice Λ is defined as the set of all integral linear combinations of basis vectors in $\mathbb{R}^n$. Let G be the generator matrix of a lattice, where the columns are the basis vectors, and a lattice point $x = Gb$ where $x \in \mathbb{R}$ and $b \in \mathbb{Z}$ are n-dimensional column vectors.

A n-dimensional lattice code is defined by a lattice Λ and a shaping region $B \subset \mathbb{R}^n$. The codewords are all lattice points that are inside of the shaping region B.

A. Low Density Lattice Codes

LDLC lattice is an n-dimensional lattice with a non-singular generator G, was introduced in [5], satisfying the condition that parity check matrix $H = G^{-1}$ is sparse. The row degree d_i, for $i = 1, 2 \ldots, n$, is the number of $nonzero$ elements in row i of H, and the column degree f_i is the number of nonzero elements in column i of H.

Note that binary LDPC codes are only defined by the locations of the nonzero elements. However, for LDLC we have to choose the values of the nonzero elements called the generator sequence so that $h_1 \geq h_2 \geq \cdots \geq h_d > 0$.

In [5], a condition for achieving the exponential convergence of the message variance in the belief propagation algorithm, was shown to be $\alpha \triangleq \frac{\sum_{i=2}^{d} h_i^2}{h_1^2} < 1$. Also a generator sequence was proposed to be of the form $\{1, \epsilon, \ldots, \epsilon\}$ where $\epsilon < 1$, where every variable will receive a single message with period 1 and $d-1$ messages with period $\frac{1}{\epsilon}$. A suitable choice is $\epsilon = \frac{1}{\sqrt{d}}$ which guarantees that $\alpha = \frac{d-1}{d} < 1$. As an example for $d = 5$, the generator sequence will be $\{1, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\}$. But how to choose the correct values for the generator sequence is an open question.

In the same way as LDPC codes the bipartite graph is defined: A graph with $check\ nodes$ on one side and $variable\ nodes$ on the other side. Each variable node corresponds to a single element of the codeword $x = Gb$. Each check node corresponds to a single element of b in the check equation of the form $\sum_k h_k x_{i_k} - integer$, where i_k denotes the location of the nonzero element at the appropriate column of H, h_k are the values of H at that location, and the $integer$ its unknown. There exists an edge that connects the variable node i and the check node j if and only if $H_{j,i} \neq 0$. This edge is assigned the value of $H_{j,i}$.

B. Desired Conditions for the parity check matrix H

In order to construct the parity check matrix H we need the following conditions:

1) Sparseness, the row and column weight is less or equal to d.
2) 4-cycle free, to improve the performance of the BP decoding algorithm.
3) Triangular structure, in order to apply shaping operation for practical applications.

The construction in the next section satisfies these conditions.

IV. Proposed Construction

In this section we describe the proposed lattice design for construction of LDLC parity check matrix based on modified array codes.

Since the LDLC parity check matrix is square, the parameters j, k are the same and denote the maximum row and column degree, that is $d = j = k$. Generating the modified array code as in equation (6) gives a sparse, 4-cycle free and triangular binary matrix. We modify the non zero elements in two ways:

1) Elements on the main diagonal are multiplied by $\frac{1}{c_i}$, where c_i is called the diagonal factor for $i = 1, \ldots, d$. This gives reliability to elements with low row degree.
2) Off-diagonal elements are modified to be elements of the generator sequence $h_2 \geq \cdots \geq h_d > 0$ with random sign, as with [5].

A. Reliability for low degree message

Having a lower triangular structure, it is evident that the rows and columns do not all contain d nonzero elements. The codeword components whose row degree is low are less protected. For example, a row with only one non-zero element is uncoded, since it only participates in a single check equation. For this reason, the information integers that are less protected should have a smaller amount of information, for example belong to a smaller constellation, as was observed by Sommer et al. [11].

To achieve the same effect, in the proposed construction, the elements of the main diagonal, with similar column weight are scaled by a factor $\frac{1}{c_i}$ for $i = 1, 2 \ldots, d$, where $c_1 > c_2 > \cdots > c_d$, $c_i \in \mathbb{Z}$. The c_i values are called diagonal factors, and modify the parity check matrix as:

$$\begin{pmatrix} \frac{h_1}{c_1}I & \frac{h_2}{c_1}I & \frac{h_3}{c_1}I & \cdots & \frac{h_d}{c_1}I \\ & \frac{h_1}{c_2}I & \frac{h_2}{c_2}P^1 & \cdots & \frac{h_{d-1}}{c_2}P^{d-1} \\ & & \frac{h_1}{c_3}I & \vdots & \frac{h_{d-2}}{c_3}P^{d-2} \\ & \mathbf{0} & & \ddots & \vdots \\ & & & & \frac{h_1}{c_d}I \end{pmatrix} \tag{7}$$

$$
\begin{bmatrix}
\frac{1}{8} & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{8} & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & \frac{1}{16} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Fig. 1. Parity check matrix of LDLC lattice based on array codes, constructed with $d = 4$, $p = 5$ and the generator sequence $h = \{1, 0.5, 0.5, 0.5, \}$. An the elements with low column degree were multiplying by $c_1 = 8$, $c_2 = 4$, $c_3 = 2$. and $c_4 = 1$.

We can see an example in Figure 1, where a LDLC based on array codes parity check matrix is given with $d = 4$, $p = 5$, generator sequence $h = \{1, 0.5, 0.5, 0.5\}$, and diagonal factors $c = \{8, 4, 2, 1\}$. It has been proven that there no exist a length 4 cycles in array codes, the LDLC lattices parity check matrix is also free of 4 cycles.

B. *LDLC Based on Array Codes Construction Algorithm*

Here we describe the construction in an algorithm form.

1) Input:
   - Prime number p.
   - Maximum number of nonzero elements d
   - Generator sequence $h_1 \geq \cdots \geq h_d > 0$.
   - Diagonal factors $c_1 > c_2 > \cdots > c_d$.
2) Construct the $dp \times dp$ modified array code parity check matrix H as matrix (6) , using $j = k = d$.
3) For each diagonal block i multiply by $\frac{1}{c_i}$.
4) Change the off-diagonal elements, with the elements in the generator sequence h with random sign.
5) Output : A $dp \times dp$ parity check matrix H, which is lower triangular, sparse and 4-cycle free.

V. Results

A LDLC lattice based on array codes was simulated for the AWGN channel. The proposed design parity check matrix H was generated using the parameters shown in Table I. In addition the diagonal factors $c = \{2^6, 2^5, 2^4, 2^3, 2^2, 2^1, 1\}$ (for $n = 91$, $n = 49$) and $c = \{2^3, 2^2, 2^1, 1\}$ (for $n = 20$) were used. The parity check matrix was further normalized such that $\sqrt[n]{\det(H)} = 1$ for a fair comparison.

A comparison is made with LDLC constructed as shown in [11]. Decoding is performed using the BP algorithm. And a different constellation is used, depends on the row degree. If we could use a constellation size of 8 for all integers, the information rate was 3 bits/integer. Practically, the information rate will be lower, due the decrease in the constellation size as shown in Table II.

In all simulations the all-zero codeword was used, AWGN with variance σ^2 was added to each codeword. Approaching channel capacity is equivalent to $\sigma^2 \rightarrow \frac{1}{2\pi e}$ [15]. The performance is measured in symbol error rate (SER), versus the distance to the noise variance σ^2 from capacity in dB. In Figure 2 we observe a gain of 0.5 dB close to channel capacity for long lattice dimension $n = 91$.

Lattice Dim. $n = dp$	Prime number p	Max degree d	Generator Sequence h	Effective Rate bits/integer
91	13	7	$\{1, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\}$	2.19
49	7	7	$\{1, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\}$	2.14
20	5	4	$\{1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$	1.5

TABLE I. SIMULATION PARAMETER FOR CONSTRUCTING A LOW DENSITY LATTICE CODE BASED ON ARRAY CODES

Column Degree	Constelation Size
1	2
2	2
3	4
4	4
5	8
6	8
7	8

TABLE II. ROW/COLUMN DEGREE AND CONSTELLATION SIZE

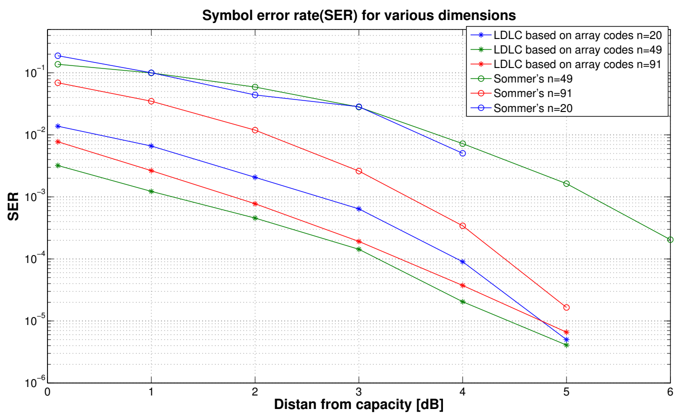

Fig. 2. Simulation results for various lattice dimensions

VI. CONCLUSION

LDLC lattices are close to optimal, practical lattice codes. However methods to construct these codes present a high computational complexity. In this paper we show a method to construct a $dp \times dp$ LDLC lattice parity check matrix based on array codes, which can be defined by 4 parameters: the maximum degree d, a prime number p, the diagonal factors c_i and the value of the non zero elements (generator sequence) h_i. The LDLC lattice based on array codes parity check matrix is sparse, lower triangular and 4-cycle free by construction.

The 4-cycle free property, by construction, eliminates tedious computations, and having a lower triangular structure is an important property that aids perform shaping, for power constrained AWGN channel in practical use. High dimensional LDLC lattices based on array codes, this implies the parameter p must be large.

How to choose the most suitable values of generator sequence h_i and the diagonal factor c_i in order to increase the reliability of those positions which are less protected and to maximize the coding gain is still an open problem.

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