Shaping Low-Density Lattice Codes Using Voronoi Integers

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Nested Lattice Codes Achieve Capacity

Lattice codes can achieve the capacity of AWGN channel [Erez and Zamir '04]

Nested lattice codes: $\Lambda / M\Lambda$

Want Λ which is simultaneously good for coding and shaping

Other *information theoretic* results using lattices:

- Lattices for relay channel e.g. [Song-Devroye '13]
- Two-way (Bidirectional) relay channel e.g. [Wilson et al.]
- Compute-forward relaying [Nazer-Gastpar '11]

How to move from information theory to practical lattice codes?

Capacity-Approaching Lattice Constructions

Recent high-dimension lattice constructions approach capacity

- Construction A with LDPC codes
- Construction D with turbo codes, spatially coupled LDPC
- Lattices based on polar codes
- Low-Density Lattice Codes [Sommer et al. 2008]

Common claim: within few tenth of dB of **unconstrained capacity:**

$$\frac{V(\Lambda)^{2/n}}{\sigma^2} \geq 2\pi e$$

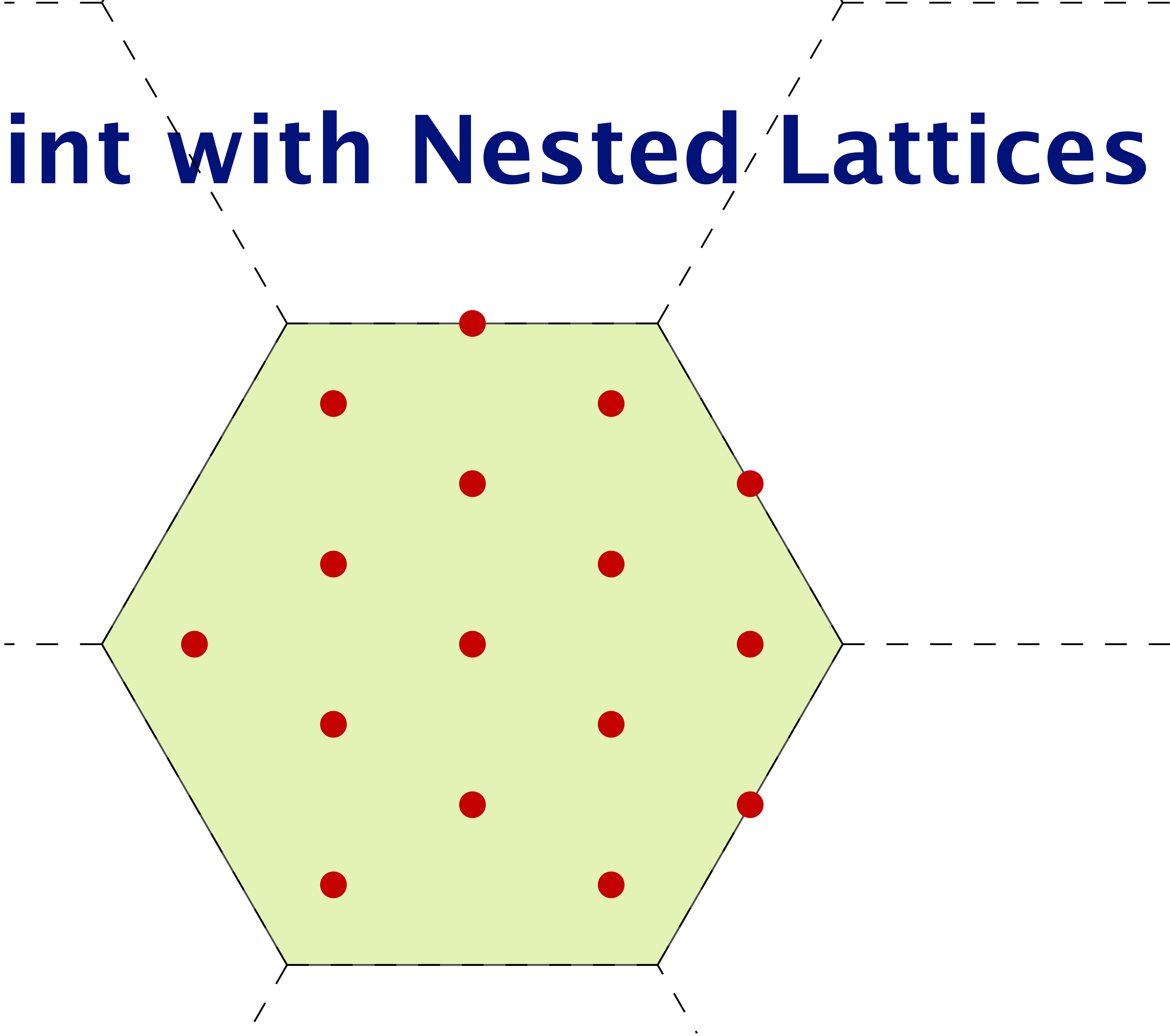
No assumption about the channel power constraint.

Satisfy Power Constraint with Nested Lattices

Good for correcting errors

$$\Lambda_{\mathrm{c}} / \Lambda_{\mathrm{s}}$$

Good for quantization
(satisfy the power constraint)



$$\mathbf{x} \bmod \Lambda_{\mathrm{s}} = \mathbf{x} - Q_{\Lambda_{\mathrm{s}}}(\mathbf{x})$$ high complexity!

1.53 dB Shaping Gain of Sphere over Cube

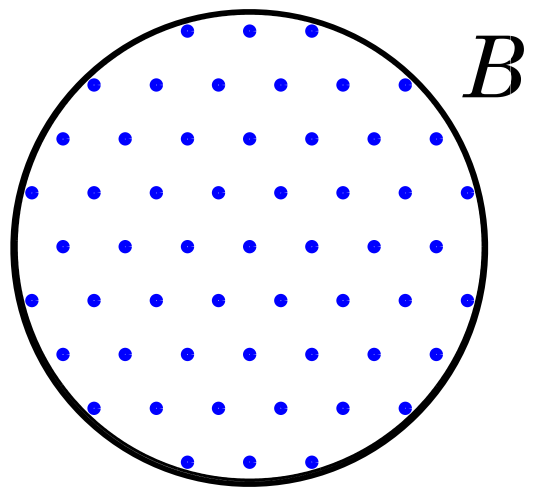
Separate lattice Λ and shaping region B contribution to signal power:

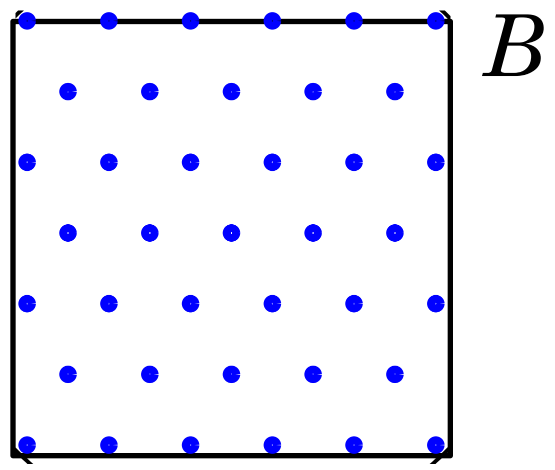
$$\text{Average Power} \quad \approx \quad \underbrace{\frac{\int_B ||\mathbf{x}||^2 d\mathbf{x}}{nV(B)^{\frac{2}{n}+1}}}_{G(B)} \cdot M^n V(\Lambda)$$

Depends only on *shape* of B
(normalized second moment)

Depends only
on coding lattice $\boldsymbol{\Lambda}$

Shaping Gain



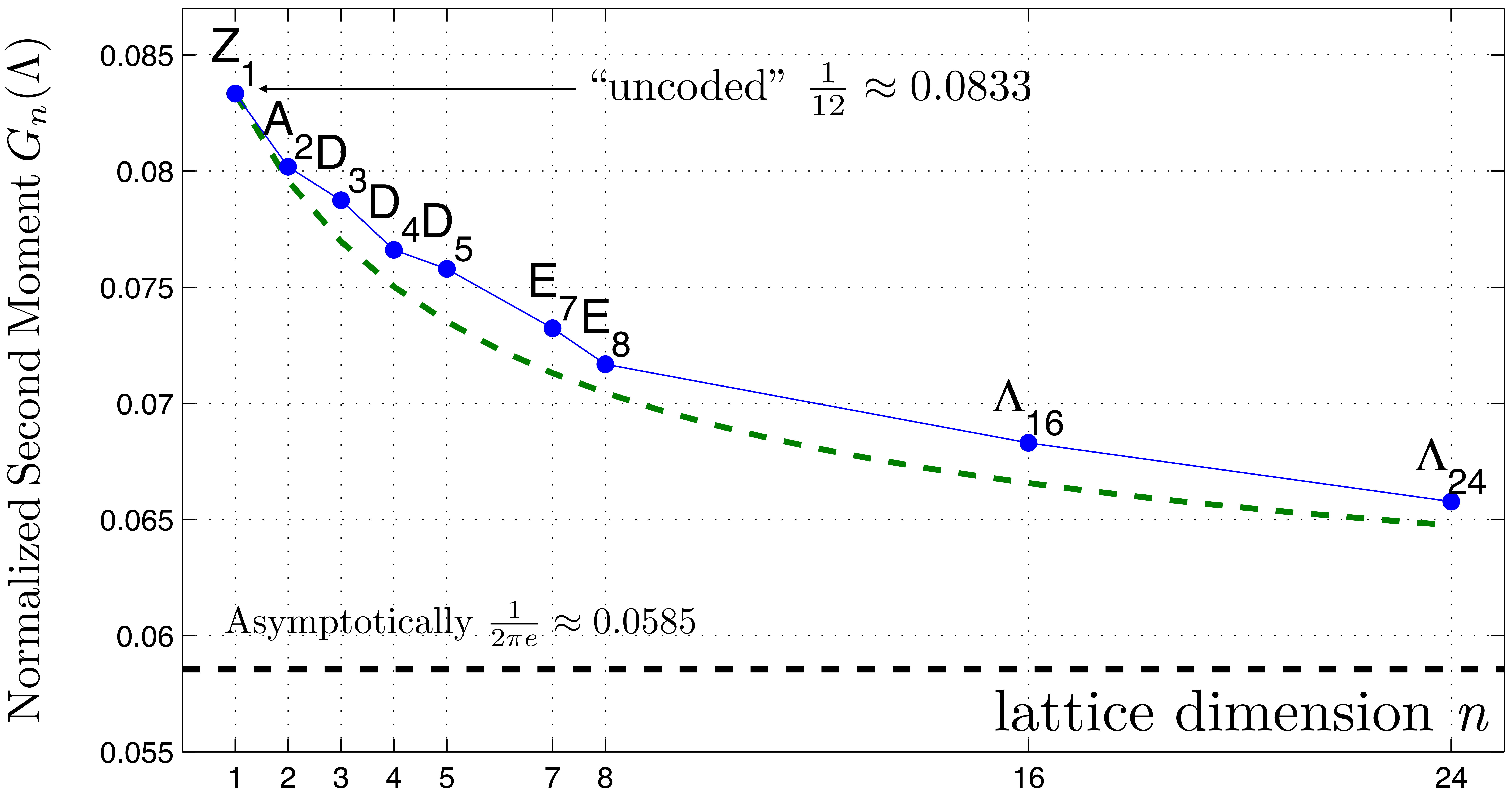


$$\lim_{n\to\infty} \frac{G(\text{cube})}{G(n\text{-sphere})} = \frac{\pi e}{6}$$
$$= 1.53 \text{ dB}$$

$$\lim_{n\to\infty} G(n\text{-sphere}) = \frac{1}{2\pi e}$$

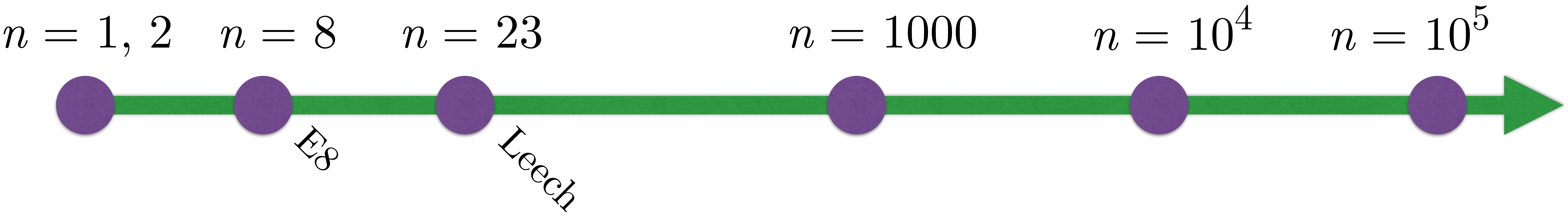
$$G(\text{cube}) = \frac{1}{12}$$

$G_n(\Lambda)$ for Well-Known Lattices



Satisfy Power Constraint with Nested Lattices

Λ_{c} and Λ_{s} both have dimension n



small n Well-known lattices

- Weak coding gain
- efficient shaping algorithms
- Good shaping gain (0.65~1.0 dB)

Large n BP-based lattices

- Strong coding gain
- Inefficient shaping algorithms
- Uncertain coding gains:
  two cases: 0.4 dB shaping gain

It Would Be Great If…

Find a construction that:

- Has the capacity-approaching coding gain high-dimension lattices
- Has the shaping gains and implementation complexity of a well-known lattice like E8.

Must overcome the problem of mismatch in dimensions

Outline

Key result:

- a lattice construction technique for shaping LDLC lattices

Elements of the technique:

1. "Voronoi Integers" $\mathbb{Z}^m/\Lambda_s$ Shape integers using small-dimension lattices
2. Systematic lattice encoding: lattice point is nearby corresponding integer

Results

Full 0.65 dB shaping gain of the E8 lattice. (2.1 dB from 1/2 log(SNR+1))

Competing nested LDLCs obtained only 0.4 dB, using higher complexity

First, review 1.53 shaping gain result and LDLC lattices

Low-Density Lattice Codes

LDLC lattices introduced by Sommer, Shalvi and Feder [IT 2008]

- LDLC have a sparse inverse generator matrix H
- Gaussian Belief-propagation decoding
- High dimension, $n = 100, 1000, 10000, 100000$
- Come within 0.6 dB of unconstrained capacity

LDLCs for the power-constrained channel [Sommer et al ITW 2009]

- H matrix in triangular form, use M algorithm for quantization
- Obtained 0.4 dB gain over hypercube (out of 1.53 dB)

LDLC "Latin Square" Construction

Inverse generator $H = G^{-1}$ has constant row and column weight d.

Latin square: each row/column $\{h_1, h_2, \ldots, h_d\}$ with random $\pm$, $h_1 \geq h_2 \geq \ldots \geq h_d$

- Choose $h_1 = 1$

  (forces determinant to be 1)
- Random sign changes
- $d = 7$ gives good performance
- BP convergence condition:

$$\frac{\sum_{i=2}^{d} h_i^2}{h_1^2} \leq 1$$

Example: $\{1, 1/2, 1/3\}$

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 & 1 & 0 & 0 & -1/3 \\ 0 & -1 & 0 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & -1 & 1/3 & 0 & -1/2 & 0 & 0 \\ 1/3 & 0 & -1/2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1/2 & 1/3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1 & 0 & -1/2 \\ -1 & 0 & 0 & 0 & 1/2 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & -1/2 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Nested Lattice Codes With LDLCs

Sommer [ITW 2009]: Triangular construction

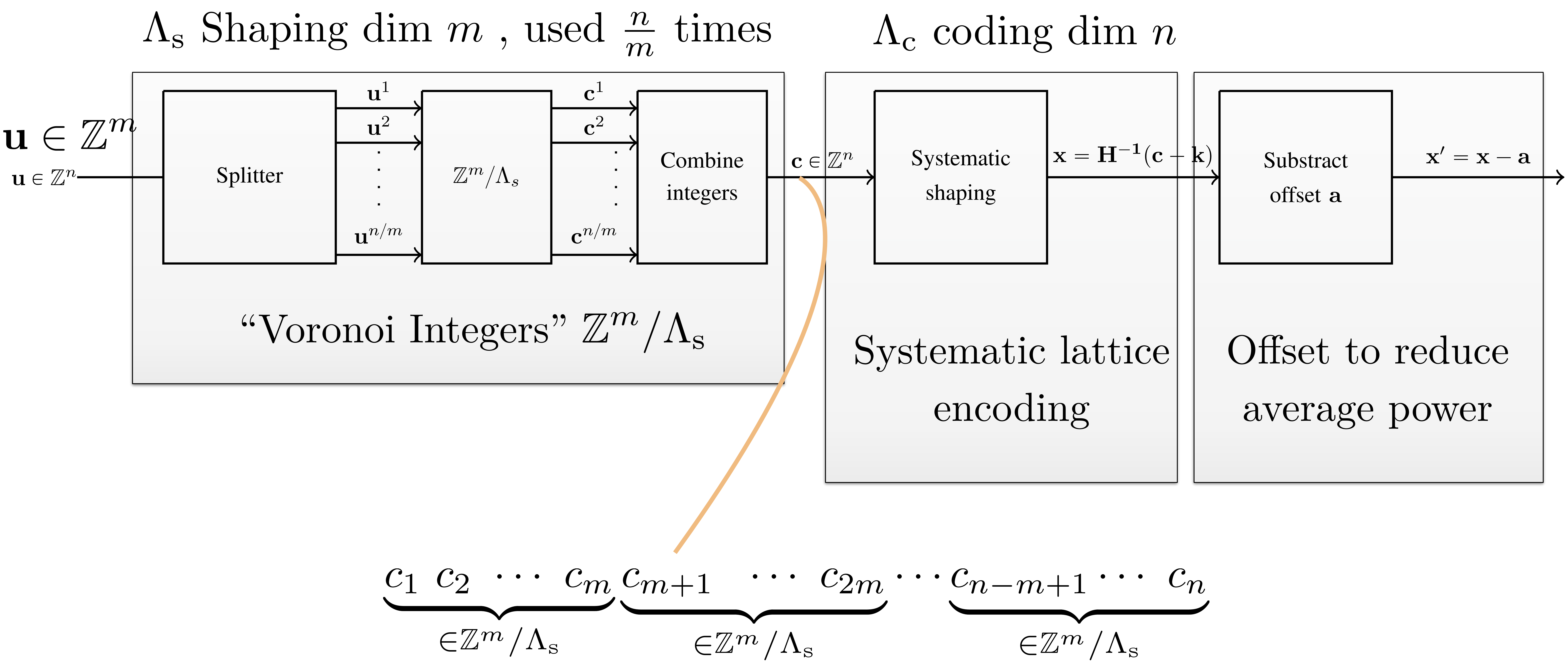
- Construct $\Lambda/M\Lambda$ using lattice quantization
- dimension $n = 10{,}000$
- Put 1's on main diagonal, make triangular
- "90% Latin square" weight d: $\{h_1, h_2, \ldots, h_d\}$
- Quantization/shaping using M-Algorithm

  Complexity is $O(ndM)$, but M is large
- Shaping gain of 0.4 dB over hypercube

  Modest shaping gain for high complexity

$$\begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 1.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.7 & 1.0 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.7 & 1.0 & 0 & 0 & 0 \\ 0 & -0.7 & 0 & 0.5 & 0 & 1.0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0.7 & 0 & 1.0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0.7 & 0 & 1.0 \end{pmatrix}$$

Proposed Construction

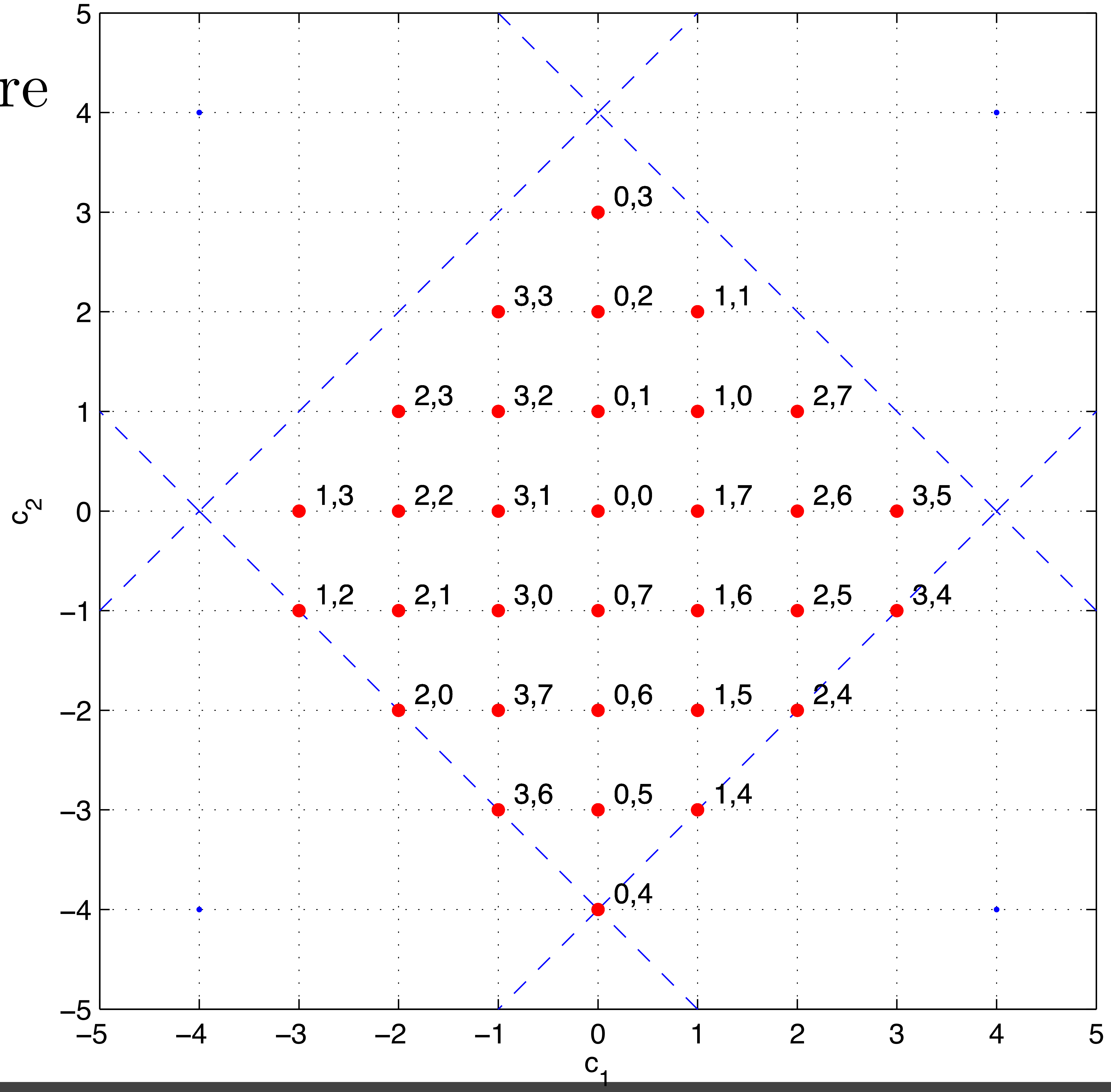


“Voronoi Integers” $\mathbb{Z}^m/\Lambda_{\text{shape}}$

Under systematic shaping, if the integers are “shaped,” then lattice code will be shaped.

L is a small-dimensional lattice quantization, i.e. shaping, is easy

Define “Voronoi Integers” $\mathbb{Z}^m/\Lambda_{\text{shape}}$ set of integers inside fundamental region



Voronoi Integers Example

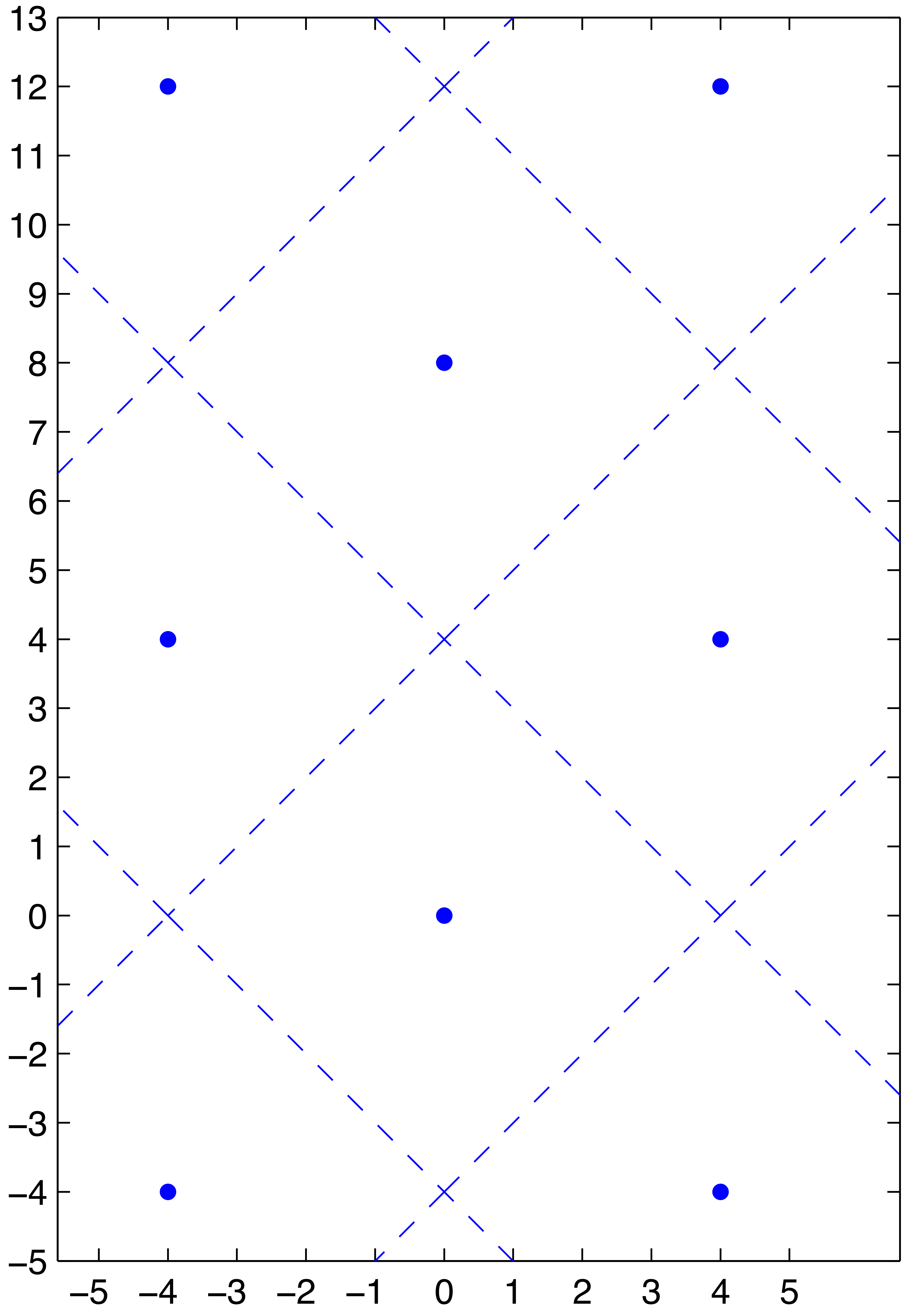
Use $4D_2$ lattice for shaping.

$$D_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$4D_2 = \begin{bmatrix} 4 & 0 \\ 4 & 8 \end{bmatrix}$$

Note that $\det 4D_2 = 32$ so the rate is:

$$R = \frac{1}{n} \log_2 \frac{|\det G|}{1} = 2.5 \text{ bits/dim}$$



Voronoi Integers Example

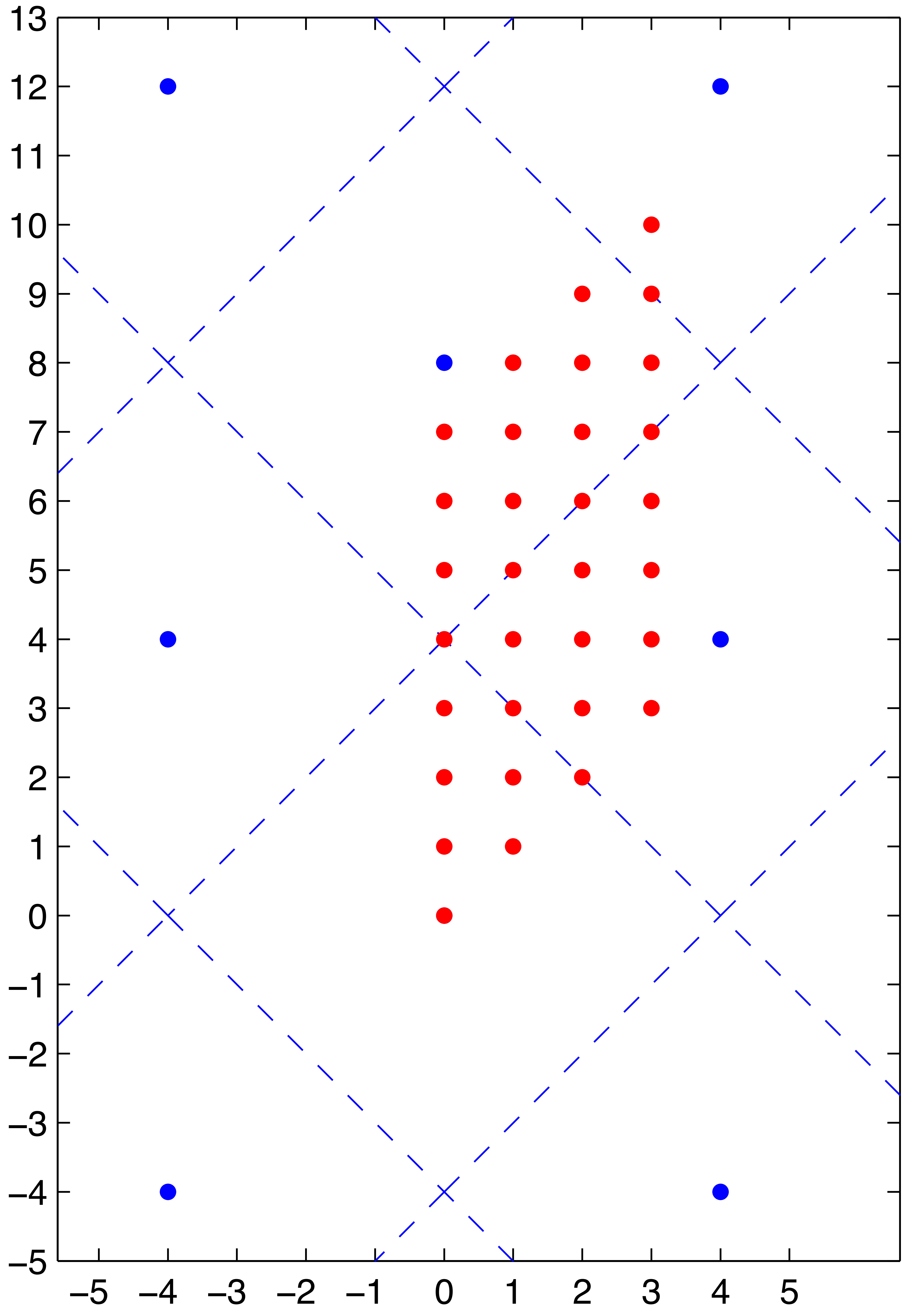
Map integers:

$$u_1 \in \{0, 1, 2, 3\}$$
$$u_2 \in \{0, 1, 2, 3, 4, 5, 6, 7\}$$

to points $\mathbf{c}$:

$$G \cdot \left(\frac{u_1}{g_{11}}, \frac{u_2}{g_{22}}\right),$$

Where G is generator matrix and g_{ii} are diagonal coefficients.



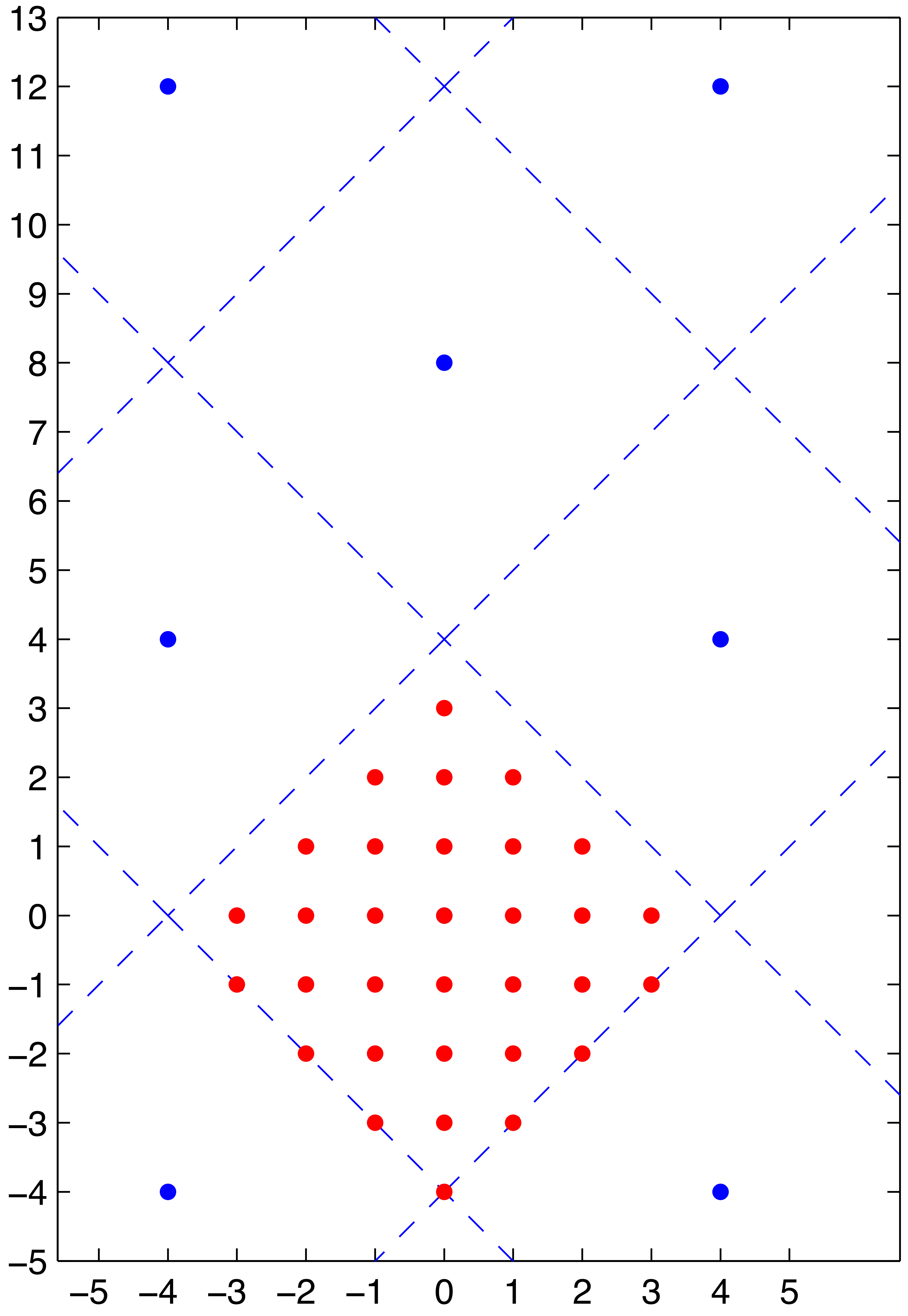
Voronoi Integers Example

Encode to $\mathbf{c} \in \mathbb{Z}^m / \Lambda_{\text{shape}}$ as:

$$\mathbf{c} = \mathbf{d} - Q_\Lambda(\mathbf{d})$$

Generalizations:

1. G is lower triangular, diagonal positive integers
2. Each column j, g_{ij}/g_{jj} is an integer
3. Use E_8 lattice. Has 0.65 dB shaping gain and efficient quantization.



Systematic Lattice Encoding

Encode integers $\mathbf{c}$ to lattice point $\mathbf{x}$ such that:

$$\mathbf{c} = \mathrm{round}(\mathbf{x}) \tag{1}$$

That is, $|x_i - c_i| \leq \frac{1}{2}$

Normal Encoding:

$$\mathbf{x} = G\mathbf{c} \tag{2}$$

Systematic lattice encoding. Find $\mathbf{k}$:

$$\mathbf{x} = G(\mathbf{c} - \mathbf{k}) \tag{3}$$

such that $\mathbf{c} = \mathrm{round}(\mathbf{x})$ holds.

Requirement

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & e & \ddots & 0 \\ c & d & & 1 \end{bmatrix}$$

Triangular H with 1's on diagonal

Systematic Lattice Encoding

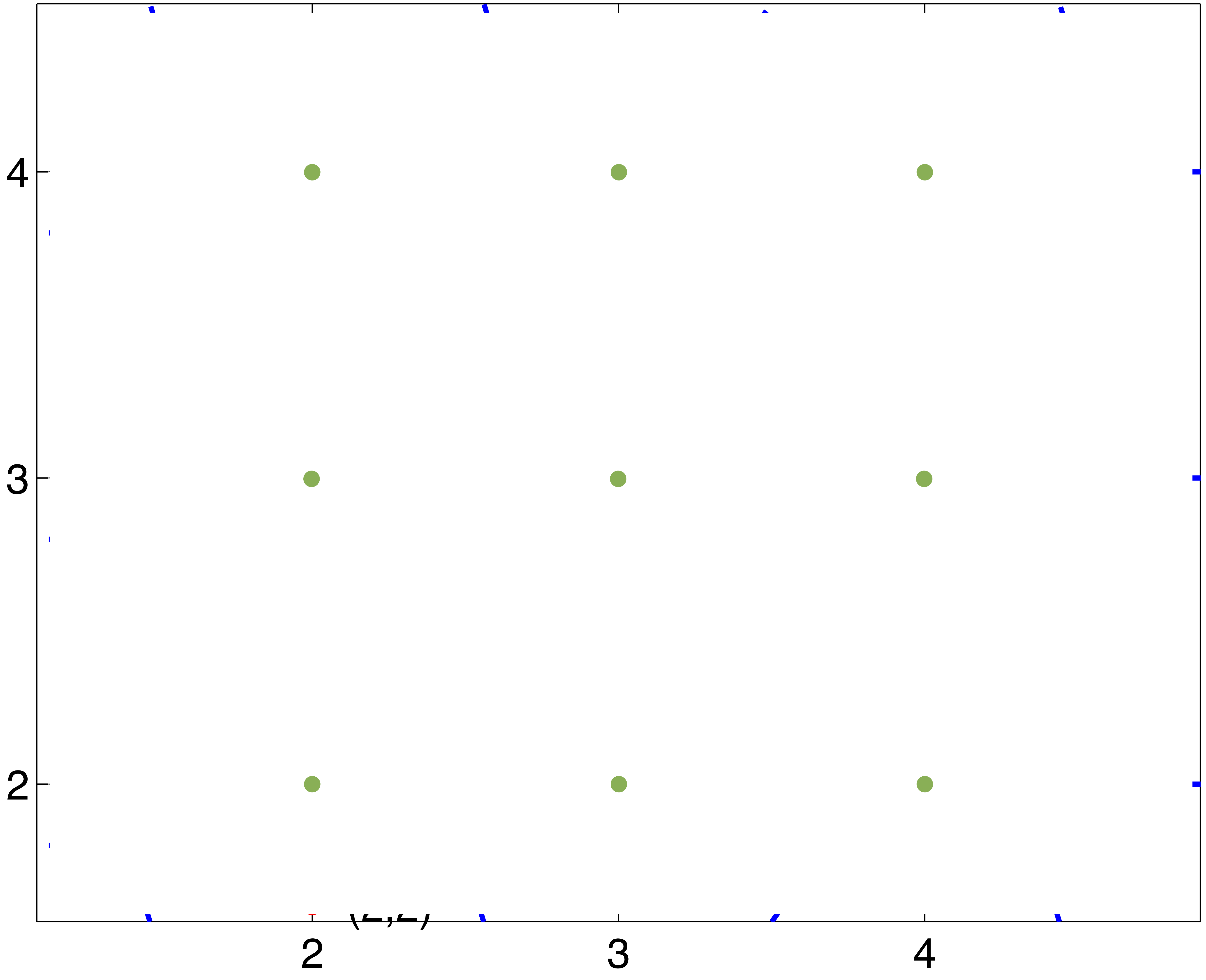
Example using $H = \begin{bmatrix} 1 & 0 \\ -0.3 & 1 \end{bmatrix}$

Recall: $\mathbf{c} = \mathrm{round}(\mathbf{x})$

Note Voronoi volume $\det(H) = 1$

and the integer grid also has vol. 1

No "gaps"



Systematic Lattice Encoding

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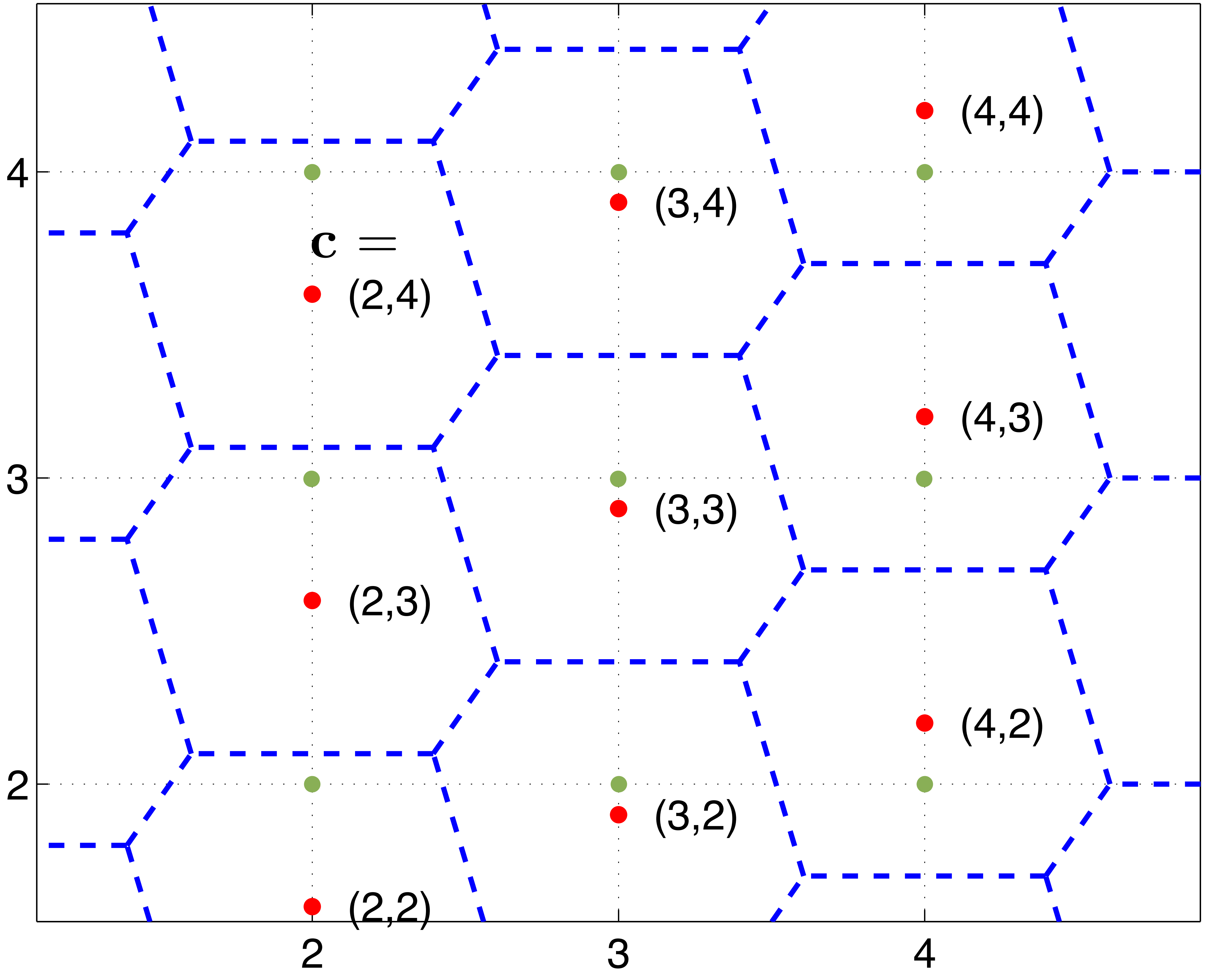


Find integers $\mathbf{k} = (k_1, k_2, \ldots, k_n)^t$ such that:

$$\mathbf{H}\mathbf{x} = \mathbf{c} - \mathbf{k} \text{ and} \tag{1}$$

$$|x_i - c_i| \leq \frac{1}{2} \text{ for all } i = 1, \ldots, n. \tag{2}$$

Note that line i of (1) is given by:

$$x_i + \sum_{j=1}^{i-1} H_{i,j} x_j = c_i - k_i \tag{3}$$

Triangular structured $\mathbf{H}$. Find k_i and x_i recursively. $x_1 = c_1$ and $k_1 = 0$. Continuing:

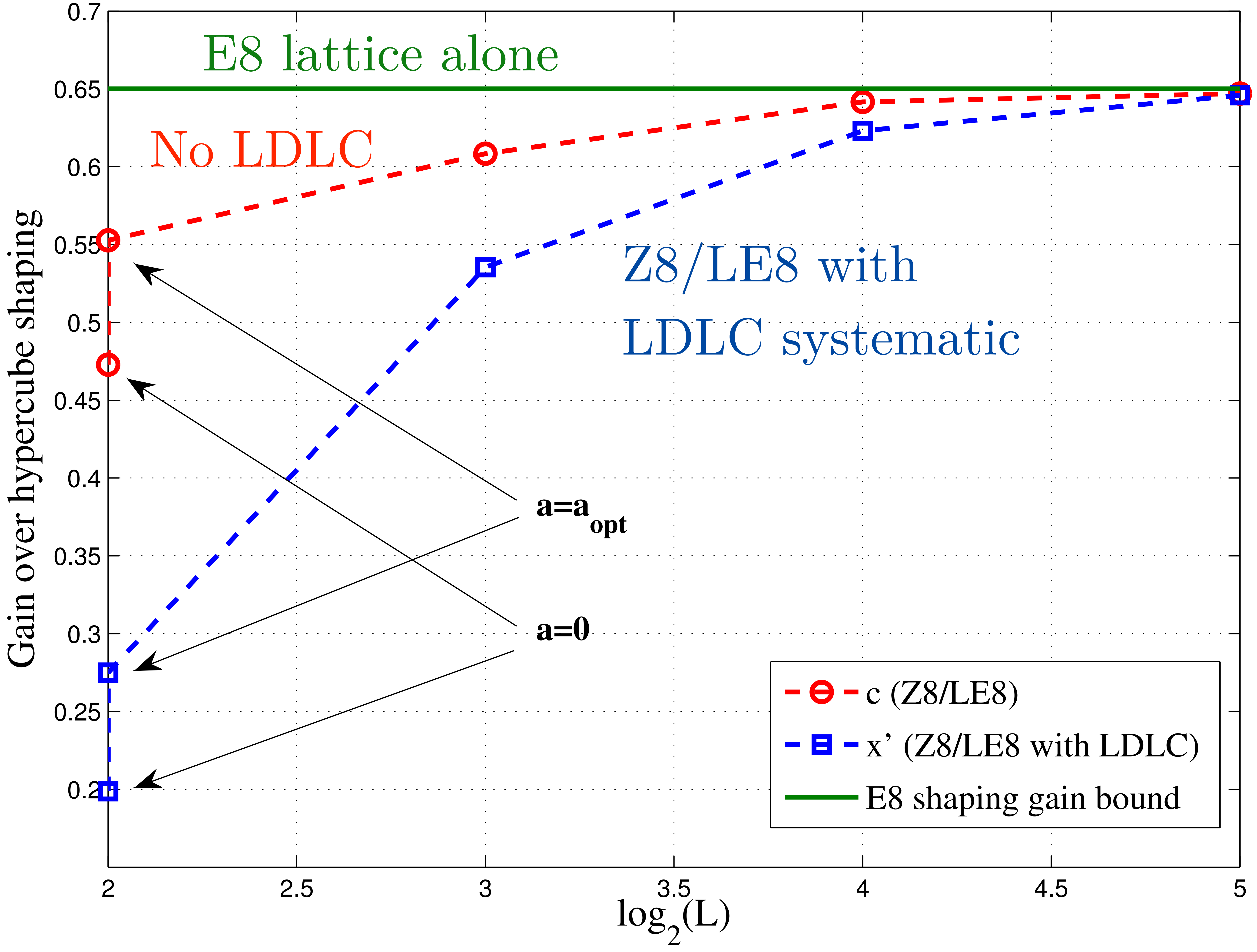
$$k_i = -\left\lfloor \sum_{j=1}^{i-1} H_{i,j} x_j \right\rceil, \tag{4}$$

and

$$x_i = c_i - \left(\sum_{j=1}^{i-1} H_{i,j} x_j - \left\lfloor \sum_{j=1}^{i-1} H_{i,j} x_j \right\rceil \right). \tag{5}$$

Average Transmit Power

Transmit power,
Gain over
hypercube

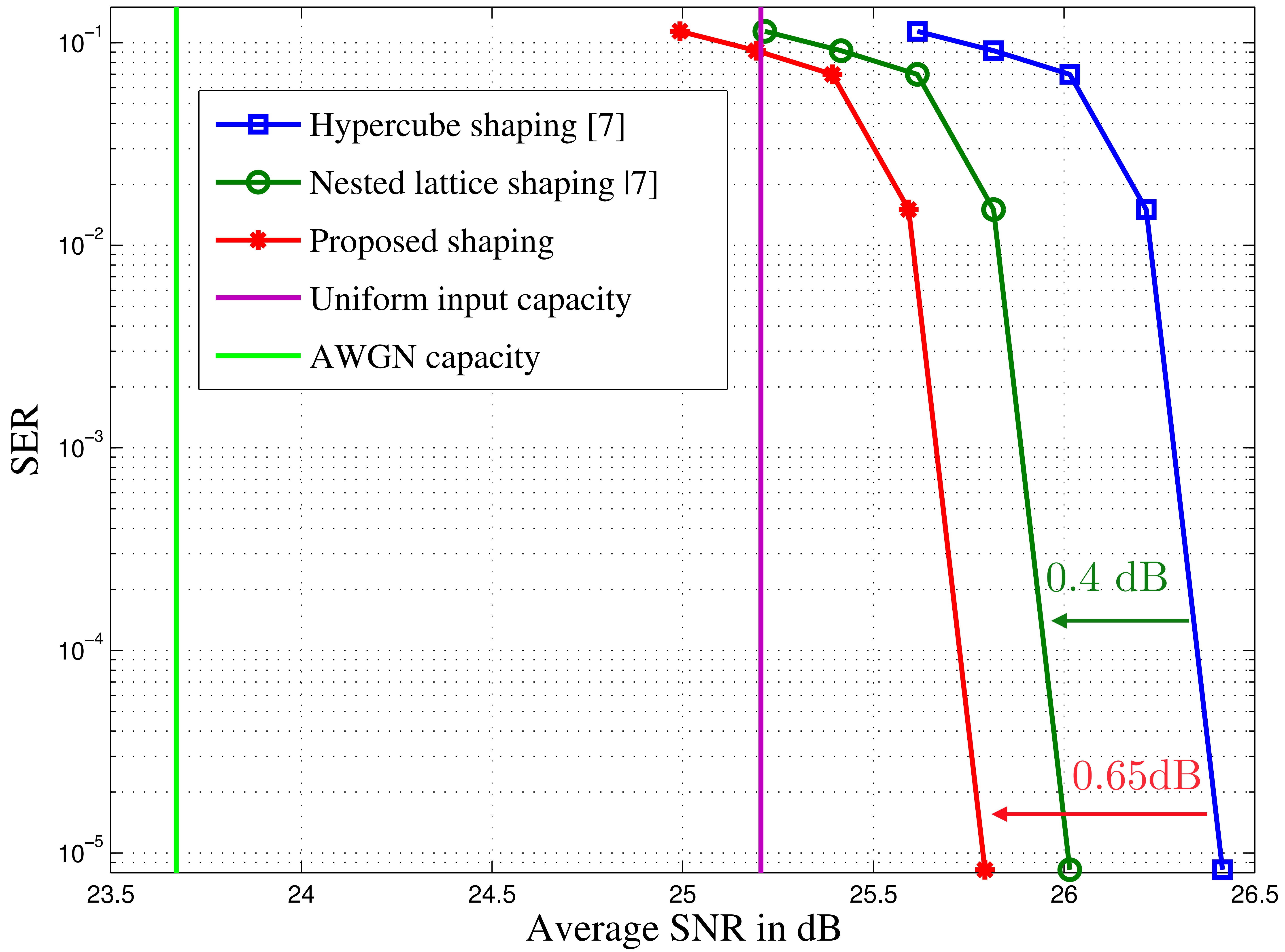


Power-Constrained AWGN Channel

AWGN channel with average power constraint

- 5 bits/dimension
- coding: LDLC lattice dimension $n = 10{,}000$
- shaping: E8 lattice with $m = 8$
- Compare with M-Algorithm LDLC shaping of Sommer et al

0.65 dB Gain Over Hypercube Shaping!



0.15 dB better than
M algorithm,
and much lower complexity

Conclusion

Lattices are an alternative to finite-field codes for AWGN

Shaping techniques to obtain 1.53 dB are "accessible"

- Coset codes/nested lattice codes, high complexity

We proposed:

- "Voronoi integers" using low-dimension lattices
- Systematic lattice shaping for LDLCs

High coding gain of LDLCs, good shaping gain of E8 lattice